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19.11.2024

Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 6

Exercise 1 (1+1+1+2 points)

Let $\Omega \subset \mathbb{R}^3$ be open and let $v : \Omega \to \mathbb{R}^3$ and $f : \Omega \to \mathbb{R}$ be twice continuously (partially) differentiable. We denote with $\partial_i = \partial_{x_i}$ the i-th partial derivative. Use the definitions

$$\operatorname{div}\, v = \sum_i \partial_i v_i, \quad \operatorname{grad}\, f = [\partial_1 f, \partial_2 f, \partial_3 f]^\top, \quad \Delta f = \sum_i \partial_i^2 f,$$
$$\operatorname{rot}\, v = [\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1]^\top, \quad \Delta v = [\Delta v_1, \Delta v_2, \Delta v_3]^\top,$$

to prove the following identities:

(i) $\operatorname{div}\operatorname{grad} f = \Delta f$, (ii) $\operatorname{rot}\operatorname{grad} f = 0$, (iii) $\operatorname{div}\operatorname{rot} v = 0$, (iv) $\operatorname{rot}\operatorname{rot} v = \operatorname{grad}\operatorname{div} v - \Delta v$.

Exercise 2 (2+3 points)

(i) For a domain $\Omega \subset \mathbb{R}^d$ let n be the outer unit normal on $\partial\Omega$. Use Gauss's theorem to show that for $u, v \in C^2(\overline{\Omega})$, we have

$$\int_{\partial\Omega} v \nabla u \cdot n \,\mathrm{d}s = \int_\Omega (\nabla u \cdot \nabla v + v \Delta u) \mathrm{d}x,$$
$$\int_\Omega (u\Delta v - v\Delta u)\mathrm{d}x = \int_{\partial\Omega} (u \nabla v \cdot n - v \nabla u \cdot n)\mathrm{d}s.$$

(ii) Let $u_1, u_2 \in C^2(\overline{\Omega})$ be solutions of the boundary value problem $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$. Show that

$$\int_\Omega |\nabla(u_1 - u_2)|^2 \mathrm{d}x = 0$$

and conclude that $u_1 = u_2$.

Exercise 3 (5 points)

Let S^n be the area of the n-dimensional unit sphere (e.g. we have $S^1 = 2\pi$ for the circumference of the unit circle and $S^2 = 4\pi$ for the surface area of the unit sphere). For $n \geq 2$ let the function $s_n : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ be defined by

$$s_n(x) = -\frac{1}{S^{n-1}} \begin{cases} \ln |x| & \text{if } n = 2, \\ \frac{|x|^{2-n}}{(2-n)} & \text{if } n > 2. \end{cases}$$

Show that $\Delta s_n = 0$ pointwise in $\mathbb{R}^n \backslash \{0\}$.

Exercise 4 (5 points)

Let $A \in \mathbb{R}^{n \times n}$ be the system matrix corresponding to the finite difference discretization of the Poisson problem with homogeneous boundary conditions. Use the discrete maximum principle to show that all entries of the matrix A^{-1} are non-negative.

Deadline: Tuesday, 26.11.2024, 10 am (in the postbox).