Prof. Dr. Sören Bartels, Tatjana Schreiber June 18th, 2025

# Algorithmic Aspects of Data Analytics and Machine Learning

SS 2025 — Sheet 8

`https://aam.uni-freiburg.de/agba/lehre/ss25/algml/index.html`

**Due:** June 27, 2025, 2 p.m.

**Task 1** (4 Points)

Let $A \in \mathbb{R}^{m\times n}$ be a real matrix with $\operatorname{rank}(A) = r$ and singular value decomposition $A = U\Sigma V^\top$.

(i) The Frobenius inner product is defined by

$$\langle A, B\rangle_F \;:=\; \operatorname{trace}\bigl(A^\top B\bigr) \;=\; \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}.$$

Prove that $\langle\cdot,\cdot\rangle_F$ is an inner product on $\mathbb{R}^{m\times n}$.

(ii) Show that the induced norm $\|A\|_F = \sqrt{\langle A, A\rangle_F}$ is unitarily invariant, i.e. $\|UAV^\top\|_F = \|A\|_F$ for all orthogonal $U$ and $V$.

Conclude that the Frobenius norm of $A$ can be expressed through its singular values as

$$\|A\|_F \;=\; \sqrt{\sum_{i=1}^{r}\sigma_i^2}.$$

(iii) Show, that

$$\|A\|_2 \;\le\; \|A\|_F \;\le\; \sqrt{\operatorname{rank}(A)}\,\|A\|_2.$$

**Task 2** (4 Points)

Let $A \in \mathbb{R}^{m\times n}$ be a real matrix with $\operatorname{rank}(A) = r$ and singular value decomposition $A = U\Sigma V^\top$.

(i) Denote by $u_i$ and $v_i$ the $i^{\text{th}}$ columns of $U$ and $V$, respectively. Show that the rank-one matrices $u_i v_i^\top$ are pairwise orthogonal with respect to the Frobenius inner product and that

$$\bigl\|u_i v_i^\top\bigr\|_F = 1 \quad \text{for every } i.$$

(ii) For $k < r$ define the truncated matrix

$$A_k \;:=\; \sum_{i=1}^{k}\sigma_i\, u_i v_i^\top.$$

Prove the error identity

$$\|A - A_k\|_F^2 \;=\; \sum_{i=k+1}^{r}\sigma_i^2,$$

and explain briefly why $A_k$ is the *best* rank-$k$ approximation of $A$ with respect to the Frobenius norm.

**Task 3** (4 Points)

Given the data matrix

$$C \;=\; \begin{bmatrix} 1 & 2 & 3 & 4 & 5\\ 1 & 2 & 3 & 5 & 5\end{bmatrix}^\top.$$

(i) Compute the column-wise means and the centered matrix $A_c = A - \mathbf{1}\bar{a}$. Compute the singular value decomposition $A_c = U\Sigma V^\top$.

(ii) Denote by $v_1$ the first singular vector. Project all three centered points onto $v_1$ and form the rank-1 reconstruction $A^{(1)} = U_1\Sigma_1 V_1^\top$. Show that the Frobenius error satisfies $\|A_c - A^{(1)}\|_F^2 = \sigma_2^2$. Discuss when it is sensible to retain only the first principal component.

(iii) Sketch the original data points, the centered data cloud (with the origin as new centroid) and the direction $v_1$ and the projected points.

**Task 4** (4 Points)

Consider the following matrix of mathematics course ratings, where each row corresponds to a student and each column to a course:

| | Logic | Numerics | Linear Algebra | Probability | Analysis |
|---|---|---|---|---|---|
| Anne | 1 | 0 | 2 | 0 | 0 |
| Berti | 0 | 2 | 0 | 2 | 2 |
| Christoph | 4 | 3 | 5 | 1 | 2 |
| Diana | 1 | 4 | 1 | 3 | 4 |
| Elif | 3 | 2 | 3 | 0 | 1 |
| Finn | 0 | 5 | 0 | 5 | 5 |
| Gianni | 2 | 4 | 2 | 1 | 3 |

Let $A \in \mathbb{R}^{7\times 5}$ be the rating matrix. Suppose a new student, Hailey, rates *Logic* with 2 and *Numerics* with 5. Based on the SVD approximation of $A$, which of the other courses would you recommend to Hailey? Justify your recommendation using the SVD structure.

What latent factors might explain the structure of the rating matrix?

**Hint:** You may use (without proof) the SVD of $A$, given by

$$A \approx \begin{bmatrix} -0.07 & -0.26 & -0.46 & -0.61 & -0.07 & -0.30 & 0.49 \\ -0.22 & 0.21 & -0.12 & 0.08 & -0.15 & -0.81 & -0.45 \\ -0.43 & -0.60 & -0.35 & 0.03 & -0.07 & 0.30 & -0.49 \\ -0.46 & 0.20 & 0.06 & -0.18 & 0.84 & 0.00 & 0.00 \\ -0.26 & -0.44 & 0.17 & 0.61 & 0.08 & -0.30 & 0.49 \\ -0.56 & 0.53 & -0.29 & 0.19 & -0.37 & 0.26 & 0.28 \\ -0.41 & -0.12 & 0.73 & -0.42 & -0.33 & 0.00 & 0.00 \end{bmatrix} \cdot \begin{bmatrix} 13.85 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 7.26 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.53 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.49 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.25 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix} \cdot \begin{bmatrix} -0.28 & -0.55 & 0.12 & 0.69 & 0.37 \\ -0.62 & 0.10 & 0.50 & 0.06 & -0.59 \\ -0.31 & -0.67 & -0.41 & -0.49 & -0.19 \\ -0.40 & 0.41 & -0.74 & 0.35 & -0.11 \\ -0.54 & 0.26 & 0.14 & -0.40 & 0.68 \end{bmatrix}^{\top}$$

# Practical exercise

The following exercise is not mandatory; the points are bonus points that you can collect. Please submit your solutions as a MATLAB or Python file by June 27, 2 p.m., via email to tatjana.stiefken@mathematik.uni-freiburg.de. Please comment your code and your results.

**Project** (4* Points)

Download the $320 \times 240$ image matrix $A \in \mathbb{R}^{320\times 240}$ from the course web page (the matrix contains gray–scale values, one pixel per entry). Compute the singular value decomposition (SVD) $A = U\Sigma V^\mathsf{T}$ and, for $k \in \{1, 5, 10, 50\}$, form the truncated matrices

$$A_k \;=\; U_{[:,1:k]}\,\Sigma_{1:k,1:k}\,V^\mathsf{T}_{[:,1:k]}.$$

(i) Plot the original matrix $A$ and the four approximations $A_k$ as gray-scale images.

(ii) For each $k$ compute the retained Frobenius-energy fraction

$$\rho_k \;=\; \frac{\left\|\left[\sigma_1,\cdots,\sigma_k\right]^\top\right\|_2}{\left\|\left[\sigma_1,\cdots,\sigma_r\right]^\top\right\|_2}, \qquad 0 \le \rho_k \le 1,$$

where $r = \operatorname{rank}(A)$. Interpret the numerical value of $\rho_k$ in terms of the visual quality of the reconstruction.