Abteilung für
Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 5

Exercise 1 (3+2 points)

(i) Determine functions $u_n(t,x) = v_n(t)w_n(x)$, $n \in \mathbb{N}$, that satisfy the wave equation in $(0,T) \times (0,1)$ subject to homogeneous Dirichlet boundary conditions.

(ii) Assume that $u_0, v_0 \in C([0,1])$ satisfy

$$u_0(x) = \sum_{n\in\mathbb{N}} \alpha_n \sin(n\pi x), \quad v_0(x) = \sum_{n\in\mathbb{N}} \beta_n \sin(n\pi x),$$

with given sequences $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$. Derive a representation formula for the solution of the wave equation $\partial_t^2 u - c^2\partial_x^2 u = 0$ in $(0,T) \times (0,1)$ with homogeneous Dirichlet boundary conditions and initial conditions $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = v_0(x)$ for all $x \in [0,1]$.

Exercise 2 (5 points)

Show that the implicit difference scheme for the wave equation has a consistency error of order $\mathcal{O}(\Delta t^2 + \Delta x^2)$.

Exercise 3 (2+3 points)

For $J \in \mathbb{N}$, let $\Delta x = 1/J$ and let $V, W \in \mathbb{R}^{J+1}$.

(i) Prove the discrete product rule

$$\partial_x^-(W_jV_j) = W_j(\partial_x^- V_j) + (\partial_x^+ W_{j-1})V_{j-1}.$$

(ii) Deduce the summation-by-parts formula

$$\Delta x \sum_{j=0}^{J-1} (\partial_x^+ W_j)V_j = -\Delta x \sum_{j=1}^{J} W_j(\partial_x^- V_j) + W_J V_J - W_0 V_0,$$

and explain its relation to the integration-by-parts formula.

Exercise 4 (5 points)

Let $(\xi_k)_{k\in\mathbb{N}_0}$ be a sequence of real numbers that satisfies the recursion

$$\begin{bmatrix} \xi_k \\ \xi_{k+1} \end{bmatrix} = A \begin{bmatrix} \xi_{k-1} \\ \xi_k \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix}$$

for $\alpha, \beta \in \mathbb{R}$ and for all $k \in \mathbb{N}$. Assume the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ of A satisfy $|\lambda_i| < 1$, $i = 1,2$. Show that there exists a $c > 0$ such that $|\xi_k| \le c$ for all $k \in \mathbb{N}_0$.

Deadline: Tuesday, 19.11.2024, 10 am (in the postbox).