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15.10.2024

Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 1

Lecture homepage:

`https://aam.uni-freiburg.de/agba/lehre/ws24/tun0`

Exercise 1 (5 points)

Derive a partial differential equation that describes the transport of a substance through a long, thin tube that allows for the injection of a substance at any time $t \in [0, T]$ and any position $x \in \mathbb{R}$ described through a function $f(t, x)$ that specifies the number of injected particles per unit volume.

Exercise 2 (2+3 points)

Let u solve the partial differential equation

$$\partial_t u + a(t, x)\partial_x u = 0.$$

(i) Show that u is constant along curves $(t, y(t))$ for solutions of the initial boundary value problems $y'(t) = a(t, y(t))$, $y(0) = x_0$, called characteristics.

(ii) Determine the characteristics for the equation for $a(t, x) = tx$ and for $a(t, x) = 2t$, sketch them, and determine the solution for the initial condition $u_0(x) = \cos(x)$.

Exercise 3 (2+3 points)

(i) Show that $\partial^+\partial^- = \partial^-\partial^+$.

(ii) Let $I \subset \mathbb{R}$ be a closed interval and $u \in C^k(I)$, with $k \geq 0$ sufficiently large. Prove the following estimates for difference quotients:

$$|\hat{\partial} u(x_j) - u'(x_j)| \leq \frac{\Delta x^2}{6} \|u'''\|_{C([0,1])},$$
$$|\partial^+\partial^- u(x_j) - u''(x_j)| \leq \frac{\Delta x^2}{12} \|u^{(4)}\|_{C([0,1])}.$$

Show that these estimates do not hold if u does not satisfy the required differentiability properties.

Exercise 4 (5 points)

Show that the upwinding scheme for the transport equation is equivalent to the scheme

$$\partial_t^+ U_j^k + a_j^k \hat{\partial}_x U_j^k = |a_j^k| \frac{\Delta x}{2} \partial_x^+ \partial_x^- U_j^k.$$

Deadline: Tuesday, 22.10.2024, 10 am (in the postbox).