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Algorithmic Aspects of Data Analytics and Machine Learning

SS 2025 — Sheet 3

`https://aam.uni-freiburg.de/agba/lehre/ss25/algml/index.html`

Due: May 16, 2025, 2 p.m.

Task 1 (6 points)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be twice continous differentiable, convex and $L > 0$. Show that the following three statements are equivalent.

(i) For all $x, y \in \mathbb{R}^d$,
$$\|\nabla f(x) - \nabla f(y)\| \;\le\; L\,\|x - y\|.$$

(ii) For all $x, y \in \mathbb{R}^d$,
$$f(y) \;\le\; f(x) \;+\; \langle \nabla f(x),\, y - x\rangle \;+\; \frac{L}{2}\,\|y - x\|^2.$$

(iii) For every $x \in \mathbb{R}^d$ and for all $v \in \mathbb{R}^d$:
$$\langle \nabla^2 f(x)v, v\rangle \le \; L\|v\|^2.$$

Task 2 (3 points)

(i) Let $f : [a, b] \to \mathbb{R}$ be a convex function. Show that the maximum of f over the interval $[a, b]$ is attained at one of the endpoints, i.e.,
$$\max_{x\in[a,b]} f(x) \in \{f(a), f(b)\}.$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex and non-decreasing function, and let $g : \mathbb{R} \to \mathbb{R}$ be convex. Show that the composition $h(x) := f(g(x))$ is convex.

(iii) Show, that the function $f : \mathbb{R}^d \to \mathbb{R}$ defined by
$$f(x_1, \dots, x_d) = \max_{i=1,\dots,d} x_i$$
is convex.

Task 3 (4 points)

Let the functions $f_i : \mathbb{R} \to \mathbb{R}$ be defined as follows:

- $f_1(x) = x^2$
- $f_2(x) = e^x$
- $f_3(x) = |x| + x^2$
- $f_4(x) = \ln(1 + e^x)$

Which of these functions are convex? Which are $\mu-$ strongly convex and which are $L-$smooth?

Task 4 (3 points)

Let $x, y \in \mathbb{R}^2_{\ge 0} \setminus \{0\} := \{(x, y) \in \mathbb{R}^2 \,|\, x, y \ge 0 \text{ and } (x, y) \ne (0, 0)\}$. The cosine distance between x and y is defined as
$$d_{\cos}(x, y) := 1 - \frac{\langle x, y\rangle}{\|x\|\,\|y\|}.$$

(i) Show that $d_{\cos}$ is not positive definite on $\mathbb{R}^2_{\ge 0} \setminus \{0\}$.

(ii) Show that $d_{\cos}$ does not satisfy the triangle inequality.