Abteilung für
Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 7

Exercise 1 (5 points)

Let $w : \mathbb{R}^2 \to \mathbb{R}$ be a quadratic polynomial and $\Delta x = 1/J$ for some $J \in \mathbb{N}$. For $j, m \in \mathbb{Z}^2$, let $x_{j,m} = (j, m)\Delta x$ and $W_{j,m} = w(x_{j,m})$. Show that

$$\Delta_h W_{j,m} = \partial^+_{x_1}\partial^-_{x_1} W_{j,m} + \partial^+_{x_2}\partial^-_{x_2} W_{j,m} = \Delta w(x_{j,m})$$

for all $j, m \in \mathbb{Z}^2$.

Exercise 2 (1+2+2 points)

We consider a swimming pool that has the horizontal shape of an annulus and assume that the stationary temperature distribution is independent of the vertical direction (i.e. the depth of the pool). Moreover, we assume that the temperature u_r is prescribed at the inner boundary and u_R is prescribed at the outer boundary by a sophisticated heating system.

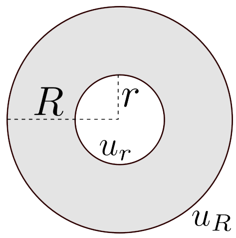
(i) Formulate a two-dimensional boundary value problem which determines the temperature distribution u in the pool.

(ii) Show that for $g \in C^2(\mathbb{R}_{\geq 0})$ and $\rho(x_1, x_2) = (x_1^2 + x_2^2)^{1/2}$, we have

$$\Delta(g \circ \rho) = g''(\rho) + \rho^{-1} g'(\rho) = \rho^{-1}(\rho g'(\rho))'.$$

Justify the assumption $u = \tilde{u} \circ \rho$ for the solution u of the problem in (i).

(iii) Solve the problem for a swimming pool with $r = 4$, $R = 8$, $u_r = 30$ and $u_R = 20$. On which radius do you have to swim to be surrounded by water of 25°C?



Exercise 3 (5 points)

For an open set $U \subset \mathbb{C}$, let $f : U \to \mathbb{C}$ be complex differentiable, i.e., for every $z \in U$ there exists $f'(z_0) \in \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0),$$

where $h \to 0$ represents an arbitrary sequence of complex numbers that converges to zero. Show that the functions $u, v : U \to \mathbb{R}$ defined by $f(x + iy) = u(x, y) + iv(x, y)$ satisfy the equations

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$$

in U and that they are harmonic, i.e., satisfy $-\Delta u = 0$ and $-\Delta v = 0$ in U.

Exercise 4 (2+2+1 points)

Determine the type of the following partial differential equations and give a short explanation:

$$\partial_t u + \Delta u = f \quad \text{in } (0, T) \times \Omega \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^d,$$

$$\partial^2_{x_1} u - 3\partial_{x_1}\partial_{x_2} u + \partial^2_{x_2} u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2,$$

$$\partial_t u - \partial^2_{x_1} u + \partial_{x_2} u = f \quad \text{in } (0, T) \times \Omega \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^2.$$

Deadline: Tuesday, 03.12.2024, 10 am (in the postbox).