Abteilung für
Angewandte Mathematik

Wintersemester 2024/25
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Practical Course: Introduction to the Theory and Numerics of Partial Differential Equations

Exercise Sheet 6 – 15.01.2024

Submission: via E-Mail to the tutor until Wednesday, 29.01.2025

Projekt 1 (15 Punkte)**.** We consider a beverage can in a refrigerator and aim to determine the time required to cool the beer inside the can to a given temperature. We assume that the thin aluminum sheet from which the can is made permanently maintains the same temperature as its surroundings in the refrigerator during the time under consideration.
To derive a model that describes the temperature change of the beer, we use the following principles: the heat density w is proportional to the temperature θ, i.e., $w = \rho c_p \theta$; the heat flux q is proportional to the temperature gradient $\nabla\theta$, i.e., $q = -\kappa\nabla\theta$; and the thermal energy is conserved, i.e., $\partial_t w + \operatorname{div} q = 0$ (see Sheet 5, Project 1).
Use the following values for the density, thermal conductivity, and specific heat capacity of the beer:

$$\rho = 1.009 \times 10^3\,\mathrm{kg/m^3}, \quad \kappa = 0.597\,\mathrm{W/(m\,K)}, \quad c_p = 4.186 \times 10^3\,\mathrm{J/(kg\,K)}.$$

A typical beer can (500 ml) has a diameter of $0.067\,\mathrm{m}$ and a height of $0.168\,\mathrm{m}$. For the simulation, we assume that the can is exactly a cylinder with these dimensions.
Assume that the can stands upright in the refrigerator with a linearly varying surrounding temperature of 1.5° C at the top and 0.5° C at the bottom of the can.
Implement a P1 finite element method with a Crank-Nicolson scheme for time discretization and experimentally determine the time required to cool the beer from an initial temperature of 20° C to a drinking temperature of 3° C.
Discuss the reliability of the result as well as the limitations of the mathematical model.
Note: On the lecture homepage, you can find a MATLAB file with a triangulation of a cylinder of the appropriate size.

Projekt 2 (5 Punkte)**.** Modify the $P1$ finite element method for the Poisson problem from Sheet 5, Project 2, so that it now solves the Robin boundary value problem

$$-\Delta u = f \text{ in } \Omega, \quad u + \alpha\partial_n u = g \text{ on } \partial\Omega,$$

Test your code with $\Omega = (0,1)^2$, $\alpha = 2$, and $u(x,y) = x^2 + y^2$.