Abteilung für
Angewandte Mathematik

Wintersemester 2024/25
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Practical Course: Introduction to the Theory and Numerics of Partial Differential Equations

Exercise Sheet 4 – 04.12.2021

Submission: via E-Mail to the tutor until Wednesday, 18.12.24, 9:00

Projekt 1 (5 points)**.** Compute approximate solutions $u_m \in \mathcal{P}_m([0,1])$ to the one-dimensional Poisson problem $-u'' = f$ in $\Omega = (0,1)$ with boundary conditions $u(0) = u(1) = 1$ by numerically solving the system of equations

$$-u_m''(x_i) = f(x_i), \quad i = 1, 2, \dots, m-1, \quad u_m(x_0) = u_m(x_m) = 1,$$

where $x_i = i/m$ for $i = 0, 1, \dots, m$. Test the method for $f(x) = 1$ and $f(x) = \mathrm{sign}(x - 1/2)$. Analyze the behavior of the error $\max_{i=0,\dots,m} |u(x_i) - u_m(x_i)|$ as well as the condition number of the system of equations as $m \to \infty$.

Projekt 2 (10 points)**.** Define functions f, g, and u_D such that $u(x,y) = \sin(\pi x)\sin(\pi y)$ is a solution of the boundary value problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega = (0,1)^2, \\ u &= u_D && \text{on } \Gamma_D = \{0\} \times [0,1], \\ \partial_n u &= g && \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D \end{aligned}$$

Solve the problem numerically using a finite difference method. Implement the Neumann boundary conditions by introducing suitable ghost points and approximating the normal derivatives on Γ_N with central difference quotients. Compare your numerical solutions with the exact solution u and verify the quadratic convergence of the method.

Projekt 3 (10 points)**.**

(1) For the domain $\Omega = (0,1)^2$, a triangulation is given by $\mathcal{T}_h = \{T_1, T_2\}$ with $T_1 = \mathrm{conv}\{(0,0), (1,0), (1,1)\}$ and $T_2 = \mathrm{conv}\{(0,0), (0,1), (1,1)\}$. Approximate the integral $\int_\Omega f_i \,\mathrm{d}x$ for the functions $f_1(x) = x_1^2 x_2^2$ and $f_2(x) = \mathrm{e}^x + \mathrm{e}^y$ using quadrature over the triangle midpoints, the triangle vertices, and with a 3-point Gaussian quadrature.

(2) Using the program `red_refine`, refine the triangulation up to five times. Approximate the integral with the refined triangulation and determine the experimental convergence orders.