Abteilung für
Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 11

Exercise 1 (5 points)
Let $a : V \times V \to \mathbb{R}$ be symmetric, bilinear, and positive semidefinite. Prove that

$$a(v, w) \leq \big(a(v, v)\big)^{1/2} \big(a(w, w)\big)^{1/2}$$

for all $v, w \in V$.

Exercise 2 (2+3 points)
Show the chain and product rule for weak derivatives:
(i) If $u, v \in W^{1,2}(\Omega)$, then $uv \in W^{1,1}(\Omega)$ and $\nabla(uv) = u\nabla v + v\nabla u$.
(ii) Let Ω be bounded. If $g \in C^1(\mathbb{R})$ with $|g'| < C$ and $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, then $\tilde{u} = g \circ u \in W^{1,p}(\Omega)$ with $\nabla\tilde{u} = g'(u)\nabla u$.

Exercise 3 (3+2 points)
(i) Derive a weak formulation for the boundary value problem

$$\begin{cases} -\operatorname{div}(K\nabla u) + b \cdot \nabla u + cu &= f \quad \text{in } \Omega, \\ u &= u_D \quad \text{on } \Gamma_D, \\ (K\nabla u) \cdot n &= g \quad \text{on } \Gamma_N. \end{cases}$$

(ii) Specify conditions on the coefficients that lead to the existence of a unique weak solution $u \in H^1(\Omega)$.

Exercise 4 (2+3 points)
For a triangle $T \subset \mathbb{R}^2$ with vertices $z_0, z_1, z_2 \in \mathbb{R}^2$, let $z_3, z_4, z_5 \in \mathbb{R}^2$ be the midpoints of the sides of T.
(i) Show that $(T, \mathscr{P}_2(T), \mathscr{K})$ with $\mathscr{K} = \{\chi_j : j = 0, 1, \dots, 5\}$ for $\chi_j(\phi) = \phi(z_j)$, $j = 0, 1, \dots, 5$, is a finite element.
(ii) Construct the dual basis for the finite element $(T, \mathscr{P}_2(T), \mathscr{K})$.

Deadline: Tuesday, 14.01.2025, 10 am (in the postbox).