Abteilung für
Angewandte Mathematik

Prof. Dr. Sören Bartels
M.Sc. Vera Jackisch

10.12.2024

Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 9

Exercise 1 (5 points)

Show that the functional

$$I(u) = \int_1^1 x^2 \big(u'(x)\big)^2 \mathrm{d}x$$

has no minimizer in $C^1((-1,1))$ subject to the boundary conditions $u(-1) = -1$ and $u(1) = 1$.

Exercise 2 (5 points)

Let $\Omega \subset \mathbb{R}^d$ be open, bounded, and connected, and let $\Gamma_D \subset \partial\Omega$ be nonempty. Prove that

$$\|v\| = \Big(\int_\Omega |\nabla v|^2 \mathrm{d}x \Big)^{1/2}$$

defines a norm on $V = \{v \in C^1(\overline{\Omega}) : v|_{\Gamma_D} = 0\}$ and conclude that the weak formulation of the Poisson equation can have at most one solution.

Exercise 3 (3+2 points)

Let $I \subset \mathbb{R}$ be a closed interval and $V = C^1(I)$.

(i) Show that $(V, \|\cdot\|_{V,\infty})$ with

$$\|v\|_{V,\infty} = \sup_{x\in I} |v(x)| + \sup_{x \in I} |v'(x)|$$

is a Banach space.

(i) Show that $(V, \|\cdot\|_{V,1})$ with

$$\|v\|_{V,1} = \int_I |v(x)| + |v'(x)| \mathrm{d}x$$

is not a Banach space.

Exercise 4 (2+3 points)

Determine all matrices $M \in \mathbb{R}^{n\times n}$ such that the bilinear map $a : (x, y) \mapsto x^\top My$ fulfills the conditions of (i) the Riesz representation theorem and (ii) the Lax-Milgram lemma.

Deadline: Tuesday, 17.12.2024, 10 am (in the postbox).