Abteilung für Angewandte Mathematik

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Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 12

Exercise 1 (5 points)

Let $w = (w_1, w_2, \dots, w_d) : \mathbb{R}^d \to \mathbb{R}^d$ be a polynomial vector field of degree $m-1$ on $\mathbb{R}^d$, and assume that $w = \nabla v$ for some function $v \in C^1(\mathbb{R}^d)$. Show that v is a polynomial of degree m.

Exercise 2 (5 points)

Let $\Phi_T : \hat{T} \to T$ be an affine diffeomorphism. Show that

$$D\Phi_T^{-1} = (D\Phi_T)^{-1}$$

and that both matrices are independent of $x \in T$ and $\hat{x} \in \hat{T}$.

Exercise 3 (2+3 points)

Let $T = \operatorname{conv}\{z_0, z_1, \dots, z_d\}$ be a simplex with positively oriented vertices $z_0, z_1, \dots, z_d \in \mathbb{R}^d$ and define

$$X_T = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_d \end{bmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}.$$

(i) Prove that the volume of T is given by $|T| = (1/d!) \det X_T$.
(ii) Prove that the gradients of the nodal basis functions on T satisfy

$$\left[\nabla\varphi_{z_0}|_T, \dots, \nabla\varphi_{z_d}|_T\right]^\top = X_T^{-1} \begin{bmatrix} 0 \\ I_d \end{bmatrix}.$$

The nodal basis functions $(\varphi_{z_i})_{i=0,\dots,d}$ on T are defined by $\varphi_{z_i}(z_j) = \delta_{ij}$.
Hint: Use Cramer's rule and the fact that the nodal basis function belonging to node z_j is given by

$$\varphi_{z_j}(x) = \frac{1}{d!|T|} \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ x & z_{j+1} & \dots & z_{j+d} \end{bmatrix}$$

for $x \in T$, where the indices are understood modulo d.

Exercise 4 (5 points)

Comment every line of the following Matlab implementation of the Poisson problem `p1_poisson(d,red)` with an explanation of what is happening in that line of code. You can do this exercise on paper or directly in the code, available at `https://aam.uni-freiburg.de/agba/prof/books.html`, where you can download a zip-file also containing the relevant subroutines for context.

```matlab
function p1_poisson(d,red)
[c4n,n4e,Db,Nb] = triang_cube(d);
for j = 1:red
        [c4n,n4e,Db,Nb,P0,P1] = red_refine(c4n,n4e,Db,Nb);
end
[nC,d] = size(c4n); nE = size(n4e,1); nNb = size(Nb,1);
dNodes = unique(Db); fNodes = setdiff(1:nC,dNodes);
u = zeros(nC,1); tu_D = zeros(nC,1); b = zeros(nC,1);
ctr = 0; ctr_max = (d+1)^2*nE;
I = zeros(ctr_max,1); J = zeros(ctr_max,1); X = zeros(ctr_max,1);
for j = 1:nE
        X_T = [ones(1,d+1);c4n(n4e(j,:),:)'];
        grads_T = X_T\[zeros(1,d);eye(d)];
        vol_T = det(X_T)/factorial(d);
        mp_T = sum(c4n(n4e(j,:),:),1)/(d+1);
        for m = 1:d+1
                b(n4e(j,m)) = b(n4e(j,m))+(1/(d+1))*vol_T*f(mp_T);
                for n = 1:d+1
                ctr = ctr+1; I(ctr) = n4e(j,m); J(ctr) = n4e(j,n);
                X(ctr) = vol_T*grads_T(m,:)*grads_T(n,:)';
                end
        end
end
s = sparse(I,J,X,nC,nC);
for j = 1:nNb
        if d == 1
                vol_S = 1;
        elseif d == 2
                vol_S = norm(c4n(Nb(j,1),:)-c4n(Nb(j,2),:));
        elseif d == 3
                vol_S = norm(cross(c4n(Nb(j,3),:)-c4n(Nb(j,1),:),...
                        c4n(Nb(j,2),:)-c4n(Nb(j,1),:)),2)/2;
        end
        mp_S = sum(c4n(Nb(j,:),:),1)/d;
        for k = 1:d
                b(Nb(j,k)) = b(Nb(j,k))+(1/d)*vol_S*g(mp_S);
        end
end
for j = 1:nC
        tu_D(j) = u_D(c4n(j,:));
end
b = b-s*tu_D; u(fNodes) = s(fNodes,fNodes)\b(fNodes); u = u+tu_D;
if d == 1; plot(c4n(n4e),u(n4e));
elseif d == 2; trisurf(n4e,c4n(:,1),c4n(:,2),u);
elseif d == 3; trisurf([Db;Nb],c4n(:,1),c4n(:,2),c4n(:,3),u);
end

function val = f(x); val = 1;
function val = g(x); val = 1;
function val = u_D(x); val = sin(2*pi*x(:,1));
```

Deadline: Tuesday, 21.01.2025, 10 am (in the postbox).