DERIVATION OF THE WAVE EQUATION

SÖREN BARTELS

We consider the planar motion of an elastic string of finite length that occupies the interval $[\alpha, \beta] \times \{0\}$ in its undeformed but stressed reference configuration. The position of the particle occupying initially the position x is at time $t > 0$ given by $y(t, x) \in \mathbb{R}^2$ with a function

$$y : [0, T] \times [\alpha, \beta] \to \mathbb{R}^2,$$

We imagine the string as a chain of elastic springs. Then, the net force acting on a segment $[x_1, x_2] \subset [\alpha, \beta]$ equals the difference of the tension vectors $T(x_2)$ and $T(x_1)$, cf. Fig. 1.

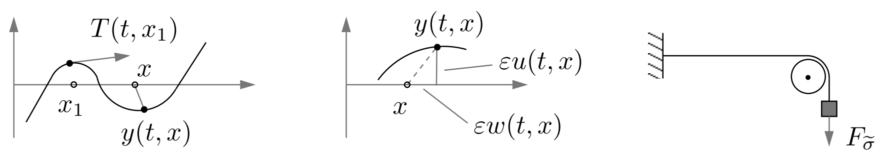

FIGURE 1. Elastic string with tangential tension vector (left); small transversal and longitudinal displacements εu and εw (middle); pre-stressed reference configuration (right).

By Newton's second law $F = ma$ this implies that

$$T(t, x_2) - T(t, x_1) = \int_{x_1}^{x_2} \varrho \, \partial_t^2 y(t, x) \, \mathrm{d}x,$$

where ϱ denotes the constant density, i.e., mass per unit length, of the string. Differentiating this identity with respect to x_2 and subsequently replacing x_2 by x yields that

$$\partial_x T(t, x) = \varrho \partial_t^2 y(t, x).$$

The stress vector T is directed tangentially to the string and its length depends on the elongation $|\partial_x y(t, x)| > 0$, i.e., we have Hooke's law

$$T(t, x) = \sigma\big(|\partial_x y(t, x)|\big) \frac{\partial_x y(t, x)}{|\partial_x y(t, x)|}.$$

We consider the situation that the motion of the string is small and thus assume that with a parameter $0 < \varepsilon \ll 1$ and the scaled transversal and longitudinal displacements $u, w : [0, T] \times [\alpha, \beta] \to \mathbb{R}$, cf. Fig. 1, we have

$$y(t, x) = \begin{bmatrix} y_1(t, x) \\ y_2(t, x) \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} w(t, x) \\ u(t, x) \end{bmatrix}.$$

Date: November 7, 2018.

With the canonical basis vectors $e_1, e_2 \in \mathbb{R}^2$ it follows that

$$\partial_x y(t,x) = e_1 + \varepsilon \begin{bmatrix} \partial_x w(t,x) \\ \partial_x u(t,x) \end{bmatrix}.$$

We linearize the euclidean length and the tangent vector using Taylor expansions at e_1, i.e., for every $z \in \mathbb{R}^2$ we have

$$|e_1 + \varepsilon z| = \big(1 + 2\varepsilon z_1 + \varepsilon^2 |z|^2\big)^{1/2} = 1 + \varepsilon z_1 + \mathcal{O}(\varepsilon^2),$$

and with $f(x) = x/|x|$ expanded at $x = e_1$,

$$\frac{e_1 + \varepsilon z}{|e_1 + \varepsilon z|} = e_1 + \varepsilon (I_2 - e_1 \otimes e_1) z + \mathcal{O}(\varepsilon^2) = e_1 + \varepsilon z_2 e_2 + \mathcal{O}(\varepsilon^2).$$

We deduce, omitting the argument (t,x), assuming that σ is C^2 regular, and neglecting terms of order ε^2, that

$$\begin{aligned} T &= \sigma\big(|\partial_x y|\big) \frac{\partial_x y}{|\partial_x y|} \\ &= \big(\sigma(1) + \sigma'(1)\varepsilon \partial_x w\big)\big(e_1 + \varepsilon \partial_x u e_2\big) \\ &= \sigma(1) e_1 + \sigma'(1) \varepsilon \partial_x w e_1 + \sigma(1) \varepsilon \partial_x u e_2. \end{aligned}$$

The vectorial equation $\partial_x T(t,x) = \varrho \partial_t^2 y(t,x)$ thus reduces with $\widetilde{\sigma} = \sigma(1)$ and $\widetilde{\sigma}' = \sigma'(1)$ to the two independent equations

$$\begin{aligned} \widetilde{\sigma}' \partial_x^2 w(t,x) &= \varrho \partial_t^2 w(t,x), \\ \widetilde{\sigma} \partial_x^2 u(t,x) &= \varrho \partial_t^2 u(t,x). \end{aligned}$$

Each of these equations is called a *wave equation*. The constant $\widetilde{\sigma}$ is the initial tension of the string, cf. Fig. 1. Note that both, transversal and longitudinal motions take place, cf. [Ant80] for further details.

References

[Ant80] Stuart S. Antman, *The equations for large vibrations of strings*, Amer. Math. Monthly **87** (1980), no. 5, 359–370.