Abteilung für
Angewandte Mathematik

Wintersemester 2024/25
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Practical Course: Introduction to Theory and Numerics of Partial Differential Equations

Exercise Sheet 3 – 20.11.2024

Submission: via E-Mail to the tutor until Wednesday, 04.12.24, 09:00

Project 1 (10 points)**.**

(1) Implement the unconditionally stable scheme

$$\partial_t^+ \partial_t^- U_j^n = \frac{1}{4} \partial_x^+ \partial_x^- (U_j^{n+1} + 2U_j^n + U_j^{n-1})$$

to numerically solve the wave equation $\partial_t^2 u - \partial_x^2 u = 0$ in $(0,T) \times (0,1)$ with homogeneous Dirichlet boundary conditions and initial conditions $u(x,0) = u_0(x)$ and $\partial_t u(0,x) = v_0(x)$ for $x \in (0,1)$ and given functions $u_0, v_0 \in C([0,1])$. Use the central difference quotient and ghost points at $t_{-1} = -\Delta t$ to discretize the initial condition $\partial_t u(0,x) = v_0(x)$ so you can expect quadratic convergence. Test your program with the exact solution $u(t,x) = \cos(\pi t)\sin(\pi x)$. Create plots that show the quadratic convergence with respect to the discretization parameters.

(2) Verify experimentally that the discrete energy

$$\Gamma^k = \frac{\Delta x}{2} \sum_{j=1}^{J-1} |\partial_t^+ U_j^k|^2 + \frac{\Delta x}{2} \sum_{j=1}^{J} |\partial_x^- U_j^{k+1/2}|^2$$

is independent of $k = 0, \ldots K-1$, where $U_j^{k+1/2} = (U_j^{k+1} + U_j^k)/2$.

Project 2 (10 points)**.** The sound of a string instrument is characterized by the appearance of various overtones. To experimentally verify that the wave equation describes such effects, we consider a string of length $\ell > 0$ that is plucked at a point $x_p \in (0,\ell)$ at time $t = 0$ and is thereby deflected from its initial position by a distance $H > 0$. The initial state of the string is therefore given by

$$u_0(x) = \begin{cases} Hx/x_p & \text{for } x \le x_p, \\ H(\ell - x)/(\ell - x_p) & \text{for } x > x_p. \end{cases}$$

We assume that the sound is picked up at a point $x_s \in (0,\ell)$, for example through the sound hole in the resonator of an acoustic guitar or an electromagnetic pickup in the case of an electric guitar. Assuming that the initial velocity of the string is 0, we can separate the variables in the wave equation to obtain the exact solution

$$u(t,x) = \sum_{m=1}^{\infty} \beta_m \cos(\omega_m t) \sin(m\pi x)$$

with $\omega_m = m\pi c$ and $c = (\sigma/\varrho)^{1/2}$, where ϱ is the density and σ the tension of the string. Use your program from Project 1 to compute the numerical solution to the wave equation with $c = 2$, $T = 2$, $x_p = 1/8$ and $H = 1/100$. Use this approximation and the MATLAB-function `dct` to compute the coefficients α_m, $m = 1, 2, \ldots, K$, of the discrete cos transform such that

$$U_{j_s}^k = \sum_{m=1}^{K} \alpha_m \cos(\omega_m t_k),$$

where j_s is the index corresponding to the node $x_s = 1/4$. Create plots of the oscillations $w_m(t) = \alpha_m \cos(\omega_m t)$, $m = 1, 2, \ldots, 6$, as functions of $t \in [0, T]$. Illustrate the dominant overtones by plotting the distribution of amplitudes in the form of the function $m \mapsto |\alpha_m|$. Compare these results with corresponding results for other values of x_p and x_s.