Abteilung für
Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 8

Exercise 1 (5 points)

Let $\Omega = (0,1)^2$ and let $f \in C(\overline{\Omega})$ be given by

$$f(x_1, x_2) = \sum_{m,n\in\mathbb{N}} \alpha_{m,n} \sin(m\pi x_1)\sin(n\pi x_2).$$

Compute $-\Delta u_{m,n}$ for $u_{m,n}(x_1,x_2) = \sin(\pi m x_1)\sin(\pi n x_2)$ and construct the solution of the Poisson problem $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$.

Exercise 2 (3+2 points)

(i) Show that the function

$$\phi(x) = \begin{cases} e^{-1/(1-|x|^2)} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

satisfies $\phi \in C^\infty(\mathbb{R}^d)$.

(ii) The support of a function is defined by $\operatorname{supp}(v) = \overline{\{x \in \Omega : v(x) \neq 0\}}$. Let $u \in C(\Omega)$ and assume that

$$\int_\Omega uv \,\mathrm{d}x = 0$$

for all $v \in C^\infty(\Omega)$ with $\operatorname{supp}(v) \subset \Omega$. Prove that $u = 0$ in Ω.

Exercise 3 (3+2 points)

For $\Omega \subset \mathbb{R}^2$ and $u \in C^2(\Omega)$, let $\tilde{u}(r,\phi) = u(r\cos\phi, r\sin\phi)$.

(i) Show that the Laplace operator in polar coordinates can be written as

$$\Delta u(r\cos\phi, r\sin\phi) = \partial_r^2 \tilde{u}(r,\phi) + r^{-1}\tilde{u}(r,\phi) + r^{-2}\partial_\phi^2\tilde{u}(r,\phi).$$

(ii) Verify that the function $\tilde{u}(r,\phi) = r^{2/3}\sin(\frac{2}{3}\phi)$ is harmonic. Is it a classical solution of the Poisson equation on $\Omega = \{r(\cos\phi, \sin\phi) : 0 \leq r \leq 1,\ 0 \leq \phi \leq \frac{3}{2}\pi\}$? Justify your answer.

Exercise 4 (2+3 points)

Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.

(i) Show that $x \in \mathbb{R}^n$ satisfies $Ax = b$ if and only if

$$(Ax)^\top y = b^\top y$$

for all $y \in \mathbb{R}^n$.

(ii) Assume that A is symmetric and positive definite. Show that there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that x is the unique minimizer of the mapping

$$z \mapsto \frac{1}{2}|Bz|^2 - b \cdot z$$

if and only if x is a solution of $Ax = b$.

Deadline: Tuesday, 10.12.2024, 10 am (in the postbox).