Abteilung für Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 3

Exercise 1 (5 points)
Using Fourier stability analysis, show that the finite difference method for approximating the transport equation $\partial_t u + a\partial_x u = 0$ with $a \neq 0$ when using the forward difference quotient in time and the central difference quotient in space is always (regardless of the stepsizes) *unstable*.

Exercise 2 (3+2 points)
The construction of a solution via a *separation of variables* consists in finding functions $u_n(t,x) = v_n(t)w_n(x)$ that solve the heat equation and the prescribed boundary conditions. A solution of the initial boundary value problem is then obtained by determining coefficients $(\alpha_n)_{n\in\mathbb{N}}$ such that

$$u(t,x) = \sum_{n=1}^{\infty} \alpha_n v_n(t) w_n(x)$$

converges in an appropriate sense and satisfies $u(0,x) = u_0(x)$.
(i) Construct pairs (v_n, w_n) such that $u_n(t,x) = v_n(t)w_n(x)$ satisfies $\partial_t u_n - \partial_x^2 u_n = 0$ in $(0,T) \times (0,1)$ and $u_n(t,0) = u_n(t,1) = 0$ for all $t \in (0,T)$.
(ii) Assume that the function $u_0 \in C([0,1])$ is given as

$$u_0(x) = \sum_{n=1}^{\infty} \gamma_n \sin(n\pi x).$$

Construct the solution of the corresponding initial boundary value problem for the heat equation.
Remark: It can be shown that every function $u_0 \in C([0,1])$ can be represented in the specified form.

Exercise 3 (5 points)
Show formally that the function

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{|x-y|^2}{4t}\right) u_0(y)\mathrm{d}y$$

solves the heat equation $\partial_t u - \partial_x^2 u = 0$ in $(0,T) \times \mathbb{R}$ for every $T > 0$.

Exercise 4 (5 points)
Let $u \in C^2([0,T] \times [0,1])$ solve the heat equation $\partial_t u - \partial_x^2 u = 0$ with homogeneous Dirichlet boundary conditions. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^1 \left(\partial_x u(t,x)\right)^2 \mathrm{d}x\right) \leq 0$$

and deduce the uniqueness of solutions for the heat equation with general (inhomogeneous) Dirichlet boundary conditions.

Deadline: Tuesday, 05.11.2024, 10 am (in the postbox).