**Abteilung für**
**Angewandte Mathematik**

Prof. Dr. Sören Bartels
M.Sc. Vera Jackisch

21.01.2025

Exercises for the lecture

# Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 13

**Exercise 1** (5 points)

Let $W = V + V_h$, and assume that $a : W \times W \to \mathbb{R}$ is bilinear and continuous with respect to a norm $\|\cdot\|_h$, and assume that $a_h$ is coercive on $V_h$. Let $u_h \in V_h$ satisfy $a_h(u_h, v_h) = \ell_h(v_h)$ for all $v_h \in V_h$, and let $u \in V$ be such that $a(u, v) = \ell(v)$ for all $v \in V$. Show there exists $c > 0$ such that

$$c^{-1}\|u - u_h\|_h \leq \inf_{v_h \in V_h} \|u - v_h\|_h + \|a_h(u, \cdot) - \ell_h\|_{V_h'}.$$

**Exercise 2** (5 points)

Let $V$ be a Hilbert space, $a : V \times V \to \mathbb{R}$ a bounded and coercive bilinear form, which is also symmetric, and let $b : V \to \mathbb{R}$ be a continuous, linear functional. Further assume that $V_h \subset V$ is a finite-dimensional subspace and $u_h \in V_h$ is the unique Galerkin approximation of $u \in V$ with $a(u, v) = b(v)$ for all $v \in V$. Show that in this case, the quasi best-approximation property in Céa's Lemma holds with the constant $(k_a/\alpha)^{\frac{1}{2}}$, i.e.

$$\|u - u_h\|_V \leq (k_a/\alpha)^{\frac{1}{2}} \inf_{w_h \in V_h} \|u - w_h\|_V.$$

The variables $k_a$ and $\alpha$ are the boundedness and coercivity constant of bilinear form $a$, respectively.

**Exercise 3** (5 points)

Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex Lipschitz domain. Provide a constructive proof that there exists a constant $c_p > 0$ such that

$$\|v\|_{L^2(\Omega)} \leq c_p \|\nabla v\|_{L^2(\Omega)}$$

for all $v \in H^1(\Omega)$ with $\int_\Omega v \,\mathrm{d}x = 0$.

**Exercise 4** (5 points)

Let $S \in \mathcal{S}_h$ be an inner side in a triangulation $\mathcal{T}_h$ with endpoints $z, y \in \mathcal{N}_h$ and neighboring triangles $T_1, T_2 \in \mathcal{T}_h$. Let $\alpha_1$ and $\alpha_2$ be the inner angles of $T_1$ and $T_2$ opposite to $S$, respectively. Prove that

$$A_{zy} = \int_{T_1 \cup T_2} \nabla\varphi_z \cdot \nabla\varphi_y \mathrm{d}x = -\frac{1}{2}(\cot\alpha_1 + \cot\alpha_2) = -\frac{1}{2}\frac{\sin(\alpha_1 + \alpha_2)}{\sin(\alpha_1)\sin(\alpha_2)}$$

and formulate precise conditions which imply $A_{zy} \leq 0$.

**Deadline:** Tuesday, 28.01.2025, 10 am (in the postbox).