Abteilung für
Angewandte Mathematik

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Algorithmic Aspects of Data Analytics and Machine Learning

SS 2025 — Sheet 11

https://aam.uni-freiburg.de/agba/lehre/ss25/algml/index.html

Due: July 18, 2025, 2 p.m.

Task 1 (6 Points)

Let $\emptyset \neq X \subseteq \mathbb{R}$ and let $D = \{(x_i, y_i) \mid i = 1, \ldots, n\} \subseteq X \times \{-1, 1\}$ be a dataset such that $x_i \neq x_j$ for $i \neq j$.

(i) Show that for any injective function $\psi : X \to \mathbb{R}^n$ with $\psi(x_i) = e_i$, the mapped dataset $\hat{D} := \{(\psi(x_i), y_i) \mid i = 1, \ldots, n\} \subseteq \mathbb{R}^n \times \{-1, 1\}$ is linearly separable by guessing the parameters of a classifier $h = \operatorname{sign}(\langle w, \cdot \rangle + b)$ for $\hat{D}$.

(ii) Think of a way to achieve the same using $\mathbb{R}$ instead of $\mathbb{R}^n$.

(iii) For $\sigma > 0$, define the mapping

$$\psi : X \to \mathbb{R}^n, \quad x \mapsto (k(x, x_1), \ldots, k(x, x_n)),$$

where $k(x, x') := \exp\left(-\frac{(x-x')^2}{2\sigma^2}\right)$ is the Gaussian kernel.
Show that there exists $\sigma > 0$ such that the mapped dataset

$$\hat{D} := \{(\psi(x_i), y_i) \mid i = 1, \ldots, n\} \subseteq \mathbb{R}^n \times \{-1, 1\}$$

is linearly separable.

Task 2 (3 Points)

Show that for a kernel $k : X \times X \to \mathbb{R}$, neither the Hilbert space $(H, \langle \cdot, \cdot \rangle)$ nor the feature map $\psi : X \to H$ is unique.

Task 3 (4 Points)

In this exercise, we consider neurons where the activation function is the Heaviside function.

(i) Show that the exclusive-or function $\mathrm{XOR} : \{0, 1\}^2 \to \{0, 1\}$ cannot be represented by a single neuron with two inputs using the Heaviside activation function.

(ii) Provide a representation of XOR using a neural network with (at most) three neurons and Heaviside activation.

Task 4 (3 Points)

Show, using the Stone–Weierstrass theorem, that the space of shallow neural networks with exponential activation,

$$\mathcal{S}^{\exp}(\mathbb{R}) \subseteq C(\mathbb{R}),$$

is uniformly dense in the space of continuous functions on compact subsets of $\mathbb{R}$.
Hint: To prove that $\mathcal{S}^{\exp}(\mathbb{R})$ is an algebra, it is sufficient to check, that $1 \in \mathcal{S}^{\exp}(\mathbb{R})$ and if $f, g \in \mathcal{S}^{\exp}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

$$\alpha f + \beta g \ \in \ \mathcal{S}^{\exp}(\mathbb{R}) \quad \text{and} \quad f \cdot g \ \in \ \mathcal{S}^{\exp}(\mathbb{R}).$$