Abteilung für
Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 10

Exercise 1 (3+2 points)

Let $1 < p, q < \infty$ with $1/p + 1/q = 1$.
(i) Derive the inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

for all $a, b \in \mathbb{R}_{\geq 0}$ by comparing integrals of $f(x) = x^{p-1}$ and $f^{-1}(y)$ with the area of the rectangle $(0, a) \times (0, b)$. Discuss geometrically under which conditions equality holds.
(ii) Prove Hölder's inequality for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, i.e. $\int_\Omega |uv| \mathrm{d}x \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$.
Hint: First, consider the case $\|u\|_{L^p(\Omega)} = \|v\|_{L^q(\Omega)} = 1$.

Exercise 2 (3+2 points)

(i) Assume that $(\Omega_j)_{j=1,\ldots,J}$ is an open partition of Ω, $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_J$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Show that every continuous, piecewise differentiable function $u \in C(\overline{\Omega})$ with $u|_{\Omega_j} \in C^1(\overline{\Omega}_j)$ for $j = 1, \ldots, J$ is weakly differentiable.
(i) Draw a sketch of the weak derivative of the function

$$u(x) = \begin{cases} x^2, & \text{if } x \in [-2, 0), \\ x^{1/2}, & \text{if } x \in [0, 1), \\ 3x - 2, & \text{if } x \in [1, 2], \end{cases}$$

and determine all p such that $u \in W^{1,p}((-2, 2))$.

Exercise 3 (10 points)

Decide for each of the following statements whether it is true or false. You should be able to justify your decision in the tutorial group.

For a C^2 function u, the central difference quotient $\hat{\partial}$ provides a more accurate approximation of the derivative than the one-sided difference quotients $\partial^{\pm}$.	
The implementation of the difference scheme $\partial_t^+ U_j^k + a\partial_x^- U_j^k = 0$ requires the solution of linear systems of equations in every time step.	
The CFL condition is a necessary and sufficient condition for stability of a finite difference scheme.	
The θ-method for the heat equation is explicit for $\theta < 1/2$ and implicit for $\theta \geq 1/2$.	
The implicit scheme for the wave equation unconditionally satisfies a discrete maximum principle.	
If $f = 0$, then the solution of the Poisson problem $-\Delta u = f$ in Ω, $u|_{\partial\Omega} = 0$, is constant.	
For $f \in C^1(\overline{\Omega})$, $\Gamma_D = \partial\Omega$ and $u_D = 0$, the Poisson problem has a classical solution.	
Every finite-dimensional subspace of a Banach space is closed.	
Every linear operator between finite-dimensional spaces is bounded.	
The function $\phi\colon [-1, 1] \to \mathbb{R}, x \mapsto \operatorname{sign}(x)$ is weakly differentiable.	

Deadline: Tuesday, 07.01.2025, 10 am (in the postbox).
Merry Christmas and a happy new year!