Abteilung für
Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 4

Exercise 1 (5 points)

For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, let $A \in \mathbb{R}^{n\times n}$ be the bandmatrix

$$A = \begin{bmatrix} a & b & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & b & a \end{bmatrix}.$$

Show that A has the eigenvalues $\lambda_p = a + 2b\cos(p\pi/(n+1))$, $p = 1, 2, \ldots, n$ and that corresponding eigenvectors $v_p \in \mathbb{R}^n$ are given by $(v_p)_j = \sin(pj\pi/(n+1))$, $j = 1, 2, \ldots, n$.

Exercise 2 (2+3 points)

(i) Show that the θ-method is well defined for every choice of θ and every choice of Δt, $\Delta x > 0$.

(ii) Show that the θ-method is unstable if $\theta < 1/2$ and $\lambda = \Delta t/\Delta x^2 > \frac{1}{2}\frac{1}{1-2\theta}$.

Exercise 3 (3+2 points)

Let $u \in C^2([0,T]\times\mathbb{R})$ solve the wave equation $\partial_t^2 u - c^2\partial_x^2 u = 0$ with initial conditions $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = v_0(x)$ for all $x \in \mathbb{R}$.

(i) By introducing the variables $\xi = x + ct$ and $\eta = x - ct$, show that the function $\tilde{u}(\xi,\eta) = u(t,x)$ satisfies $\partial_\xi\partial_\eta\tilde{u} = 0$.

(ii) Deduce that $\tilde{u}(\xi,\eta) = f(\xi) + g(\eta)$ and conclude that there exist functions $f, g \in C(\mathbb{R})$ such that $u(t,x) = f(x+ct) + g(x-ct)$. Determine f and g in terms of u_0 and v_0.

Exercise 4 (3+2 points)

(i) Prove the energy conservation principle for the wave equation with homogeneous Neumann boundary conditions.

(ii) Deduce uniqueness for solutions of the wave equation with homogeneous Dirichlet boundary conditions and for solutions of the wave equation with homogeneous Neumann boundary conditions.

Deadline: Tuesday, 12.11.2024, 10 am (in the postbox).