Abteilung für
Angewandte Mathematik

Wintersemester 2024/25
Prof. Dr. Sören Bartels
Vera Jackisch, Stefan Kater, Dominik Schneider

Practical Course: Introduction to Theory and Numerics of Partial Differential Equations

Exercise Sheet 1 – 23.10.2024

Submission: via e-mail to the tutor until Wednesday, 06.11.24, 09:00 Uhr

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Projekt 1 (10 points)**.**

(1) Implement a numerical method for solving the transport equation

$$\begin{aligned} \partial_t u + \partial_x u &= 0 \qquad \text{in } (0,T)\times(0,1) \\ u(t,0) &= 0 \\ u(0,x) &= u_0(x) \quad \text{für } x \in [0,1], \end{aligned}$$

where $T = 1$ and $u_0(x) = 1$ for $0.4 \leq x \leq 0.6$ and $u_0(x) = 0$ else. Use the forward difference quotient in time and the backward difference quotient in space. Test the discretization parameters

$$(\Delta t, \Delta x) = 1/80(2,2),\ (\Delta t, \Delta x) = 1/80(2,1),\ (\Delta t, \Delta x) = 1/80(1,2).$$

In each case, check whether the CFL condition is fulfilled and compare the numerical solution with the exact solution of the transport equation.

(2) Modify your code so that you can use the numerical method for

$$\partial_t u + a(x)\partial_x u = 0,$$

where $a : (0,1) \to \mathbb{R}_{\geq 0}$ is a given function. How should the CFL condition be formulated for non-constant functions a? Test your code in the case $a(x) = (1+4x^2)^{1/2}$ and initial conditions $u_0(x) = 1$ for $0.05 \leq x \leq 0.25$ and $u_0(x) = 0$ else. Compare the numerical solutions for different discretization parameters.

(3) Test the program for $a(x) = -1$ and initial conditions as in (1). Are there pairs of discretization parameters that satisfy the CFL condition?

(4) Change your code so that only forward difference quotients are used. Let the boundary condition be given as $u(t,1) = 0$ for $t \in [0,T]$. Derive the CFL condition for this method and try out different values for Δt and Δx.

Projekt 2 (10 points)**.**
The *Upwind method* for the transport equation is defined by

$$U_j^{k+1} = \begin{cases} (1-\mu_j^k)U_j^k + \mu_j^k U_{j-1}^k, & \mu_j^k \geq 0, \\ (1+\mu_j^k)U_j^k - \mu_j^k U_{j+1}^k, & \mu_j^k < 0, \end{cases}$$

with $\mu_j^k = a(t_k.x_j)\Delta t/\Delta x$. Implement this method and test it for different initial conditions, different discretization parameters, the function $a(x) = \sin(x)$ and boundary conditions defined by $u(t,0) = u(t,1) = 0$. Discuss your results and the validity of a CFL condition.