Abteilung für
Angewandte Mathematik

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Algorithmic Aspects of Data Analytics and Machine Learning

SS 2025 — Sheet 6

`https://aam.uni-freiburg.de/agba/lehre/ss25/algml/index.html`

Due: June 06, 2025, 2 p.m.

Task 1 (5 Points)

Let $A \in \mathbb{R}^{m\times n}$ be of rank r.

(i) Prove that $A^\top A \in \mathbb{R}^{n\times n}$ and $AA^\top \in \mathbb{R}^{m\times m}$ are symmetric and positive semi-definite. Show that all their eigenvalues are non-negative and that they share the same eigenvalues (including multiplicities, but not necessarily with the same eigenvectors).

(ii) Show that there exists an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ such that

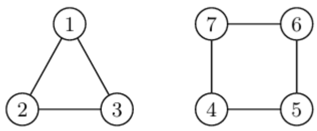
$$V^\top A^\top A V = \Lambda,$$

where Λ is a diagonal matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ on the diagonal.

(iii) Show, that there exist $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ orthogonal, such that $U\Sigma V^\top$, where $\Sigma \in \mathbb{R}^{m\times n}$ is a diagonal matrix with entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and $\sigma_i = 0$ for $i = r+1, \cdots, \min\{m, n\}$.

Task 2 (3 Points)

Determine the Laplacian L, its eigenvalues, and an associated orthonormal system for the following disconnected graph G. Can you deduce any structural properties (e.g., connected components, connectivity strength, etc.) of the Graph from the eigenvectors and the eigenvalues?



Task 3 (5 Points)

We study the steady-state Poisson equation with homogeneous Neumann boundary condition in $\Omega \subset \mathbb{R}^2$:

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

(i) Show that a solution exists only if

$$\int_\Omega f(x)\, dx = 0.$$

Provide a physical interpretation of this condition.

(ii) Now consider the time-dependent heat equation with homogeneous Neumann boundary condition and initial condition $u_0 \in L^2(\Omega)$:

$$\frac{\partial u}{\partial t} - \Delta u = f(x) \quad \text{in } \Omega \times (0,T), \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad u(x,0) = u_0(x) \quad \text{in } \Omega.$$

Prove that the spatial average of the temperature is conserved over time

$$\frac{d}{dt}\int_\Omega u(x,t)\, dx = 0.$$

What does this imply in the context of an insulated plate?

(iii) Show that there exists a constant $\lambda > 0$ such that

$$\|u(\cdot,t) - u_\infty\|_{L^2(\Omega)} \leq C e^{-\lambda t},$$

for some constant $C > 0$ depending on the initial data, where u_∞ denotes a solution of the steady-state Poisson equation.

Task 4 (3 Points)

(i) Let $a, b, c, d \in \mathbb{R}$ such that $a, c \geq 0$ and $b, d > 0$. Show, that

$$\min\left\{\frac{a}{b}, \frac{c}{d}\right\} \leq \frac{a+c}{b+d} \leq \max\left\{\frac{a}{b}, \frac{c}{d}\right\}.$$

(ii) Let $u, v \in \mathbb{R}$. Prove the following inequality

$$(u+v)^2 \leq 2(u^2+v^2).$$

When does equality hold?

Practical exercise

The following exercise is not mandatory; the points are bonus points that you can collect. Please submit your solutions as a MATLAB or Python file by June 6, 2 p.m., via email to tatjana.stiefken@mathematik.uni-freiburg.de. Please comment your code and your results.

Project (4* Points)

(i) Download the provided graph from the course website. Visualize the graph and compute its Laplacian matrix.

(ii) Use the power method with a small $\mu > 0$ to compute the eigenvector corresponding to the second smallest eigenvalue of L:

$$v_{k+1} = \frac{(L - \mu I)^{-1} v_k}{\|(L - \mu I)^{-1} v_k\|}.$$

Ensure that v_k stays orthogonal to the first eigenvector (the constant vector), e.g., by subtracting the mean from v_k in each step. You can use $\mu = 10^{-3}$. Visualize the graph by coloring the nodes according to the entries in the computed eigenvector.

(iii) Write a function that recursively splits the graph based on the sign of the second eigenvector. After removing edges between nodes with different signs, compute the connected components and apply the algorithm to each of them. What are useful stopping criteria for the algorithm?

Hint: You can use the following code snippets:

Listing 1: Import the graph in Matlab

```matlab
edges = readtable('graph_edges.csv');
G = graph(edges.Source, edges.Target);
```

Listing 2: Import the graph in Python

```python
import pandas as pd
import networkx as nx

nodes_df = pd.read_csv("nodes.csv")
edges_df = pd.read_csv("edges.csv")
G = nx.Graph()
G.add_nodes_from(nodes_df['Id'])
G.add_edges_from(edges_df[['Source', 'Target']].values)
```