Abteilung für
Angewandte Mathematik

Wintersemester 2024/25
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Practical Course: Introduction to Theory and Numerics of Partial Differential Equations

Exercise Sheet 2 – 06.11.2024

Submission: via E-Mail to the tutor until Wednesday, 20.11.24, 09:00

Projekt 1 (10 points). For $T > 0$ we consider the heat equation $\partial_t u - \kappa \partial_x^2 u = 0$ on $(0,T) \times (-1,1)$ with initial value $u(0,x) = \cos((\pi/2)x)$, boundary conditions $u(t,-1) = u(t,1) = 0$ and thermal diffusivity $\kappa = 1/200$.
(i) Implement the θ-method to solve the heat equation. Choose $\Delta x = 1/20$ and experimentally determine the larges time step size Δt, for which the θ-method is stable with $\theta = 0$.
(ii) Prove that the exact solution to the problem is given by $u(t,x) = \cos((\pi/2)x)e^{-(\kappa\pi^2/4)t}$. Compute the approximation error of the θ-method at $(t,x) = (1,0)$ for $\theta = 1/2,\ 3/4,\ 1$ and $\Delta x = \Delta t = 2^{-j}/10$, $j = 2,3,\ldots,5$. Plot all the results in one window. Use `semilogy` instead of `plot`, to get logarithmic scaling of the y-axis. Interpret the results.
(iii) Modify your code to solve the heat equation with source term f on the right-hand side, i.e. $\partial_t u - \kappa \partial_x^2 u = f$ on $(0,T) \times (-1,1)$, where $f(t,x) = (1/20)x^2$. Compute the approximate solutions for homogeneous Dirichlet boundary conditions and starting value $u_0(x) = 1$, if $-0.1 \le x \le 0.1$, and $u_0(x) = 0$, else. Compare the numerical solutions for various discretization parameters and for $\theta = 0$, $\theta = 1/2$ and $\theta = 1$.

Projekt 2 (10 Punkte). (i) Modify your MATLAB-program from Projekt 1 to solve the homogeneous heat equation $\partial_t u - \kappa \partial_x^2 u = 0$ on $(0,5) \times (-1,1)$ with $\kappa = 1/10$, starting value

$$u_0(x) = \begin{cases} \exp\left(-\frac{1}{4(0.5+x)(0.5-x)}\right), & \text{if } |x| < 0.5, \\ 0, & \text{else,} \end{cases}$$

and Neumann boundary conditions $\partial_x u(t,-1) = g_l(t)$ and $\partial_x u(t,1) = g_r(t)$ for $t \in (0,5]$ using the Crank-Nicolson scheme ($\theta = 1/2$). Use the difference quotients $\partial_x^+ U_0^{k+1}$ and $\partial_x^- U_J^{k+1}$ to approximate the partial derivatives $\partial_x u(t_{k+1},-1)$ and $\partial_x u(t_{k+1},1)$. Test your program with homogeneous Neumann boundary conditions and discretization parameters $\Delta x = \Delta t = 2^{-j}/10$ for $j = 2,3,\ldots,5$. Does the numerical scheme yield a sensible solution? Compute the initial total mass $\int_{-1}^{1} u_0(x)\,\mathrm{d}x$ using the MATLAB-function `trapz` with $10^3 + 1$ grid points. Compare it with the total mass of the discrete solutions at $t = 5$. For this use the display format `long`.
(ii) Now use the central difference quotients $\widehat{\partial}_x U_0^{k+1}$ and $\widehat{\partial}_x U_J^{k+1}$ to approximate the derivatives $\partial_x u(t_{k+1},-1)$ and $\partial_x u(t_{k+1},1)$. For this approximation to be well defined on the boundary, so-called *ghost points* $x_{-1} = -1 - \Delta x$ and $x_{J+1} = 1 + \Delta x$ have to be introduced. The values U_{-1}^0 and U_{J+1}^0 are calculated from the given initial values $(U_j^0)_{j=0,\ldots,J}$ and the discrete Neumann boundary conditions at $t = 0$. Just like in (i), compute the total mass of the discrete solutions for $\Delta x = \Delta t = 2^{-j}/10$ for $j = 2,3,\ldots,5$ at $T = 5$, as well as the difference from the initial total mass. Compare the results with the results from (i). What do you notice? Why is it sensible to use $\widehat{\partial}_x U_0^{k+1}$ and $\widehat{\partial}_x U_J^{k+1}$ instead of $\partial_x^+ U_0^{k+1}$ and $\partial_x^- U_J^{k+1}$ to realize Neumann boundary conditions?