Abteilung für
Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 2

Exercise 1 (2+3 points)

(i) Show by constructing appropriate initial data that the difference scheme $U_j^{k+1} = U_j^k - \mu(U_j^k - U_{j-1}^k)$ with $\mu = a\Delta t/\Delta x$ is unstable if $\mu > 1$.

(ii) Check the CFL condition and the estimate $\sup_{j=0,\dots,J} |U_j^{k+1}| \le \sup_{j=0,\dots,J} |U_j^k|$ of the following difference schemes for the transport equation:

$$\partial_t^+ U_j^k - \partial_x^- U_j^k = 0, \quad \partial_t^+ U_j^k + \partial_x^+ U_j^k = 0, \quad \partial_t^+ U_j^k + \hat{\partial}_x U_j^k = 0.$$

Exercise 2 (2+3 points)

(i) Show that the functions $\phi_k(x) = e^{ikx}$, $x \in [-\pi, \pi]$, $k \in \mathbb{Z}$, define an orthonormal system in $L^2(-\pi, \pi)$, i.e., for all $k, \ell \in \mathbb{Z}$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_k(x)\overline{\phi_\ell(x)}\mathrm{d}x = \delta_{k\ell}.$$

(ii) For $f \in L^2(-\pi, \pi)$ and $k \in \mathbb{Z}$ set $f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{\phi_k(x)}\mathrm{d}x$. Prove *Bessel's inequality*

$$\sum_{k \in \mathbb{Z}} |f_k|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \mathrm{d}x.$$

Remark: Since the orthonormal system above is complete, even equality can be shown in this case ('*Parseval's theorem*').

Exercise 3 (5 points)

Let $u \in C^2([0,T] \times [\alpha, \beta])$ solve the heat equation $\partial_t u - \kappa \partial_x^2 u = 0$. Show that for appropriate $\tau, L, x_0 > 0$, the function $\tilde{u}(s,y) = u(\tau s, Ly + x_0)$ solves $\partial_s \tilde{u} - \partial_y^2 \tilde{u} = 0$ in $(0, T') \times (0, 1)$.

Exercise 4 (5 points)

Derive a mathematical model for a diffusion process that includes sinks and sources of the diffusing substance, described by a function $f \in C([0,T] \times [0,1])$.

Remark: 'Sinks' and 'sources' in this case mean that at points $x \in [0,1]$ additional mass can flow out or in. In the heat equation derived in the lecture, this was only the case at the ends of the interval.

Deadline: Tuesday, 29.10.2024, 10 am (in the postbox).