Abteilung für
Angewandte Mathematik

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28.01.2025

Exercises for the lecture

Introduction to Theory and Numerics of Partial Differential Equations

WS 2024/25 — Exercise Sheet 14 (bonus)

Exercise 1 (5 bonus points)

Let $(\mathcal{T}_h)_{h>0}$ be a sequence of triangulations of the domain $\Omega \subset \mathbb{R}^d$ with maximum grid size $h \to 0$ and let $\mathcal{S}^1(\mathcal{T}_h) = \{v_h \in C(\bar{\Omega}) : v_h|_T \in P^1(T)\ \forall T \in \mathcal{T}_h\}$.

(i) Show that the union $\bigcup_{h>0} \mathcal{S}^1(\mathcal{T}_h)$ is dense in the space $H^1(\Omega)$.

(ii) Prove (without determining a convergence rate) that the Galerkin approximations of the Poisson problem always converge to the exact solution even without additional regularity assumptions.

Exercise 2 (5 bonus points)

We consider the one-dimensional Poisson equation on the interval $\Omega = (0, 1)$ with zero boundary values on $\Gamma_D = \{0, 1\}$ and the right-hand side $f \in C(\bar{\Omega})$. For $n \in \mathbb{N}$ let $\mathcal{T}_h = \mathcal{T}_{1/n} = \{[\frac{k-1}{n}, \frac{k}{n}] : k = 1, \ldots, n\}$ be a triangulation of Ω. Show that in this case, the discrete solution with the finite difference method using the second-order central difference quotient coincides with the P1 finite element approximation if in the latter method the right-hand side is numerically integrated via

$$\int_\Omega f\varphi_z \mathrm{d}x \approx \int_\Omega \mathcal{I}_h^1[f\varphi_z]\mathrm{d}x.$$

Exercise 3 (5 bonus points)

Write a short summary of the lecture content. Dedicate approximately one page to each chapter. Use the following keywords and give examples if necessary:

1. Finite difference method: *CFL condition, implicit/explicit methods, consistency, stability, convergence.*
2. Elliptic partial differential equations: *Singularities, differentiability, weak formulation, Sobolev space, existence/uniqueness of weak solutions.*
3. Finite element method: *Galerkin approximation, nodal basis, interpolation, error estimation, maximum principle.*

Exercise 4 (5 bonus points)

Decide whether the following statements are true or false. You should be able to justify your decision in the tutorial group.

There exists a constant $c > 0$ such that for all piecewise polynomials $v \in H^1(\Omega)$ the inequality $\|\nabla v_h\|_{L^2(\Omega)} \leq c\|v_h\|_{L^2(\Omega)}$ holds.	
For all $v \in H^3(T)$ there exists a $q \in \mathcal{P}_2(T)$ with $\|\nabla(v - q)\|_{L^2(T)} \leq ch_T^2\|D^3 v\|_{L^2(T)}$.	
The P1 finite element method for the Poisson problem fulfills the discrete maximum principle.	
For all $v_h \in \mathcal{S}_0^1(\mathcal{T}_h)$ we have that $\|v_h\|_{L^4(\Omega)} \leq c\|\nabla v_h\|_{L^2(\Omega)}$.	
If the solution u of the Poisson problem satisfies $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then we have $\|u - u_h\|_{L^2(\Omega)} \leq ch^2\|D^2 u\|_{L^2(\Omega)}$ for the Galerkin approximation $u_h \in \mathcal{S}_0^1(\mathcal{T})$.	

Deadline: Tuesday, 04.02.2025, 10 am (in the postbox).