Abteilung für
Angewandte Mathematik

Wintersemester 2024/25
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Practical Course: Introduction to the Theory and Numerics of Partial Differential Equations

Exercise Sheet 5 - 18.12.2024

Submission: via E-Mail to the tutor until Wednesday, 08.01.25, 9:00

Projekt 1 (15 points)**.** Due to challenging exercises, a group of students have no time and must resort to eating frozen pizza. The interior of their oven is represented by the region

$$\Omega = (0, 0.4) \times (0, 0.3) \times (0, 0.4).$$

The temperature distribution in the oven is denoted by θ. At the rear side $y = 0$ of the oven, the temperature is constantly $200°$C. At the front side $y = 0.3$ is the door. If the door is open, the temperature there matches the ambient temperature ($20°$C). If it is closed, the oven is perfectly insulated at this boundary, as well as at the other two sides $x = 0$ and $x = 0.4$, i.e., $\partial_n \theta = 0$. Additionally, the oven is well preheated: at time $t = 0$, the temperature everywhere is $200°$C.

However, the students cannot agree on which procedure is more energy-efficient:

(i) The oven is opened for 30 seconds, then closed for 30 seconds, and finally opened again for 30 seconds;

(ii) The oven is closed for 30 seconds and then opened continuously for 60 seconds.

The students know that a mathematical model can be derived by leveraging the facts that the heat density w is proportional to the temperature θ, i.e., $w = \rho c_p \theta$, and the heat flux is proportional to the temperature gradient, i.e., $q = -\kappa \nabla \theta$, and that thermal energy is conserved overall, i.e., $\partial_t w + \operatorname{div} q = 0$. Using the internet, they discover that reasonable parameters for the model can be approximated using constants corresponding to air, namely the density $\rho = 1.2041\,\text{kg/m}^3$, the thermal conductivity $\kappa = 0.0262\,\text{W/(m K)}$, and the specific heat capacity $c_p = 1.005 \times 10^3\,\text{J/(kg K)}$.

Together, they deduce that a dimensional reduction can be performed by replacing θ with the averaged value

$$\widetilde{\theta}(t, x, y) = 0.4^{-1} \int_0^{0.4} \theta(t, x, y, z)\,\mathrm{d}z.$$

However, none of them are able to use this information to resolve the dispute.

Formulate an initial-boundary value problem to describe the averaged temperature distribution $\widetilde{\theta}$ in $\widetilde{\Omega} = (0, 0.4) \times (0, 0.3)$. Implement a Crank-Nicolson scheme to solve the problem and simulate scenarios (i) and (ii). Based on your simulation, decide whether it is more energy-efficient to open the oven once for a longer period or twice for shorter periods. Discuss the weaknesses of the model and the numerical method.

Projekt 2 (10 points)**.** Modify the MATLAB program using the $P1$ finite element method for the Poisson problem so that the boundary value problem

$$-\operatorname{div}(K \nabla u) = f \text{ in } \Omega, \quad u = u_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}}, \quad (K \nabla u) \cdot n = g \text{ on } \Gamma_{\mathrm{N}},$$

is solved. Here, let $K : \Omega \to \mathbb{R}^{d \times d}$ be a piecewise continuous mapping such that $K(x)$ is symmetric and positive definite for almost all $x \in \Omega$. Test your code with $\Omega = (0, 1) \times (0, 2)$, $\Gamma_{\mathrm{N}} = \{1\} \times (0, 2)$, $\Gamma_{\mathrm{D}} = \partial\Omega \setminus \Gamma_{\mathrm{N}}$, $u(x, y) = x^2 y$ and

$$K(x, y) = \begin{bmatrix} 2 & \sin(x) \\ \sin(x) & 2 \end{bmatrix}.$$