A CONVERGENT AND CONSTRAINT-PRESERVING FINITE ELEMENT METHOD FOR THE P-HARMONIC FLOW INTO SPHERES

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Abstract. An explicit fully discrete finite element method, which satisfies the nonconvex side constraint at every node, is developed for approximating the p-harmonic flow for $p \in (1,\infty)$. Convergence of the method is established under certain conditions on the domain and mesh. Computational examples are presented to demonstrate finite-time blow-ups and qualitative geometric changes of weak solutions of the p-harmonic flow.

Key words. p-harmonic map, singular & degenerate PDE, finite element method, convergence analysis

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1. Introduction and Summary. Minimizing the energy

$$E_p(\mathbf{u}) := \frac{1}{p}\int_\Omega |\nabla \mathbf{u}|^p \,\mathrm{d}\mathbf{x}\,, \quad 1 \le p < \infty\,, \tag{1.1}$$

for maps $\mathbf{u} : \Omega \to S^{m-1}\,(m \ge 2)$, where $\Omega \subset \mathbb{R}^n\,(n \ge 1)$ is bounded and $S^{m-1} \subset \mathbb{R}^m$ is the unit sphere, gives rise to p-harmonic maps. Such maps have natural applications such as micromagnetics [11, 27], liquid crystal theory [1, 23, 28, 5] ($p = 2$) or color image denoising [32, 33, 34, 10, 21, 4] ($p = 1$). At present, there are not many schemes available to reliably approximate such maps. The main numerical difficulties are the nonconvexity of the constraint, $|\mathbf{u}| = 1$ a.e. in Ω, and the limited regularity and nonuniqueness of minimizers.

The first numerical schemes to approximate (1.1) in the case $p = 2$ were proposed in [16, 17, 22, 28]. The idea is in each search direction, first to reduce the energy functional ignoring the sphere constraint; then renormalize this solution $\mathbf{V}^j$ to obtain $\mathbf{U}^j = \frac{\mathbf{V}^j}{|\mathbf{V}^j|}$. However, the question is then if the energy is still decreased during the renormalization step. This problem has been elegantly solved in [1], where an interesting convergent algorithm is proposed. Given an admissible $\mathbf{U}^j$, the strategy there is to decrease the energy $E_2(\mathbf{U}^j - \mathbf{V}^j)$ for $\mathbf{V}^j$ belonging to the tangential plane $\{\mathbf{w} \in H^1(\Omega,\mathbb{R}^m) :\ \langle \mathbf{w}, \mathbf{U}^j\rangle_{\mathbb{R}^m} = 0 \text{ a.e. in } \Omega\}$; then perform the renormalization $\mathbf{U}^{j+1} = \frac{\mathbf{U}^j - \mathbf{V}^j}{|\mathbf{U}^j - \mathbf{V}^j|} \in H^1(\Omega, S^{m-1})$. By construction, it follows that $E_2(\mathbf{U}^j - \mathbf{V}^j) = \min_{\mathbf{w}} E_2(\mathbf{U}^j - \mathbf{w}) \le E_2(\mathbf{U}^j)$, since $\mathbf{w} = \mathbf{0}$ is admissible. Moreover, $|\mathbf{U}^j - \mathbf{V}^j| \ge 1$ a.e. in Ω, which is sufficient to guarantee decrease of the energy in the renormalization step. Recently, convergence of a finite element realization of this algorithm has been verified for restricted (acute) mesh partitions [8]. A generalization of this (Alouges') strategy to the degenerate regime $p \ne 2$ is easily possible, but convergence behavior seems unclear to the authors for the singular cases $p < 2$.

Another discretization approach is based on the convergent penalization strategy, see [29]. Here the nonconvex constraint is approximated by adding the penalty term $\varepsilon^{-1}\int_\Omega (|\mathbf{u}|^2 - 1)^2\,\mathrm{d}\mathbf{x}$ to $E_p(\mathbf{u})$, leading to the unconstrained Ginzburg-Landau energy $E_{p,\varepsilon}(\mathbf{u})$ for an $\varepsilon > 0$. However, a numerical approximation of $E_{p,\varepsilon}(\mathbf{u})$ requires the penalization parameter ε and the mesh parameter h to be tuned. In [34] a different approach is proposed. This is based on the unconstrained minimization of

$$F_p(\mathbf{v}) := \int_\Omega \Big|\nabla\Big(\frac{\mathbf{v}}{|\mathbf{v}|}\Big)\Big|^p \,\mathrm{d}\mathbf{x}\,, \quad 1 \le p < \infty\,, \tag{1.2}$$

for maps $\mathbf{v} : \Omega \to \mathbb{R}^m \setminus \{0\}$. A parametrization of the sphere then yields an efficient unconstrained numerical scheme, which is consistent with the nonconvex side-constraint and leads to energy decay. However this approach restricts possible minimizers of (1.1), and leaves convergence properties of (1.2) unclear.

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An alternative strategy to study minimizers of (1.1) is to consider the long-time behavior of the p-harmonic flow into spheres:

$$\mathbf{u}_t - \Delta_p \mathbf{u} = |\,\nabla \mathbf{u}\,|^p \mathbf{u} \quad \text{on } \Omega_T\,, \qquad \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega_T\,, \tag{1.3}$$

$$|\,\mathbf{u}(\cdot,\cdot)\,| = 1 \quad \text{a.e. in } \Omega_T\,, \qquad \mathbf{u}(0,\cdot) = \mathbf{u}_0 \quad \text{on } \Omega\,, \tag{1.4}$$

for any $T > 0$. Here $\Omega_T := (0,T) \times \Omega$, $\partial\Omega_T := (0,T) \times \partial\Omega$ with $\partial\Omega$ being the boundary of Ω with normal $\mathbf{n}$. The system (1.3)–(1.4) characterizes the L^2-gradient flow of (1.1) with $\Delta_p \mathbf{u} \equiv \nabla \cdot (|\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u})$. Solutions to this problem have been studied intensively over the last fifteen years, starting with the case $p = 2$ [13], and followed by $p > 2$ (existence [14, 26], nonuniqueness [24]), and $1 < p < 2$ (existence and nonuniqueness [18, 30]). Weak solutions to (1.3)–(1.4) satisfy (1.3) in a distributional sense, the initial condition in (1.4) in the sense of traces for $\mathbf{u}_0 \in W^{1,p}(\Omega, S^{m-1})$, and the energy inequality (cf. [31])

$$\int_0^t \|\,\mathbf{u}_t(s)\,\|_{L^2}^2 \,\mathrm{d}s + E_p\big(\mathbf{u}(t)\big) \le E_p(\mathbf{u}_0) \quad \text{for a.e. } t \in (0,T)\,. \tag{1.5}$$

This motivates the conjecture that there exists a subsequence $\{t_{k'}\} \subset \{t_k\}$, for $t_k \to \infty$, such that $\mathbf{u}^* = \lim_{k' \to \infty} \mathbf{u}(t_{k'}, \cdot)$ is a p-harmonic map, which is known for the case $p = 2$, and for any $p > 1$ in the case of small initial data [19].

In order to verify existence of a weak solution to (1.3)–(1.4), the problem is modified to first finding a solution $\mathbf{u}^\varepsilon : \Omega_T \to \mathbb{R}^m$ to the following unconstrained penalized formulation for $\varepsilon > 0$ and $T > 0$:

$$\mathbf{u}_t^\varepsilon - \Delta_p \mathbf{u}^\varepsilon + \frac{1}{2\varepsilon}\big(|\,\mathbf{u}^\varepsilon\,|^2 - 1\big)\mathbf{u}^\varepsilon = 0 \quad \text{on } \Omega_T\,, \qquad \frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega_T\,, \tag{1.6}$$

$$\mathbf{u}^\varepsilon(0,\cdot) = \mathbf{u}_0 \quad \text{on } \Omega\,, \tag{1.7}$$

cf. [13, 15]. Tracing the limit as $\varepsilon \to 0$ for solutions to (1.6)–(1.7) then leads to weak solutions of (1.3)–(1.4) satisfying (1.5) for the cases $1 < p < \infty$. When $p = 1$, which is the most difficult case, local strong solutions are proved by Giga et al. in [21], whereas global weak solutions are established in [4]. Apart from its use as an analytical tool, problem (1.6)–(1.7) is often the starting point to construct convergent discretizations for which the computed (discrete) solutions $\mathbf{U}_{k,h}^\varepsilon$ converge to solutions of (1.3)–(1.4), see [4], as the time step k, the mesh parameter h and the penalization parameter ε tend to zero. Popularization of this approach is partially due to the fact that the direct construction of a convergent discretization of (1.3)–(1.4) is a nontrivial task.

The goal of this paper is to propose a convergent fully discrete finite element approximation of (1.3)–(1.4). Its construction is inspired by the recent work [2] for the Landau Lifshitz equations. Our numerical scheme is based on the following equivalent reformulation of (1.3)–(1.4): find $\mathbf{u}$ satisfying the constraint and the initial condition such that

$$\int_0^T (\mathbf{u}_t(t), \mathbf{w}) \,\mathrm{d}t + \int_0^T (|\,\nabla \mathbf{u}(t)\,|^{p-2} \nabla \mathbf{u}(t), \nabla \mathbf{w}) \,\mathrm{d}t = 0 \quad \forall T > 0\,, \tag{1.8}$$

for all $\mathbf{w} \in L^2((0,T); W^{1,p}(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega_T, \mathbb{R}^m)$, such that $\langle \mathbf{w}, \mathbf{u} \rangle_{\mathbb{R}^m} = 0$ a.e. in Ω_T, where $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) := \int_\Omega \langle \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \rangle_{\mathbb{R}^{\ell_1 \times \ell_2}} \,\mathrm{d}\mathbf{x}$ for $\boldsymbol{\eta}_i(t,\cdot) \in \mathbb{R}^{\ell_1 \times \ell_2}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{\ell_1 \times \ell_2}}$ is the standard inner product on $\mathbb{R}^{\ell_1 \times \ell_2}$.

To introduce our finite element scheme and state the main convergence result, we need to make the following assumptions on the finite element partitioning:

(A1) Assuming that Ω is either polygonal ($n = 2$) or polyhedral ($n = 3$), let $\mathcal{T}_h$ be a quasi-uniform partitioning of Ω into disjoint open simplices K with $h_K := \mathrm{diam}(K)$ and $h := \max_{K \in \mathcal{T}_h} h_K$, so that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} \overline{K}$.

We require the quasi-uniformity constraint on the partitioning, as many of the proofs in this paper use the inverse inequalities on functions in $\boldsymbol{\mathcal{V}}_h$. The convergence proof of our finite element approximation for $p > n$ or $p = 2$ is fairly straightforward. In order to prove convergence if $p \le n$ and $p \ne 2$; our proof requires the denseness of $C^\infty(\overline{\Omega}, S^{m-1})$ in $W^{1,p}(\Omega, S^{m-1})$, which imposes the restrictions of either $p = n$ or

$p < m-1$; see [9]. Moreover, in this case we have to place a further restriction on the partitioning for a monotonicity argument to hold:

(A2) In addition to the assumptions (A1) above, we assume that all simplices $K \in \mathcal{T}_h$ are right-angled. (For $n = 3$ this means that all tetrahedra have one vertex with exactly one right angle, one vertex with exactly two right angles and all other angles are strictly acute, see Section 4 for more details. We note that a cube is easily partitioned into such tetrahedra. Sufficient for our analysis is to assume that each element has n mutually perpendicular edges; the case that a tetrahedron has a vertex with three right angles is unrealistic in practice and therefore, for ease of exposition, excluded.)

Let $\mathcal{P}_1$ be the space of linear polynomials. We then introduce the following sets of functions:

$$\begin{aligned}
\boldsymbol{\mathcal{V}}_h &:= \big\{ \mathbf{W} \in C(\overline{\Omega}, \mathbb{R}^m);\ \mathbf{W}\big|_K \in \boldsymbol{\mathcal{P}}_1(K, \mathbb{R}^m)\ \forall\, K \in \mathcal{T}_h \big\},\\
\boldsymbol{\mathcal{M}}_h &:= \big\{ \mathbf{W} \in \boldsymbol{\mathcal{V}}_h;\ |\, \mathbf{W}(\mathbf{q}_i)\,| = 1\ \forall \text{ nodes } \mathbf{q}_i \text{ of } \mathcal{T}_h \big\},\\
\boldsymbol{\mathcal{F}}_h(\boldsymbol{\chi}) &:= \big\{ \mathbf{W} \in \boldsymbol{\mathcal{V}}_h;\ \big\langle \mathbf{W}(\mathbf{q}_i), \boldsymbol{\chi}(\mathbf{q}_i) \big\rangle_{\mathbb{R}^m} = 0 \quad \forall \text{ nodes } \mathbf{q}_i \text{ of } \mathcal{T}_h \big\}\,, \quad \text{where } \boldsymbol{\chi} \in \boldsymbol{\mathcal{M}}_h.
\end{aligned}$$

Let $I_h : C(\overline{\Omega}, \mathbb{R}) \to V_h$ be the linear interpolation operator, where $V_h \equiv \boldsymbol{\mathcal{V}}_h$ with $m = 1$, such that $(I_h\, v)(\mathbf{q}_i) = v(\mathbf{q}_i)$ for all nodes $\mathbf{q}_i$ of $\mathcal{T}_h$. We then set $(\cdot,\cdot)_h : C(\overline{\Omega}, \mathbb{R}^m) \times C(\overline{\Omega}, \mathbb{R}^m) \to \mathbb{R}$ to be

$$(\boldsymbol{\chi}, \boldsymbol{\mathfrak{Z}})_h := \int_\Omega I_h\big(\langle \boldsymbol{\chi}, \boldsymbol{\mathfrak{Z}} \rangle_{\mathbb{R}^m}\big)\, \mathrm{d}\mathbf{x} \equiv \sum_{K \in \mathcal{T}_h} \frac{|K|}{n+1} \sum_{\mathbf{q}_i \in K} \langle \boldsymbol{\chi}(\mathbf{q}_i), \boldsymbol{\mathfrak{Z}}(\mathbf{q}_i) \rangle_{\mathbb{R}^m}\,, \tag{1.9}$$

where $|K|$ is the area/volume of K.

Let k be the time step such that $Jk = T$ and $d_t v^j = k^{-1}\,(v^j - v^{j-1})$. Then a fully discrete implicit approximation of (1.8) reads: For $j = 0 \to J-1$, given $\widehat{\mathbf{U}}^j \in \boldsymbol{\mathcal{M}}_h$, find $\widehat{\mathbf{U}}^{j+1} \in \boldsymbol{\mathcal{M}}_h$ such that

$$(d_t \widehat{\mathbf{U}}^{j+1}, \mathbf{W}) + (|\, \nabla \widehat{\mathbf{U}}^{j+1}\,|^{p-2} \nabla \widehat{\mathbf{U}}^{j+1}, \nabla \mathbf{W}) = 0 \quad \forall\, \mathbf{W} \in \boldsymbol{\mathcal{F}}_h(\widehat{\mathbf{U}}^j)\,, \tag{1.10}$$

where $\widehat{\mathbf{U}}^0$ is an approximation of $\mathbf{u}_0 \in W^{1,p}(\Omega, S^{m-1})$. This problem is clearly too difficult to solve because of the imposed nonconvex constraint on $\boldsymbol{\mathcal{M}}_h$. However, since $\langle \mathbf{u}_t, \mathbf{u} \rangle_{\mathbb{R}^m} = 0$, we may assume that $d_t \widehat{\mathbf{U}}^{j+1}$ is almost an element of $\boldsymbol{\mathcal{F}}_h(\widehat{\mathbf{U}}^j)$. This motivates our explicit scheme, which adapts the algorithm in [2] for the Landau Lifshitz equations to the p-harmonic flow with $p \in (1, \infty)$.

SCHEME

Step 1: Start with an initial vector field $\mathbf{U}^0 \in \boldsymbol{\mathcal{M}}_h$.

Step 2: For $j = 0 \to J-1$, given $\mathbf{U}^j \in \boldsymbol{\mathcal{M}}_h$, find $\mathbf{V}^j \in \boldsymbol{\mathcal{F}}_h(\mathbf{U}^j)$ which solves

$$(\mathbf{V}^j, \mathbf{W})_h = -(|\, \nabla \mathbf{U}^j\,|^{p-2} \nabla \mathbf{U}^j, \nabla \mathbf{W}) \qquad \forall\, \mathbf{W} \in \boldsymbol{\mathcal{F}}_h(\mathbf{U}^j)\,.$$

Step 3: Define $\mathbf{U}^{j+1} \in \boldsymbol{\mathcal{M}}_h$ via

$$\mathbf{U}^{j+1}(\mathbf{q}_i) = \frac{\mathbf{U}^j(\mathbf{q}_i) + k\mathbf{V}^j(\mathbf{q}_i)}{|\, \mathbf{U}^j(\mathbf{q}_i) + k\mathbf{V}^j(\mathbf{q}_i)\,|} \qquad \forall \text{ nodes } \mathbf{q}_i \text{ of } \mathcal{T}_h\,.$$

We note that Step 2 is explicit, due to the use of numerical integration on the left-hand side, but remark that our analysis also holds if exact integration is used. For the fully discrete finite element solution $\{\mathbf{U}^j\}_{j\ge 1}$ we define its constant and linear interpolations in time as follows:

$$\begin{aligned}
\underline{\mathbf{U}}(t,\cdot) &:= \mathbf{U}^{j-1}(\cdot) && \forall\, t \in [t_{j-1}, t_j)\,, \quad 1 \le j \le J\,,\\
\mathbf{U}(t,\cdot) &:= \frac{t - t_{j-1}}{k} \mathbf{U}^j(\cdot) + \frac{t_j - t}{k} \mathbf{U}^{j-1}(\cdot) && \forall\, t \in [t_{j-1}, t_j]\,, \quad 1 \le j \le J\,.
\end{aligned} \tag{1.11}$$

In this paper we will prove the following theorem.

THEOREM 1.1. *If $p=2$ or $p\in(n,\infty)$, let the assumptions (A1) hold. If $p=n$ or $p\in(1,m-1)$, let the assumptions (A2) hold. In addition, we assume that $\mathbf{u}_0\in W^{1,p}(\Omega,S^{m-1})$ and $\mathbf{U}^0\in\boldsymbol{\mathcal{M}}_h$ satisfies $\mathbf{U}^0\to\mathbf{u}_0$ strongly in $W^{1,p}(\Omega,\mathbb{R}^m)$ as $h\to 0$, and*

$$k\le\begin{cases} o\big(\min\{h^{\frac{p}{p-1}},h^{p+\frac{n}{2}}\}\big) & \text{for } 1<p<2\,,\\ o\big(\min\{h^{p},h^{1+n\,(1-\frac{1}{p})}\}\big) & \text{for } 2\le p<\infty\,.\end{cases}\tag{1.12}$$

Then there exists a subsequence of $\{\mathbf{U}\}_h$ such that as $h\to 0$

$$\mathbf{U}\rightharpoonup\mathbf{u}\quad \text{weakly* in } L^\infty\big(0,T;W^{1,p}(\Omega,\mathbb{R}^m)\big)\,,\qquad \mathbf{U}_t\rightharpoonup\mathbf{u}_t\quad\text{weakly in } L^2(\Omega_T,\mathbb{R}^m)\,,$$

where $\mathbf{u}\in H^1\big((0,T);L^2(\Omega,\mathbb{R}^m)\big)\cap L^\infty\big((0,T);W^{1,p}(\Omega,\mathbb{R}^m)\big)$ is a weak solution to (1.3)–(1.4).

To summarize: we prove convergence of our finite element approximation when

$$\begin{aligned}&n=2,\text{ if either (i) } m=2 \text{ and } p\in[2,\infty) \text{ or (ii) } m\ge 3 \text{ and } p\in(1,\infty)\,;\\ &n=3,\text{ if either (i) } m=2 \text{ and } p\in\{2\}\cup[3,\infty),\text{ (ii) } m=3 \text{ and } p\in(1,2]\cup[3,\infty)\\ &\qquad\text{or (iii) } m\ge 4 \text{ and } p\in(1,\infty).\end{aligned}\tag{1.13}$$

We remark also that the above theorem does not hold for $p=1$, in which case the weak solutions are only BV-functions, instead of Sobolev functions [4]. Moreover, computational experiments suggest that the constraint on the time step k is sharp as $p\to 1$. Convergence of a space-time discretization of a Ginzburg-Landau penalization, (1.6)–(1.7), which also handles the case $p=1$ is proposed in [4]. That approach requires an additional penalization parameter ε, which, as is common with any penalization process, poses some additional constraint on the mesh parameters.

The remainder of this paper is organized as follows. In Section 2, we give a precise weak formulation of problem (1.3)–(1.4). In Section 3, we establish the stability of the numerical solution and the mesh conditions on k described in Theorem 1.1 above. In Section 4, we prove the convergence result of Theorem 1.1. Finally, in Section 5, we present some numerical experiments, which motivate finite-time blow-up and other qualitative behaviors of solutions of the p-harmonic flow for various values of p.

2. Preliminaries. For $\Omega\subset\mathbb{R}^n$ be bounded, we define the nonlinear Sobolev space

$$W^{1,p}(\Omega,S^{m-1})=\big\{\mathbf{v}\in W^{1,p}(\Omega,\mathbb{R}^m)\big|\ \mathbf{v}\in S^{m-1}\quad\text{a.e. in }\Omega\big\}\,,\quad 1<p<\infty\,.$$

Critical points $\mathbf{u}\in W^{1,p}(\Omega,S^{m-1})$ of $E_p(\mathbf{u})$ for $p\in(1,\infty)$ can be characterized as solutions to the Euler-Lagrange equation

$$-\Delta_p\mathbf{u}=|\,\nabla\mathbf{u}\,|^p\mathbf{u}\quad\text{on }\Omega\,,\qquad \frac{\partial\mathbf{u}}{\partial\mathbf{n}}=0\quad\text{on }\partial\Omega\,.\tag{2.1}$$

If a map $\mathbf{u}\in W^{1,p}(\Omega,S^{m-1})$ satisfies (2.1) in the sense of distributions, $\mathbf{u}$ is called a weakly p-harmonic map. The p-harmonic flow (1.3)–(1.4) was first studied in [14, 25]. We now make precise what we mean by a weak solution to (1.3)–(1.4).

DEFINITION 2.1. *Let $\mathbf{u}_0\in W^{1,p}(\Omega,\mathbb{R}^m)$, $p>1$, then $\mathbf{u}$ is a weak solution to (1.3)–(1.4) if $\mathbf{u}$ is a function defined a.e. on $\Omega\times\mathbb{R}^+$, such that*

1. $\mathbf{u}\in L^\infty\big((0,T);W^{1,p}(\Omega,\mathbb{R}^m)\big)\cap H^1\big((0,T);L^2(\Omega,\mathbb{R}^m)\big)$, *for all $T>0$,*
2. *$\mathbf{u}$ is weakly continuous for $t>0$ with values in $W^{1,p}(\Omega,\mathbb{R}^m)$, i.e. for any test function $\mathbf{g}\in C^\infty(\Omega,\mathbb{R}^m)$,*

$$f_1(t)=\int_\Omega\langle\mathbf{u},\mathbf{g}\rangle_{\mathbb{R}^m}\,\mathrm{d}\mathbf{x}\,,\quad f_2(t)=\int_\Omega\langle\nabla\mathbf{u},\nabla\mathbf{g}\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}$$

   are continuous for $t>0$, with possible modification on a set of measure zero on $(0,\infty)$,
3. *$|\,\mathbf{u}\,|=1$ a.e. on $\Omega\times\mathbb{R}^+$,*
4. *(1.3) holds in the sense of distributions,*

5. *the initial condition holds in the sense of traces.*

Verification of the existence of a weak solution to (1.3)–(1.4) uses monotonicity arguments for a penalization approach to approximate the p-harmonic flow on the space $W^{1,p}(\Omega,\mathbb{R}^m)$. A parabolic version of Murat's lemma then gives enough compactness to identify limits of terms of a wedged version of the penalized problem as a wedged version of (1.3), which holds in distributional sense. The weak solution is known to satisfy the energy law (1.5), We refer to [31, 23] for further details in this direction. Also, weak solutions to (1.3)–(1.4) are not unique, see e.g. [30] and [24]. Of course, the subsequent proof of Theorem 1.1 can be considered as an alternative way to construct weak solutions to (1.3)–(1.4).

REMARK 2.1. *In [30] (see also [26], for $p>2$), Misawa demonstrates existence of weak solutions to* (1.3)–(1.4) *by the Rothe method: set $\mathbf{u}^0=\mathbf{u}_0$, then for $j\geq 1$ minimizers $\mathbf{u}^j=\mathrm{argmin}_{W^{1,p}(\Omega,S^{m-1})}E_p(\mathbf{v})$, of $E_p(\mathbf{v}):=E_p(\mathbf{v})+\frac{1}{2k}\int_\Omega|\,\mathbf{v}-\mathbf{u}^{j-1}\,|^2\,\mathrm{d}\mathbf{x}$, exist, and solve*

$$d_t\mathbf{u}^j-\Delta_p\mathbf{u}^j=\big(|\,\nabla\mathbf{u}^j\,|^p+\frac{k}{2}\,|\,d_t\mathbf{u}^j\,|^2\big)\mathbf{u}^j\quad on\ \Omega\,,\qquad \frac{\partial\mathbf{u}^j}{\partial\mathbf{n}}=0\quad on\ \partial\Omega. \tag{2.2}$$

In addition, they satisfy a semidiscrete version of energy inequality (1.5) *on the equidistant time mesh $\{t_j\}_{j\geq 0}$. Then a compactness argument as in [14], together with a parabolic version of Murat's lemma (cf. [26]) proves subsequence convergence to a weak solution of* (1.3)–(1.4) *as $k\to 0$. Unfortunately, the scheme* (2.2) *is not practically useful, due to the nonconvex constraint.*

We end this section by introducing some notation and stating a few useful results. Let $1<p<\infty$. For all $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{m\times n}$, m, $n\geq 1$, and $\delta\geq 0$ there exist positive constants $C_i(p,m,n)$ such that

$$\begin{aligned}&\text{(i)}\quad \big|\,|\,\mathbf{P}\,|^{p-2}\mathbf{P}-|\,\mathbf{Q}\,|^{p-2}\mathbf{Q}\big|\leq C_1\big(|\,\mathbf{P}\,|+|\,\mathbf{Q}\,|\big)^{p-2+\delta}|\,\mathbf{P}-\mathbf{Q}\,|^{1-\delta}\,,\\ &\text{(ii)}\quad \big\langle|\,\mathbf{P}\,|^{p-2}\mathbf{P}-|\,\mathbf{Q}\,|^{p-2}\mathbf{Q},\mathbf{P}-\mathbf{Q}\big\rangle_{\mathbb{R}^{m\times n}}\geq C_2\big(|\,\mathbf{P}\,|+|\,\mathbf{Q}\,|\big)^{p-2-\delta}|\,\mathbf{P}-\mathbf{Q}\,|^{2+\delta}\,.\end{aligned} \tag{2.3}$$

For example, these results were proved in [7] for the case $\mathbb{R}^{n\times n}$, and that proof easily transfers to the present case. We recall the following well-known results concerning $(\cdot,\cdot)_h$:

$$\|\boldsymbol{\chi}\|_{L^2}^2\leq|\boldsymbol{\chi}|_h^2:=(\boldsymbol{\chi},\boldsymbol{\chi})_h\leq(n+2)\|\boldsymbol{\chi}\|_{L^2}^2\qquad\forall\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h\,; \tag{2.4}$$

$$|(\boldsymbol{\chi},\boldsymbol{\mathfrak{z}})-(\boldsymbol{\chi},\boldsymbol{\mathfrak{z}})_h|\leq Ch\|\boldsymbol{\chi}\|_{L^2}\|\nabla\boldsymbol{\mathfrak{z}}\|_{L^2}\leq C\|\boldsymbol{\chi}\|_{L^2}\|\boldsymbol{\mathfrak{z}}\|_{L^2}\qquad\forall\boldsymbol{\chi},\,\boldsymbol{\mathfrak{z}}\in\boldsymbol{\mathcal{V}}_h\,. \tag{2.5}$$

For later purposes, we introduce also the linear interpolation operator $\mathcal{I}_h:C(\overline{\Omega},\mathbb{R}^m)\to\boldsymbol{\mathcal{V}}_h$ such that $(\mathcal{I}_h\mathbf{v})(\mathbf{q}_i)=\mathbf{v}(\mathbf{q}_i)$ for all nodes $\mathbf{q}_i$ of $\mathcal{T}_h$. Finally, throughout the paper we adopt the standard notation for Sobolev spaces and their associated norms. For notational convenience, we drop the domain from the norm subscript if the domain is Ω, that is, $\|\cdot\|_{L^2}\equiv\|\cdot\|_{L^2(\Omega)}$.

3. Stability. As a first step toward showing the convergence of our numerical scheme to a weak solution of problem (1.3)–(1.4), we shall establish a discrete version of the energy inequality (1.5).

LEMMA 3.1. *Let the assumptions (A1) hold. Let $\mathbf{u}_0\in W^{1,p}(\Omega,S^{m-1})$ and $\mathbf{U}^0\in\boldsymbol{\mathcal{M}}_h$ satisfy $\mathbf{U}^0\to\mathbf{u}_0$ strongly in $W^{1,p}(\Omega,\mathbb{R}^m)$ as $h\to 0$, and let k satisfy (1.12). Then the iterates $\{\mathbf{V}^{j-1},\mathbf{U}^j\}_{j=1}^J$ computed from our scheme satisfy for $j=1\to J$*

$$\begin{aligned}(1-c_0)k\sum_{\ell=1}^{j}\|\,d_t\mathbf{U}^\ell\,\|_{L^2}^2+\frac{1}{p}\|\,\nabla\mathbf{U}^j\,\|_{L^p}^p&\leq(1-c_1)k\sum_{\ell=0}^{j-1}\|\,\mathbf{V}^\ell\,\|_{L^2}^2+\frac{1}{p}\|\,\nabla\mathbf{U}^j\,\|_{L^p}^p\\&\leq\frac{1}{p}\|\,\nabla\mathbf{U}^0\,\|_{L^p}^p+c_2\,,\end{aligned} \tag{3.1}$$

where c_i are $o(1)$.

Proof. Firstly, we choose $\mathbf{W}=\mathbf{V}^j$ in Step 2 of the scheme. As $\mathbf{U}^j\in\boldsymbol{\mathcal{M}}_h$, on noting (2.4) and on applying an inverse inequality, we conclude that

$$\begin{aligned}\|\,\mathbf{V}^j\,\|_{L^2}^2\leq|\mathbf{V}^j|_h^2&=-\big(|\,\nabla\mathbf{U}^j\,|^{p-2}\nabla\mathbf{U}^j,\nabla\mathbf{V}^j\big)\leq\int_\Omega|\,\nabla\mathbf{U}^j\,|^{p-1}|\,\nabla\mathbf{V}^j\,|\,\mathrm{d}\mathbf{x}\\&\leq\|\,\nabla\mathbf{U}^j\,\|_{L^{2(p-1)}}^{p-1}\|\,\nabla\mathbf{V}^j\,\|_{L^2}\leq Ch^{-p}\,\|\,\mathbf{U}^j\,\|_{L^{2(p-1)}}^{p-1}\|\,\mathbf{V}^j\,\|_{L^2}\leq Ch^{-2p}\,.\end{aligned} \tag{3.2}$$

We note that if $\|\nabla \mathbf{U}^j\|_{L^p} \leq C$; then we have, via (2.4) and inverse inequalities, the improved bound

$$\begin{aligned} \| \mathbf{V}^j \|_{L^2}^2 \leq |\mathbf{V}^j|_h^2 &= -\big(| \nabla \mathbf{U}^j |^{p-2} \nabla \mathbf{U}^j, \nabla \mathbf{V}^j\big) \leq \| \nabla \mathbf{U}^j \|_{L^p}^{p-1} \| \nabla \mathbf{V}^j \|_{L^p} \leq C h^{-1} \| \mathbf{V}^j \|_{L^p} \\ &\leq \begin{cases} C h^{-2} & \text{for } 1 \leq p \leq 2\,, \\ C h^{-2-n(1-\frac{2}{p})} & \text{for } 2 \leq p < \infty\,. \end{cases} \end{aligned} \tag{3.3}$$

The following argument is adapted from [2]. On defining $\mathbf{R}^j := \mathbf{U}^{j+1} - \mathbf{U}^j - k\mathbf{V}^j \in \boldsymbol{\mathcal{V}}_h$, then Step 3 of the scheme yields for all nodes $\mathbf{q}_i$ of $\mathcal{T}_h$ that

$$| \mathbf{R}^j(\mathbf{q}_i) | = \Big| \frac{\mathbf{U}^j(\mathbf{q}_i) + k\mathbf{V}^j(\mathbf{q}_i)}{| \mathbf{U}^j(\mathbf{q}_i) + k\mathbf{V}^j(\mathbf{q}_i) |} - \mathbf{U}^j(\mathbf{q}_i) - k\mathbf{V}^j(\mathbf{q}_i) \Big| = \Big| 1 - \big| \mathbf{U}^j(\mathbf{q}_i) + k\mathbf{V}^j(\mathbf{q}_i) \big| \Big|\,,$$

and since $1 \leq | \mathbf{U}^j(\mathbf{q}_i) + k\mathbf{V}^j(\mathbf{q}_i) | = \sqrt{1 + k^2 | \mathbf{V}^j(\mathbf{q}_i) |^2} \leq 1 + \frac{k^2}{2} | \mathbf{V}^j(\mathbf{q}_i) |^2$, we conclude that

$$| \mathbf{R}^j(\mathbf{q}_i) | \leq \frac{k^2}{2} | \mathbf{V}^j(\mathbf{q}_i) |^2\,. \tag{3.4}$$

Therefore, on recalling (2.4), we have that

$$\int_\Omega |\mathbf{R}^j| \,\mathrm{d}\mathbf{x} \leq \int_\Omega I_h[\,|\mathbf{R}^j|\,] \,\mathrm{d}\mathbf{x} \leq \frac{k^2}{2} \int_\Omega I_h[\,|\mathbf{V}^j|^2\,] \,\mathrm{d}\mathbf{x} \leq \frac{k^2(n+2)}{2} \int_\Omega |\mathbf{V}^j|^2 \,\mathrm{d}\mathbf{x}\,. \tag{3.5}$$

Similarly, it follows from (2.4) and an inverse inequality that

$$\|\mathbf{R}^j\|_{L^2}^2 \leq |\mathbf{R}^j|_h^2 \leq \frac{k^4}{4} \|\mathbf{V}^j\|_{L^\infty}^2 |\mathbf{V}^j|_h^2 \leq C k^4 h^{-n} \|\mathbf{V}^j\|_{L^2}^4\,; \tag{3.6}$$

and hence we have that

$$\|d_t \mathbf{U}^{j+1}\|_{L^2}^2 \leq [\|\mathbf{V}^j\|_{L^2} + k^{-1}\|\mathbf{R}^j\|_{L^2}]^2 \leq [1 + C k h^{-\frac{n}{2}} \|\mathbf{V}^j\|_{L^2}]^2 \|\mathbf{V}^j\|_{L^2}^2\,. \tag{3.7}$$

Now, choosing $\mathbf{W} = \mathbf{V}^j = d_t \mathbf{U}^{j+1} - k^{-1}\mathbf{R}^j$ in Step 2 of our scheme, noting the convexity of $|\nabla \cdot|^p$, that $\mathbf{U}^j$, $\mathbf{U}^{j+1} \in \boldsymbol{\mathcal{M}}_h$ and applying (2.3)(i) with $\delta = 2 - p$ if $p \in (1, 2]$ and $\delta = 0$ if $p \in [2, \infty)$, together with inverse estimates and (3.5), we arrive at

$$\begin{aligned} &\| \mathbf{V}^j \|_{L^2}^2 + \frac{1}{p} d_t \| \nabla \mathbf{U}^{j+1} \|_{L^p}^p \\ &\quad \leq |\mathbf{V}^j|_h^2 + (|\nabla \mathbf{U}^{j+1}|^{p-2} \nabla \mathbf{U}^{j+1}, \nabla (d_t \mathbf{U}^{j+1})) \\ &\quad = k^{-1}\big(| \nabla \mathbf{U}^j |^{p-2} \nabla \mathbf{U}^j, \nabla \mathbf{R}^j\big) + \big(| \nabla \mathbf{U}^{j+1} |^{p-2} \nabla \mathbf{U}^{j+1} - | \nabla \mathbf{U}^j |^{p-2} \nabla \mathbf{U}^j, \nabla (d_t \mathbf{U}^{j+1})\big) \\ &\quad \leq k^{-1} \| \nabla \mathbf{U}^j \|_{L^\infty}^{p-1} \| \nabla \mathbf{R}^j \|_{L^1} + C k^{1-\delta} [\, \|\nabla \mathbf{U}^{j+1}\|_{L^\infty} + \|\nabla \mathbf{U}^j\|_{L^\infty}\,]^{p-2+\delta} \|\nabla (d_t \mathbf{U}^{j+1})\|_{L^2}^{2-\delta} \\ &\quad \leq C k^{-1} h^{-p} \| \mathbf{R}^j \|_{L^1} + C k^{1-\delta} h^{-p} \|d_t \mathbf{U}^{j+1}\|_{L^2}^{2-\delta} \\ &\quad \leq C k h^{-p} \| \mathbf{V}^j \|_{L^2}^2 + C k^{1-\delta} h^{-p} \|d_t \mathbf{U}^{j+1}\|_{L^2}^{2-\delta}\,. \end{aligned} \tag{3.8}$$

We first consider the simpler case, $p \in [2, \infty)$. It follows from our assumptions on $\mathbf{U}^{\mathbf{0}}$ that there exists a constant $C_1 > 0$ such that $\| \nabla \mathbf{U}^0 \|_{L^p} \leq C_1$ for all $h > 0$. Assuming that $\| \nabla \mathbf{U}^j \|_{L^p} \leq C_1$ and $k = O(h^{1+n(1-\frac{1}{p})})$, it then follows from (3.3) that there exists a constant $C_2 > 0$ such that $k h^{-\frac{n}{2}} \|\mathbf{V}^j\|_{L^2} \leq C_2$. Therefore, combining (3.7) and (3.8) yields in the case $p \in [2, \infty)$ that there exists a constant $C_3 > 0$ such that

$$(1 - C_3 \frac{k}{h^p})\, k \| \mathbf{V}^j \|_{L^2}^2 + \frac{1}{p} \| \nabla \mathbf{U}^{j+1} \|_{L^p}^p \leq \frac{1}{p} \| \nabla \mathbf{U}^j \|_{L^p}^p\,. \tag{3.9}$$

If the time step k satisfies $C_3 k \leq h^p$, it follows from the above inequality that $\| \nabla \mathbf{U}^{j+1} \|_{L^p} \leq C_1$. Hence, by induction, (3.9) holds for $j = 0 \to J-1$ under the above two restrictions on k. On recalling our assumptions on k, (1.12), the desired stability result (3.1) for $p \in [2,\infty)$, with no c_2 term on the right hand side, follows from summing (3.9) and noting from (3.7) that $\|d_t\mathbf{U}^{j+1}\|_{L^2}^2 \le (1+o(1))\|\mathbf{V}^j\|_{L^2}^2$.

We now consider the case $p \in (1,2)$. Firstly, there exists a constant $C_4(p) > 0$ such that

$$\|d_t\mathbf{U}^{j+1}\|_{L^2}^p \le \|d_t\mathbf{U}^{j+1}\|_{L^2}^2 + C_4\,. \tag{3.10}$$

Assuming $k = O(h^{p+\frac{n}{2}})$, it then follows from (3.2) that there exists a constant $C_5 > 0$ such that $kh^{-\frac{n}{2}}\|\mathbf{V}^j\|_{L^2} \le C_5$. Therefore combining (3.7), (3.8) and (3.10) yields in the case $p \in (1,2)$ that there exists a constant $C_6 > 0$ such that

$$(1 - C_6\frac{k^{p-1}}{h^p})\,k\|\,\mathbf{V}^j\,\|_{L^2}^2 + \frac{1}{p}\|\,\nabla\mathbf{U}^{j+1}\,\|_{L^p}^p \le \frac{1}{p}\|\,\nabla\mathbf{U}^j\,\|_{L^p}^p + C_5\frac{k^p}{h^p}\,. \tag{3.11}$$

On recalling our assumptions on k, (1.12), the desired stability result (3.1) for $p \in (1,2)$ then follows from summing (3.11) and noting (3.7). □

4. Convergence. The following lemma, where we adopt the notation (1.11), will be needed for showing the convergence of our scheme.

LEMMA 4.1. *Let the assumptions of Lemma 3.1 hold. Then for all $\mathbf{W} \in L^2\big((0,T);\boldsymbol{\mathcal{F}}_h(\underline{\mathbf{U}})\big)$ it follows that*

$$\Big|\int_0^T \big[(\mathbf{U}_t,\mathbf{W}) + (|\,\nabla\underline{\mathbf{U}}\,|^{p-2}\nabla\underline{\mathbf{U}},\nabla\mathbf{W})\big]\,\mathrm{d}t\Big| \le C\left[kh^{-(\frac{n}{2}+1+\sigma)}\|\mathbf{W}\|_{L^2(\Omega_T)} + h\|\nabla\mathbf{W}\|_{L^2(\Omega_T)}\right], \tag{4.1}$$

where $\sigma = 0$ if $p \in (1,2)$ and $\sigma = n(\frac{1}{2}-\frac{1}{p})$ if $p \in [2,\infty)$.

Proof. Write $\underline{\mathbf{V}} = \mathbf{U}_t - k^{-1}\underline{\mathbf{R}}$ in Step 2 of our scheme to obtain for any $\mathbf{W} \in L^2\big((0,T);\boldsymbol{\mathcal{F}}_h(\underline{\mathbf{U}})\big)$ that

$$\int_0^T \big[(\mathbf{U}_t,\mathbf{W}) + (|\,\nabla\underline{\mathbf{U}}\,|^{p-2}\nabla\underline{\mathbf{U}},\nabla\mathbf{W})\big]\,\mathrm{d}t = k^{-1}\int_0^T (\underline{\mathbf{R}},\mathbf{W})\,\mathrm{d}t + \int_0^T [(\mathbf{U}_t,\mathbf{W}) - (\mathbf{U}_t,\mathbf{W})_h]\,\mathrm{d}t\,. \tag{4.2}$$

¿From (3.6), (3.3) and (3.1) we have that

$$\int_0^T \|\,\underline{\mathbf{R}}\,\|_{L^2}^2\,\mathrm{d}t \le Ck^4h^{-n}\int_0^T \|\,\underline{\mathbf{V}}\,\|_{L^2}^4\,\mathrm{d}t \le Ck^4h^{-(n+2+2\sigma)}\int_0^T \|\,\underline{\mathbf{V}}\,\|_{L^2}^2\,\mathrm{d}t \le Ck^4h^{-(n+2+2\sigma)}\,. \tag{4.3}$$

Hence the desired result (4.1) follows from (4.2), (4.3), (2.5) and (3.1). □

It follows from (3.1), our assumptions on $\mathbf{U}^0$ and as $\mathbf{U} \in \boldsymbol{\mathcal{M}}_h$ that there exists a function $\mathbf{u} \in H^1((0,T);L^2(\Omega,\mathbb{R}^m)) \cap L^\infty((0,T);W^{1,p}(\Omega,\mathbb{R}^m))$, and a subsequence of $\{\mathbf{U}\}_h$ such that as $h \to 0$

$$\begin{aligned}
&\mathbf{U},\,\underline{\mathbf{U}} \rightharpoonup \mathbf{u} \quad \text{weakly* in } L^\infty\big(0,T;W^{1,p}(\Omega,\mathbb{R}^m)\big)\,, \qquad \mathbf{U}_t \rightharpoonup \mathbf{u}_t \quad \text{weakly in } L^2(\Omega_T,\mathbb{R}^m)\,,\\
&\mathbf{U},\,\underline{\mathbf{U}} \to \mathbf{u} \quad \text{strongly in } L^q\big(\Omega_T,\mathbb{R}^m\big)\,,
\end{aligned} \tag{4.4}$$

where $q < \infty$ if $p \le n$ and $q = \infty$ if $p > n$. Furthermore, we have that (1.5) holds.

As $\mathbf{U} \in \boldsymbol{\mathcal{M}}_h$ it follows that $I_h[|\mathbf{U}|] \equiv 1$, and hence for every $K \in \mathcal{T}_h$ that

$$\big\||\,\mathbf{U}\,|^2 - 1\big\|_{L^p(K)} \le Ch^2\|D^2(|\,\mathbf{U}\,|^2)\|_{L^p(K)} \le Ch^2\big\|\nabla\mathbf{U}\big\|_{L^{2p}(K)}^2 \le Ch\big\|\nabla\mathbf{U}\big\|_{L^p(K)}\,. \tag{4.5}$$

Therefore, we deduce that

$$|\,\mathbf{u}\,| = 1 \text{ a.e. in } \Omega_T\,. \tag{4.6}$$

Next, in order to identify the limit of the p-Laplacian term in (4.1), we need to establish that

$$|\,\nabla\underline{\mathbf{U}}\,|^{p-2}\nabla\underline{\mathbf{U}} \rightharpoonup |\,\nabla\mathbf{u}\,|^{p-2}\nabla\mathbf{u} \quad \text{weakly in } L^{\frac{p}{p-1}}(\Omega_T,\mathbb{R}^{m\times n}) \quad \text{as } h \to 0\,. \tag{4.7}$$

The standard employment of Minty's lemma for monotone operators (see [35], 'the decisive monotonicity trick') is not so straightforward as (4.1) is only valid for $\mathbf{W} \in L^2((0,T);\boldsymbol{\mathcal{F}}_h(\underline{\mathbf{U}}))$ and not for all $\mathbf{W} \in$

$L^2((0,T);\boldsymbol{\mathcal{V}}_h)$. Obviously, if $p=2$ then (4.7) follows immediately from (4.4). The lemma below establishes a stronger version of (4.7) in the easier case when $p \in (n,\infty)$.

LEMMA 4.2. *In addition to the assumptions of Lemma 3.1 hold, let $p \in (n,\infty)$. Then we have for the subsequence $\{\mathbf{U}\}_{\mathbf{h}}$ of (4.4) that*

$$|\nabla \underline{\mathbf{U}}|^{p-2}\nabla \underline{\mathbf{U}} \to |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \quad \textit{strongly in } L^{\frac{p}{p-1}}(\Omega_T,\mathbb{R}^{m\times n}) \qquad \textit{as } h \to 0\,. \tag{4.8}$$

Proof. As $p \in (n,\infty)$, it follows that $\mathcal{I}_h\mathbf{u}$ is well-defined and

$$\mathcal{I}_h\mathbf{u} \to \mathbf{u} \quad \text{strongly in } L^\infty\big((0,T);W^{1,p}(\Omega,\mathbb{R}^m)\big) \text{ and hence in } L^\infty\big(\Omega_T,\mathbb{R}^m\big)\,. \tag{4.9}$$

We deduce from (2.3)(ii) with $\delta = p-2$ that

$$\begin{aligned} C\int_{\Omega_T} |\nabla(\mathbf{u}-\underline{\mathbf{U}})|^p \,\mathrm{d}\mathbf{x}\mathrm{d}t \le& \int_{\Omega_T} |\nabla\mathbf{u}|^{p-2}\langle\nabla\mathbf{u},\nabla(\mathbf{u}-\underline{\mathbf{U}})\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &- \int_{\Omega_T} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}},\nabla(\mathbf{u}-\mathcal{I}_h\mathbf{u})\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &- \int_{\Omega_T} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}},\nabla(\mathcal{I}_h\mathbf{u}-\underline{\mathbf{U}})\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t =: T_1+T_2+T_3\,. \end{aligned} \tag{4.10}$$

It follows from (4.4), (3.1) and (4.9) that T_1, $T_2 \to 0$ as $h \to 0$. As $\mathcal{I}_h\underline{\mathbf{U}} \equiv \underline{\mathbf{U}}$ and $\underline{\mathbf{U}}$, $\mathcal{I}_h\mathbf{u} \in \boldsymbol{\mathcal{M}}_h$, recall (4.6), we have that $\mathcal{I}_h\mathbf{u} - \underline{\mathbf{U}} \equiv \mathbf{W}+\mathbf{Z}$, where

$$\mathbf{W} = \mathcal{I}_h[\mathbf{u} - \langle\mathbf{u},\underline{\mathbf{U}}\rangle_{\mathbb{R}^m}\underline{\mathbf{U}}] \in \boldsymbol{\mathcal{F}}_h(\underline{\mathbf{U}}) \qquad \text{and} \qquad \mathbf{Z} = \mathcal{I}_h[(\langle\mathbf{u},\underline{\mathbf{U}}\rangle_{\mathbb{R}^m}-1)\underline{\mathbf{U}}] = -\tfrac{1}{2}\mathcal{I}_h[|\mathbf{u}-\underline{\mathbf{U}}|^2\underline{\mathbf{U}}]\,. \tag{4.11}$$

It follows from (4.11), (4.1), (1.12), an inverse inequality and (3.1) that

$$\begin{aligned} |T_3| &\le C\,\left[1+\|\mathbf{U}_t\|_{L^2(\Omega_T)}\right]\|\mathcal{I}_h[\mathbf{u}-\langle\mathbf{u},\underline{\mathbf{U}}\rangle_{\mathbb{R}^m}\underline{\mathbf{U}}]\|_{L^2(\Omega_T)} \\ &\qquad\qquad + C\|\underline{\mathbf{U}}\|^{p-1}_{L^\infty(0,T;W^{1,p}(\Omega))}\|\mathcal{I}_h[(\langle\mathbf{u},\underline{\mathbf{U}}\rangle_{\mathbb{R}^m}-1)\underline{\mathbf{U}}]\|_{L^1(0,T;W^{1,p}(\Omega))} \\ &\le C\left[\|\mathbf{u}-\langle\mathbf{u},\underline{\mathbf{U}}\rangle_{\mathbb{R}^m}\underline{\mathbf{U}}\|_{L^\infty(\Omega_T)} + \|\mathcal{I}_h[|\mathbf{u}-\underline{\mathbf{U}}|^2\,\underline{\mathbf{U}}]\|_{L^1(0,T;W^{1,p}(\Omega))}\right] \\ &\le C\left[\|\mathbf{u}-\underline{\mathbf{U}}\|_{L^\infty(\Omega_T)} + \||\mathbf{u}-\underline{\mathbf{U}}|^2\,\underline{\mathbf{U}}\|_{L^1(0,T;W^{1,p}(\Omega))}\right] \le C\|\mathbf{u}-\underline{\mathbf{U}}\|_{L^\infty(\Omega_T)}\,. \end{aligned} \tag{4.12}$$

On noting (4.4), as $p>n$, we have that $T_3 \to 0$ as $h \to 0$; and hence we have that the subsequence of $\{\mathbf{U}\}_h$ in (4.4) is such that

$$\underline{\mathbf{U}} \to \mathbf{u} \quad \text{strongly in } L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^m)\big)\,, \qquad \text{as } h \to 0. \tag{4.13}$$

The above and (2.3)(i) with $\delta = 0$ immediately yields the desired result (4.8). ☐

Unfortunately, if $p \in (1,n]$ and $p \neq 2$ the proof of the desired result (4.7) is far more complicated. One difficulty occurs as $\mathcal{I}_h$ is not well-defined on $\mathbf{u}$. If one replaces $\mathcal{I}_h$ by a generalised interpolation operator, $\mathcal{I}_h^g$, then $\mathcal{I}_h^g\underline{\mathbf{U}} \not\equiv \underline{\mathbf{U}}$, $\mathcal{I}_h^g\mathbf{u} \notin \boldsymbol{\mathcal{M}}_{\mathbf{h}}$ and, moreover, a generalization of (4.11) with the second crucial identity for $\mathbf{Z}$, exploited in (4.12) above, does not hold. To overcome this difficulty, we employ a density argument by smoothing $\mathbf{u}$ and continue to work with $\mathcal{I}_h$. However, to obtain a generalization of the second identity for $\mathbf{Z}$ in (4.11) we require this smoothed $\mathbf{u}(\cdot,t)$ to belong to S^{m-1} and not just $\mathbb{R}^m$. This requires the denseness of $C^\infty(\overline{\Omega},S^{m-1})$ in $W^{1,p}(\Omega,S^{m-1})$, which imposes the restrictions of either $p=n$ or $p<m-1$; see [9]. Another difficulty occurs if $p \in (1,n]$ and $p \neq 2$ as $\underline{\mathbf{U}} \to \mathbf{u}$ in $L^q(\Omega_T,\mathbb{R}^m)$ only for $q<\infty$ and not for $q=\infty$, recall (4.4). To overcome this we require a discrete version of Theorem 2.1 in [14], which exploits a monotonicity argument to deduce that the term II' in the proof there is non-positive. To obtain a discrete analogue of this, we require the right angle constraint, (A2), on our partitioning; which we now discuss in more detail.

Let $\{\mathbf{e}_i\}_{i=1}^n$ be the standard orthonormal vectors in $\mathbb{R}^n$, such that the j^{th} component of $\mathbf{e}_i$ is δ_{ij}, $i,j = 1 \to n$. Given non-zero constants ρ_i, $i = 1 \to n$; let $\widehat{K}(\{\rho_i\}_{i=1}^n)$ be a reference simplex in $\mathbb{R}^n$ with vertices $\{\widehat{\mathbf{q}}_i\}_{i=0}^n$, where $\widehat{\mathbf{q}}_0$ is the origin and $\widehat{\mathbf{q}}_i = \widehat{\mathbf{q}}_{i-1} + \rho_i \mathbf{e}_i$, $i = 1 \to n$. Then under assumptions (A2), given a $K \in \mathcal{T}_h$ with vertices $\{\mathbf{q}_{j_i}\}_{i=0}^n$, such that $\mathbf{q}_{i_0}$ is not a right-angled vertex, there exists a rotation/reflection matrix $B_K \in \mathbb{R}^{n\times n}$ such that the mapping $\mathcal{F}_K : \widehat{\mathbf{x}} \in \mathbb{R}^n \to \mathbf{q}_{j_0} + B_K \widehat{\mathbf{x}} \in \mathbb{R}^n$ maps the vertex $\widehat{\mathbf{q}}_i$ to $\mathbf{q}_{j_i}$, $i = 0 \to n$, and hence $\widehat{K}(\{\rho_i\}_{i=1}^n)$ to K. Then for all $K \in \mathcal{T}_h$, $\phi \in C(\overline{K}, \mathbb{R})$ and $\boldsymbol{\phi} \in C(\overline{K}, \mathbb{R}^m)$, we set for all $\widehat{\mathbf{x}} \in \overline{\widehat{K}(\{\rho_i\}_{i=1}^n)}$

$$(4.14) \qquad \widehat{\phi}(\widehat{\mathbf{x}}) \equiv \phi(\mathcal{F}_K\widehat{\mathbf{x}}), \qquad (\widehat{I\phi})(\widehat{\mathbf{x}}) \equiv (I_h\phi)(\mathcal{F}_K\widehat{\mathbf{x}}); \qquad \widehat{\boldsymbol{\phi}}(\widehat{\mathbf{x}}) \equiv \boldsymbol{\phi}(\mathcal{F}_K\widehat{\mathbf{x}}), \quad (\widehat{\mathcal{I}\boldsymbol{\phi}})(\widehat{\mathbf{x}}) \equiv (\mathcal{I}_h\boldsymbol{\phi})(\mathcal{F}_K\widehat{\mathbf{x}}).$$

We have for any $\mathbf{Z} \in \boldsymbol{\mathcal{V}}_h$ and $K \in \mathcal{T}_h$ that

$$(4.15) \qquad \nabla \mathbf{Z} \equiv (\widehat{\nabla}\widehat{\mathbf{Z}})\, B_K^{-1} \qquad \text{on } K,$$

where $\mathbf{x} \equiv (x_1, \cdots, x_n)^T$, $\nabla \equiv (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$, $\widehat{\mathbf{x}} \equiv (\widehat{x}_1, \cdots, \widehat{x}_n)^T$ and $\widehat{\nabla} \equiv (\frac{\partial}{\partial \widehat{x}_1}, \cdots, \frac{\partial}{\partial \widehat{x}_n})$. It is easily deduced, see e.g. [6] for details, that for any z_1, $z_2 \in C(\overline{\Omega}, \mathbb{R})$

$$(4.16) \qquad \nabla(I_h[z_1 z_2]) = \nabla(I_h z_2) D(I_h z_1) + \nabla(I_h z_1) D(I_h z_2);$$

where for any $Z \in V^h$,

$$(4.17) \qquad D(Z)\,|_K := B_K\, \widehat{D}(\widehat{Z}) B_K^{-1} \qquad \forall\ K \in \mathcal{T}_h,$$

and $\widehat{D}(\widehat{Z})$ is the $n \times n$ diagonal matrix with diagonal entries

$$(4.18) \qquad [\widehat{D}(\widehat{Z})]_{ii} := \tfrac{1}{2}\left[\widehat{Z}(\widehat{\mathbf{q}}_i) + \widehat{Z}(\widehat{\mathbf{q}}_{i-1})\right] \qquad i = 1 \to n.$$

Lemma 4.3. *In addition to the assumptions of Lemma 3.1 hold, let either $p = n$ or $p < m - 1$, and let the assumptions (A2) hold. Then we have for the subsequence $\{\mathbf{U}\}_\mathbf{h}$ of (4.4) and for any $s \in [1, p)$ that*

$$(4.19) \qquad \nabla \underline{\mathbf{U}} \to \nabla \mathbf{u} \qquad \textit{strongly in } L^s(\Omega_T, \mathbb{R}^{m\times n}) \textit{ as } h \to 0.$$

Hence the desired result (4.7) holds.

Proof. As either $p = n$ or $p < m - 1$, it follows that $C^\infty(\overline{\Omega}, S^{m-1})$ is a dense subset of $W^{1,p}(\Omega, S^{m-1})$; see [9]. Hence for any fixed $\alpha \in (0, 1)$, there exists $\mathbf{u}_\alpha \in L^\infty(0, T; C^\infty(\overline{\Omega}, S^{m-1}))$ such that

$$(4.20) \qquad \|\mathbf{u} - \mathbf{u}_\alpha\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq \alpha^2.$$

Therefore $\mathcal{I}_h \mathbf{u}_\alpha$ is well-defined and

$$(4.21) \qquad \mathcal{I}_h \mathbf{u}_\alpha \to \mathbf{u}_\alpha \quad \text{strongly in } L^\infty\big(0, T; W^{1,p}(\Omega, \mathbb{R}^m)\big).$$

In addition, we introduce $\boldsymbol{\eta}_\alpha : \mathbb{R}^m \to \mathbb{R}^m$ and $\eta_\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$(4.22) \qquad \boldsymbol{\eta}_\alpha(\mathbf{y}) := \eta_\alpha(|\mathbf{y}|)\mathbf{y} := \begin{cases} \mathbf{y} & \text{if } |\mathbf{y}| \leq \alpha \\ \frac{\alpha}{|\mathbf{y}|}\mathbf{y} & \text{if } |\mathbf{y}| \geq \alpha \end{cases}.$$

On adopting the notation in (4.14) and (4.15), we have for all $\mathbf{Z} \in \boldsymbol{\mathcal{V}}_h$ and $K \in \mathcal{T}_h$ that

$$(4.23) \qquad \frac{\partial}{\partial \widehat{x}_k}\widehat{\mathcal{I}}[\boldsymbol{\eta}_\alpha(\widehat{\mathbf{Z}})] \equiv A_\alpha^{(k)}(\widehat{\mathbf{Z}})\frac{\partial \widehat{\mathbf{Z}}}{\partial \widehat{x}_k} \qquad \text{on } \widehat{K}, \qquad k = 1 \to n,$$

where $A_\alpha^{(k)}(\widehat{\mathbf{Z}}) \in \mathbb{R}^{m\times m}$ is such that for i, $j = 1 \to m$

$$(4.24) \qquad \begin{aligned} [A_\alpha^{(k)}(\widehat{\mathbf{Z}})]_{ij} &= \tfrac{1}{2}[\eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)|) + \eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|)]\delta_{ij} \\ &\quad + \tfrac{1}{2}\frac{[\eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)|) - \eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|)]}{|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)| - |\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|}\,\frac{([\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)]_i + [\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})]_i)([\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)]_j + [\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})]_j)}{|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)| + |\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|}. \end{aligned}$$

For any $\mathbf{y} \in \mathbb{R}^m$, we deduce from the monotonicity of η_α that

$$
\begin{aligned}
\mathbf{y}^T A_\alpha^{(k)}(\widehat{\mathbf{Z}})\mathbf{y} &\geq \tfrac{1}{2}[\eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)|) + \eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|)]|\mathbf{y}|^2 \\
&\quad + \tfrac{1}{2}\frac{[\eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)|) - \eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|)]}{|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)| - |\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|}\frac{|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k) + \widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|^2}{|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)| + |\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|}|\mathbf{y}|^2 \\
&\geq \tfrac{1}{2}\left([\eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)|) + \eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|)] + \frac{[\eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)|) - \eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|)]}{|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)| - |\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|}(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)| + |\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|)\right)|\mathbf{y}|^2 \\
&\geq \frac{[\eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)|)|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)| - \eta_\alpha(|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|)|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|]}{|\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_k)| - |\widehat{\mathbf{Z}}(\widehat{\mathbf{q}}_{k-1})|}|\mathbf{y}|^2 \geq 0\,. \qquad (4.25)
\end{aligned}
$$

Therefore $A_\alpha^{(k)}(\widehat{\mathbf{Z}})$ is symmetric positive semi-definite for any $\mathbf{Z} \in \boldsymbol{\mathcal{V}}_h$. Similarly to (4.25), we have for all $\mathbf{Z} \in \boldsymbol{\mathcal{V}}_h$ and on any $K \in \mathcal{T}^h$ that

$$
\begin{aligned}
\mathbf{y}^T A_\alpha^{(k)}(\widehat{\mathbf{Z}})\mathbf{y} &\leq \frac{\eta_\alpha(|\widehat{\mathbf{Z}}(\mathbf{q}_{k-1})|)|\widehat{\mathbf{Z}}(\mathbf{q}_k)| - \eta_\alpha(|\widehat{\mathbf{Z}}(\mathbf{q}_k)|)|\widehat{\mathbf{Z}}(\mathbf{q}_{k-1})|}{|\widehat{\mathbf{Z}}(\mathbf{q}_k)| - |\widehat{\mathbf{Z}}(\mathbf{q}_{k-1})|}|\mathbf{y}|^2 \\
&\leq [\eta_\alpha(|\widehat{\mathbf{Z}}(\mathbf{q}_k)|) + \eta_\alpha(|\widehat{\mathbf{Z}}(\mathbf{q}_{k-1})|)]|\mathbf{y}|^2 \leq 2|\mathbf{y}|^2 \qquad \forall \mathbf{y} \in \mathbb{R}^m\,. \qquad (4.26)
\end{aligned}
$$

It follows from (4.15), $B_K^{-1} \equiv B_K^T$, (4.23), (4.25) and (4.26) that for all $\mathbf{Z}$, $\mathbf{Y} \in \boldsymbol{\mathcal{V}}_h$ and on any $K \in \mathcal{T}_h$

$$
\begin{aligned}
\langle \nabla \mathbf{Z}, \nabla(\mathcal{I}_\mathbf{h}[\eta_\alpha(\mathbf{Y}-\mathbf{Z})])\rangle_{\mathbb{R}^{m\times n}} &= \langle (\widehat{\nabla}\widehat{\mathbf{Z}})B_K^{-1}, (\widehat{\nabla}(\widehat{\mathcal{I}}[\eta_\alpha(\widehat{\mathbf{Y}}-\widehat{\mathbf{Z}})]))B_K^{-1}\rangle_{\mathbb{R}^{m\times n}} \\
&= \langle \widehat{\nabla}\widehat{\mathbf{Z}}, \widehat{\nabla}(\widehat{\mathcal{I}}[\eta_\alpha((\widehat{\mathbf{Y}}-\widehat{\mathbf{Z}})])\rangle_{\mathbb{R}^{m\times n}} = \sum_{k=1}^{n}\langle \frac{\partial \widehat{\mathbf{Z}}}{\partial \widehat{x}_k}, A_\alpha^{(k)}(\widehat{\mathbf{Y}}-\widehat{\mathbf{Z}})\frac{\partial(\widehat{\mathbf{Y}}-\widehat{\mathbf{Z}})}{\partial \widehat{x}_k}\rangle_{\mathbb{R}^m} \\
&\leq C|\widehat{\nabla}\widehat{\mathbf{Z}}||\widehat{\nabla}\widehat{\mathbf{Y}}| \leq C|\nabla \mathbf{Z}||\nabla \mathbf{Y}|\,. \qquad (4.27)
\end{aligned}
$$

Hence we deduce from (4.27) that for all $\mathbf{Z}, \mathbf{Y} \in \boldsymbol{\mathcal{V}}_h$ and $K \in \mathcal{T}_h$

$$
\int_K |\nabla \mathbf{Z}|^{p-2}\langle \nabla \mathbf{Z}, \nabla(\mathcal{I}_h[\boldsymbol{\eta}_\alpha(\mathbf{Y}-\mathbf{Z})])\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x} \leq C\|\nabla \mathbf{Z}\|_{L^p(K)}^{p-1}\|\nabla \mathbf{Y}\|_{L^p(K)}\,. \qquad (4.28)
$$

It is this bound, which we use in bounding T_3 below (containing the analogue of the term II' in the proof of Theorem 2.1 in [14]), that exploits the right angle constraint, (A2), on the partitioning.

As $|\mathbf{u}_\alpha| = |\underline{\mathbf{U}}| = 1$ in Ω_T, we have from (4.22) that

$$
\langle \boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbb{R}^m} = -\tfrac{1}{2}\langle \boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \mathbf{u}_\alpha - \underline{\mathbf{U}}\rangle_{\mathbb{R}^m}\,. \qquad (4.29)
$$

It follows from (4.29) and (4.22) that

$$
\|I_h[\langle \boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbb{R}^m}]\|_{L^\infty(\Omega_T)} \leq \tfrac{1}{2}\alpha\|\mathbf{u}_\alpha - \underline{\mathbf{U}}\|_{L^\infty(\Omega_T)} \leq \alpha\,. \qquad (4.30)
$$

It follows from (4.15), (4.14), (4.29) and (4.22) that for all $K \in \mathcal{T}_h$

$$
\begin{aligned}
\|\nabla(I_h[\langle \boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbb{R}^m})]\|_{L^p(K)} &\leq C\|\widehat{\nabla}(\widehat{I}[\langle \boldsymbol{\eta}_\alpha(\widehat{\mathbf{u}}_\alpha - \widehat{\underline{\mathbf{U}}}), \widehat{\underline{\mathbf{U}}}\rangle_{\mathbb{R}^m})]\|_{L^p(\widehat{K})} \\
&\leq C\alpha\|\widehat{\nabla}|\widehat{I}(\widehat{\mathbf{u}}_\alpha) - \widehat{\underline{\mathbf{U}}}|\|_{L^p(\widehat{K})} \leq C\alpha\|\widehat{\nabla}[\widehat{I}(\widehat{\mathbf{u}}_\alpha) - \widehat{\underline{\mathbf{U}}}]\|_{L^p(\widehat{K})} \\
&\leq C\alpha\|\nabla[I_h(\mathbf{u}_\alpha) - \underline{\mathbf{U}}]\|_{L^p(K)}\,. \qquad (4.31)
\end{aligned}
$$

For a.a. $t \in (0,T)$ let

$$
\begin{aligned}
&\mathcal{J}_{h,\alpha}(t) := \{\text{nodes } \mathbf{q}_i \text{ of } \mathcal{T}_h : |(\mathcal{I}_h \mathbf{u}_\alpha)(t,\mathbf{q}_i) - \underline{\mathbf{U}}(t,\mathbf{q}_i)| \geq \alpha\}, \\
&\mathcal{T}_{h,\alpha}(t) := \{K \in \mathcal{T}_h : K \text{ has a vertex } \mathbf{q}_i \in \mathcal{J}^{h,\alpha}(t)\}, \\
\text{and}\quad &\mathcal{R}_{h,\alpha}(t) := \cup_{K \in \mathcal{T}_{h,\alpha}(t)}\overline{K}\,. \qquad (4.32)
\end{aligned}
$$

It follows from (2.4), (1.9) and (4.32) that

$$
\begin{aligned}
\frac{\alpha^2}{n+1}\int_0^T |\mathcal{R}_{h,\alpha}(t)|\,\mathrm{d}t &\leq \int_0^T |\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}}|_h^2\,\mathrm{d}t \leq (n+2)\|\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}}\|_{L^2(\Omega_T)}^2 \\
&\leq 2(n+2)\left[\|\mathbf{u}_\alpha - \mathcal{I}_h\mathbf{u}_\alpha\|_{L^2(\Omega_T)}^2 + \|\mathbf{u}_\alpha - \underline{\mathbf{U}}\|_{L^2(\Omega_T)}^2\right]. \qquad (4.33)
\end{aligned}
$$

Hence we deduce from (4.33), (4.21), (4.4) and (4.20) that

$$
\lim_{h\to 0}\int_0^T |\mathcal{R}_{h,\alpha}(t)|\,\mathrm{d}t \leq C\alpha^2\,. \qquad (4.34)
$$

In addition, it follows from (4.21) and (4.20) that

$$
\lim_{h\to 0}\int_0^T\int_{\mathcal{R}_{h,\alpha}(t)} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha)|^p\,\mathrm{d}\mathbf{x}\mathrm{d}t \leq \lim_{h\to 0}\int_0^T\int_{\mathcal{R}_{h,\alpha}(t)} |\nabla\mathbf{u}|^p\,\mathrm{d}\mathbf{x}\mathrm{d}t + C\alpha^2\,. \qquad (4.35)
$$

For any $s\in[1,p)$, we have that

(4.36)

$$
\int_0^T\left(\int_{\mathcal{R}_{h,\alpha}(t)} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}})|^s\mathrm{d}\mathbf{x}\right)\mathrm{d}t \leq \left(\int_0^T |\mathcal{R}_{h,\alpha}(t)|\,\mathrm{d}t\right)^{\frac{p-s}{p}}\left(\int_0^T\left(\int_{\mathcal{R}_{h,\alpha}(t)} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}})|^p\mathrm{d}\mathbf{x}\right)\mathrm{d}t\right)^{\frac{s}{p}}.
$$

Let $p^\star := \max\{2,p\}$. Then on applying a Hölder inequality, noting (4.20), (4.21), (4.4) and (2.3)(ii) with $\delta = p^\star - 2$ we have that

$$
\begin{aligned}
&C\left(\int_0^T\int_{\Omega\setminus\mathcal{R}_{h,\alpha}(t)} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}})|^p\mathrm{d}\mathbf{x}\mathrm{d}t\right)^{\frac{p^\star}{p}} \\
&\quad\leq C\int_0^T\left(\int_{\Omega\setminus\mathcal{R}_{h,\alpha}(t)} [\,|\nabla(\mathcal{I}_h\mathbf{u}_\alpha)| + |\nabla\underline{\mathbf{U}}|\,]^p\,\mathrm{d}\mathbf{x}\right)^{-\frac{(p^\star-p)}{p}}\left(\int_{\Omega\setminus\mathcal{R}_{h,\alpha}(t)} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}})|^p\mathrm{d}\mathbf{x}\right)^{\frac{p^\star}{p}}\mathrm{d}t \\
&\quad\leq C_2\int_0^T\int_{\Omega\setminus\mathcal{R}_{h,\alpha}(t)} [\,|\nabla(\mathcal{I}_h\mathbf{u}_\alpha)| + |\nabla\underline{\mathbf{U}}|\,]^{p-p^\star}\,|\nabla(\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}})|^{p^\star}\mathrm{d}\mathbf{x}\mathrm{d}t \\
&\quad\leq \int_0^T\int_{\Omega\setminus\mathcal{R}_{h,\alpha}(t)} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha)|^{p-2}\langle\nabla(\mathcal{I}_h\mathbf{u}_\alpha), \nabla(\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}})\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&\qquad - \int_0^T\int_{\Omega\setminus\mathcal{R}_{h,\alpha}(t)} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \nabla(\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}})\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t =: T_1 + T_2\,. \qquad (4.37)
\end{aligned}
$$

It follows from (3.1), (4.21) and (4.20) that

(4.38)

$$
|T_1| \leq \left|\int_{\Omega_T} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha)|^{p-2}\langle\nabla(\mathcal{I}_h\mathbf{u}_\alpha), \nabla(\mathcal{I}_h\mathbf{u}_\alpha - \underline{\mathbf{U}})\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t\right| + C\left(\int_0^T\int_{\mathcal{R}_{h,\alpha}(t)} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha)|^p\,\mathrm{d}\mathbf{x}\mathrm{d}t\right)^{\frac{p-1}{p}}.
$$

Hence we deduce from (4.38), (4.21), (4.4) and (4.20) that

$$
\lim_{h\to 0}|T_1| \leq C\,\alpha^2 + C\lim_{h\to 0}\left(\int_0^T\int_{\mathcal{R}_{h,\alpha}(t)} |\nabla(\mathcal{I}_h\mathbf{u}_\alpha)|^p\,\mathrm{d}\mathbf{x}\mathrm{d}t\right)^{\frac{p-1}{p}}. \qquad (4.39)
$$

Next we note from (4.32) and (4.22) that

$$
\begin{aligned}
T_2 &= -\int_0^T \int_{\Omega\setminus\mathcal{R}_{h,\alpha}(t)} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \nabla(\mathcal{I}_h[\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}})])\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&= \int_0^T \int_{\mathcal{R}_{h,\alpha}(t)} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \nabla(\mathcal{I}_h[\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}})])\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&\quad - \int_{\Omega_T} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \nabla(\mathcal{I}_h[\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}})])\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t =: T_3 + T_4\,.
\end{aligned}
\tag{4.40}
$$

It follows from (4.28) and (3.1) that

$$
T_3 \le C\|\nabla\underline{\mathbf{U}}\|_{L^p(\Omega_T)}^{p-1}\left(\int_0^T\int_{\mathcal{R}_{h,\alpha}(t)}|\nabla(\mathcal{I}_h\mathbf{u}_\alpha)|^p\,\mathrm{d}\mathbf{x}\mathrm{d}t\right)^{\frac{1}{p}} \le C\left(\int_0^T\int_{\mathcal{R}_{h,\alpha}(t)}|\nabla(\mathcal{I}_h\mathbf{u}_\alpha)|^p\,\mathrm{d}\mathbf{x}\mathrm{d}t\right)^{\frac{1}{p}}\,. \tag{4.41}
$$

Noting that $\mathcal{I}_h[\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}) - \langle\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbf{R}^m}\underline{\mathbf{U}}] \in \boldsymbol{\mathcal{F}}_h(\underline{\mathbf{U}})$, we have that

$$
\begin{aligned}
T_4 &= -\int_{\Omega_T}|\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \nabla(\mathcal{I}_h[\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}) - \langle\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbf{R}^m}\underline{\mathbf{U}}])\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&\quad - \int_{\Omega_T}|\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \nabla(\mathcal{I}_h[\langle\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbf{R}^m}\underline{\mathbf{U}}])\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t =: T_5 + T_6\,.
\end{aligned}
\tag{4.42}
$$

It then follows from (4.1), (1.12), an inverse inequality, (3.1), (2.4), (4.22) and (4.20) that

$$
\begin{aligned}
|T_5| &\le C\left[1 + \|\mathbf{U}_t\|_{L^2(\Omega_T)}\right]\left[\int_0^T \|\mathcal{I}_h[\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}) - \langle\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbf{R}^m}\underline{\mathbf{U}}]\|_{L^2}^2\,\mathrm{d}t\right]^{\frac{1}{2}} \\
&\le C\|\mathcal{I}_h[\mathbf{u}_\alpha - \underline{\mathbf{U}}]\|_{L^2(\Omega_T)} \le C\,[\,\|\mathbf{u} - \underline{\mathbf{U}}\|_{L^2(\Omega_T)} + \|\mathbf{u}_\alpha - \mathcal{I}_h\mathbf{u}_\alpha\|_{L^2(\Omega_T)} + \alpha^2\,]\,.
\end{aligned}
\tag{4.43}
$$

We note from (3.1), (4.16), (4.17), (4.18), (4.30) and (4.31) that

$$
\begin{aligned}
|T_6| &\le \|\nabla\underline{\mathbf{U}}\|_{L^p(\Omega_T)}^{p-1}\,\|\nabla(\mathcal{I}_h[\langle\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbf{R}^m}\underline{\mathbf{U}}])\|_{L^p(\Omega_T)} \le C\,\|\nabla(\mathcal{I}_h[\langle\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbf{R}^m}\underline{\mathbf{U}}])\|_{L^p(\Omega_T)} \\
&\le \|\underline{\mathbf{U}}\|_{L^\infty(\Omega_T)}\|\nabla(\mathcal{I}_h[\langle\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbf{R}^m}])\|_{L^p(\Omega_T)} + \|\mathcal{I}_h[\langle\boldsymbol{\eta}_\alpha(\mathbf{u}_\alpha - \underline{\mathbf{U}}), \underline{\mathbf{U}}\rangle_{\mathbf{R}^m}]\|_{L^\infty(\Omega_T)}\|\nabla\underline{\mathbf{U}}\|_{L^p(\Omega_T)} \\
&\le C\alpha\left[\|\nabla\mathbf{u}_\alpha\|_{L^p(\Omega_T)} + \|\nabla\underline{\mathbf{U}}\|_{L^p(\Omega_T)}\right]\,.
\end{aligned}
\tag{4.44}
$$

On combining (4.36)–(4.44), (3.1), (4.20), (4.21), (4.34), (4.35) and (4.4) we have that given any $\epsilon > 0$, there exist an $\alpha(\epsilon)$ and an $h_0(\alpha)$ such that for the subsequence $\{\mathbf{U}\}_{\mathbf{h}}$ of (4.4)

$$
\|\nabla(\mathbf{u}_\alpha - \underline{\mathbf{U}})\|_{L^s(\Omega_T)} \le \epsilon \qquad \forall h \le h_0\,. \tag{4.45}
$$

The desired result (4.19) then follows immediately from (4.45), (4.20) and (4.21). Finally, the desired result (4.7) follows immediately from (4.19) and (3.1), cf. [30, Lemma 6]. □

We now are ready to give a proof for Theorem 1.1.

Proof of Theorem 1.1: Given any $\boldsymbol{\phi} \in C^\infty(\overline{\Omega_T}, \mathbb{R}^m)$, let $\mathbf{w} = \mathbf{u}\times\boldsymbol{\phi}$, and $\mathbf{W} = \mathcal{I}_h(\underline{\mathbf{U}}\times\boldsymbol{\phi})$. Interpolation theory yields that

$$
\begin{aligned}
\|\,\mathcal{I}_h(\underline{\mathbf{U}}\times\boldsymbol{\phi}) - \underline{\mathbf{U}}\times\boldsymbol{\phi}\,\|_{L^2}^2 &\le Ch^4\sum_{K\in\mathcal{T}_h}\|\,D^2(\underline{\mathbf{U}}\times\boldsymbol{\phi})\,\|_{L^2(K)}^2 \\
&\le Ch^4\left[\|\,|\,\underline{\mathbf{U}}\,|\,|\,D^2\boldsymbol{\phi}\,|\,\|_{L^2}^2 + \|\,|\,\nabla\underline{\mathbf{U}}\,|\,|\,\nabla\boldsymbol{\phi}\,|\,\|_{L^2}^2\right] \\
&\le Ch^4\|\,\boldsymbol{\phi}\,\|_{H^2}^2 + Ch^{4-\gamma}\|\nabla\boldsymbol{\phi}\|_{L^\infty}^2\,\|\,\nabla\underline{\mathbf{U}}\,\|_{L^p}^2\,,
\end{aligned}
\tag{4.46}
$$

where $\gamma = n(2-p)/p$ if $p \in (1,2]$ and $\gamma = 0$ if $p \in (2,\infty)$. Therefore (4.46) and (4.4) yield that $\mathbf{W} \to \mathbf{w}$ strongly in $L^2(\Omega_T, \mathbb{R}^m)$, which in turn implies that

$$
\int_{\Omega_T}\langle\mathbf{U}_t, \mathbf{W}\rangle_{\mathbb{R}^m}\,\mathrm{d}\mathbf{x}\mathrm{d}t \to \int_{\Omega_T}\langle\mathbf{u}_t, \mathbf{w}\rangle_{\mathbb{R}^m}\,\mathrm{d}\mathbf{x}\mathrm{d}t \qquad \text{as } h \to 0\,. \tag{4.47}
$$

We now consider the p-Laplacian term. Similarly to (4.46), we have that

$$\|\nabla(\mathcal{I}_h(\underline{\mathbf{U}} \times \boldsymbol{\phi}) - \underline{\mathbf{U}} \times \boldsymbol{\phi})\|_{L^p}^2 \leq Ch^2 \left[\|\boldsymbol{\phi}\|_{W^{2,p}}^2 + \|\nabla\boldsymbol{\phi}\|_{L^\infty}^2 \|\nabla\underline{\mathbf{U}}\|_{L^p}^2\right] . \tag{4.48}$$

On noting the vector identity $\langle \nabla\mathbf{z}, \nabla(\mathbf{z} \times \boldsymbol{\phi})\rangle_{\mathbb{R}^{m\times n}} = \langle \nabla\mathbf{z}, \mathbf{z} \times \nabla\boldsymbol{\phi}\rangle_{\mathbb{R}^{m\times n}}$, (4.4) and (4.7) it follows that as $h \to 0$

$$\begin{aligned}\int_{\Omega_T} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \nabla(\underline{\mathbf{U}} \times \boldsymbol{\phi})\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t &= \int_{\Omega_T} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \underline{\mathbf{U}} \times \nabla\boldsymbol{\phi}\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\ \to \quad \int_{\Omega_T} |\nabla\mathbf{u}|^{p-2}\langle\nabla\mathbf{u}, \mathbf{u} \times \nabla\boldsymbol{\phi}\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t &= \int_{\Omega_T} |\nabla\mathbf{u}|^{p-2}\langle\nabla\mathbf{u}, \nabla(\mathbf{u} \times \boldsymbol{\phi})\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t\,.\end{aligned} \tag{4.49}$$

Noting (4.48), (4.49) and (4.7) we have that

$$\int_{\Omega_T} |\nabla\underline{\mathbf{U}}|^{p-2}\langle\nabla\underline{\mathbf{U}}, \nabla\mathbf{W}\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \quad \to \quad \int_{\Omega_T} |\nabla\mathbf{u}|^{p-2}\langle\nabla\mathbf{u}, \nabla\mathbf{w}\rangle_{\mathbb{R}^{m\times n}}\,\mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{as } h \to 0\,. \tag{4.50}$$

Finally if $p \in (1, 2]$, we deduce from an inverse inequality that

$$\begin{aligned}h^2\|\nabla\mathcal{I}_h(\underline{\mathbf{U}} \times \boldsymbol{\phi})\|_{L^2}^2 &\leq h^2\|\nabla\mathcal{I}_h(\underline{\mathbf{U}} \times \boldsymbol{\phi})\|_{L^\infty}^{2-p}\|\nabla\mathcal{I}_h(\underline{\mathbf{U}} \times \boldsymbol{\phi})\|_{L^p}^p \\ &\leq Ch^p\|\mathcal{I}_h(\underline{\mathbf{U}} \times \boldsymbol{\phi})\|_{L^\infty}^{2-p}\|\nabla\mathcal{I}_h(\underline{\mathbf{U}} \times \boldsymbol{\phi})\|_{L^p}^p \leq Ch^p\|\nabla\mathcal{I}_h(\underline{\mathbf{U}} \times \boldsymbol{\phi})\|_{L^p}^p\,.\end{aligned} \tag{4.51}$$

It follows from (4.47), (4.50), (4.1), (4.46), (4.4), our constraints on the time step k, (1.12), (4.51), (4.48) and (3.1) that we now can pass to the limit $h \to 0$ in (4.1) to obtain that for all $\boldsymbol{\phi} \in C^\infty(\overline{\Omega_T}, \mathbb{R}^m)$

$$\int_0^T \left[(\mathbf{u}_t, \mathbf{u} \times \boldsymbol{\phi}) + (|\nabla\mathbf{u}|^{\mathrm{p}-2}\nabla\mathbf{u}, \nabla(\mathbf{u} \times \boldsymbol{\phi}))\right]\,\mathrm{d}t = 0. \tag{4.52}$$

However, as (4.6) holds, the above equation implies that $\mathbf{u} : \Omega_T \to S^{m-1}$ satisfies (1.3)–(1.4) in the weak sense, see Lemma 1.8 in [31], or the proof of Theorem 2.2 in [14]. Hence, we have proved Theorem 1.1. ☐

5. Numerical experiments: finite-time blow-up and geometric changes. The global existence and the nonuniqueness of weak solutions to (1.3)–(1.4) for $p > 1$, and the local existence of smooth solutions motivate finite-time blow-up studies. We say that $\mathbf{u}$ blows up at $t^\star$ if

$$\limsup_{t \nearrow t^\star} \|\nabla\mathbf{u}(t)\|_{L^\infty} = \infty\,.$$

We employ our convergent numerical scheme to compute such phenomenon. Throughout these numerical experiments, we set $\Omega := (-1, 1)^2 \subset \mathbb{R}^2$, i.e. $n = 2$, and $m = 3$; recall (1.13). We choose a uniform right-angled triangulation of Ω with $h = \sqrt{2}/2^3$ and set $\mathbf{U}^0 \equiv \mathcal{I}_h\mathbf{u_0}$. Unless otherwise stated, we choose $k = h^{s+1/2}/10$ for $s = \max\{p/(p-1), p\}$. In all of the experiments reported below we observed that $E_p(\mathbf{U}^{j+1}) \leq E_p(\mathbf{U}^j)$ for all $j \geq 0$ for this choice of k; recall the stability requirements of Theorem 1.1 and that for $p \in (1, 2)$ we computed with $p = 3/2$ and $5/4 \Rightarrow p/(p-1) \geq p+1 \equiv p + \frac{n}{2}$. Finally, as $m = 3$, below we plot at each node $\mathbf{q}_i$ of $\mathcal{T}_h$ a vector based on the first two components of $\mathbf{U}^j(\mathbf{x}_i)$.

EXAMPLE 5.1. *Let $b > 0$, and define $\mathbf{u}_0 : \overline{\Omega} \to S^2$ by*

$$\mathbf{u}_0(\mathbf{x}) := \big(\frac{\mathbf{x}}{|\mathbf{x}|}\sin\phi(|\mathbf{x}|), \cos\phi(|\mathbf{x}|)\big), \quad \textit{where} \quad \phi(r) := \begin{cases} br^2 & \textit{for } r \leq 1, \\ b & \textit{for } r \geq 1. \end{cases}$$

According to the results in [12, 31] we expect finite time blow-up for $p = 2$ if $b > \pi$. We choose
(ai) $p = 2$ and $b = \pi/2$ and (aii) $p = 2$ and $b = 3\pi/2$,
(bi) $p = 3/2$ and $b = \pi/2$ and (bii) $p = 3/2$ and $b = 3\pi/2$,
(ci) $p = 5/2$ and $b = \pi/2$ and (cii) $p = 5/2$ and $b = 3\pi/2$.

Figure 5.1 displays the numerical solution in Example 5.1 (ai) at various times. As expected we do not observe finite time blow-up; at $t = 0.9180$ all vectors point in the same direction. We observe a similar behaviour in (bi) and (ci).

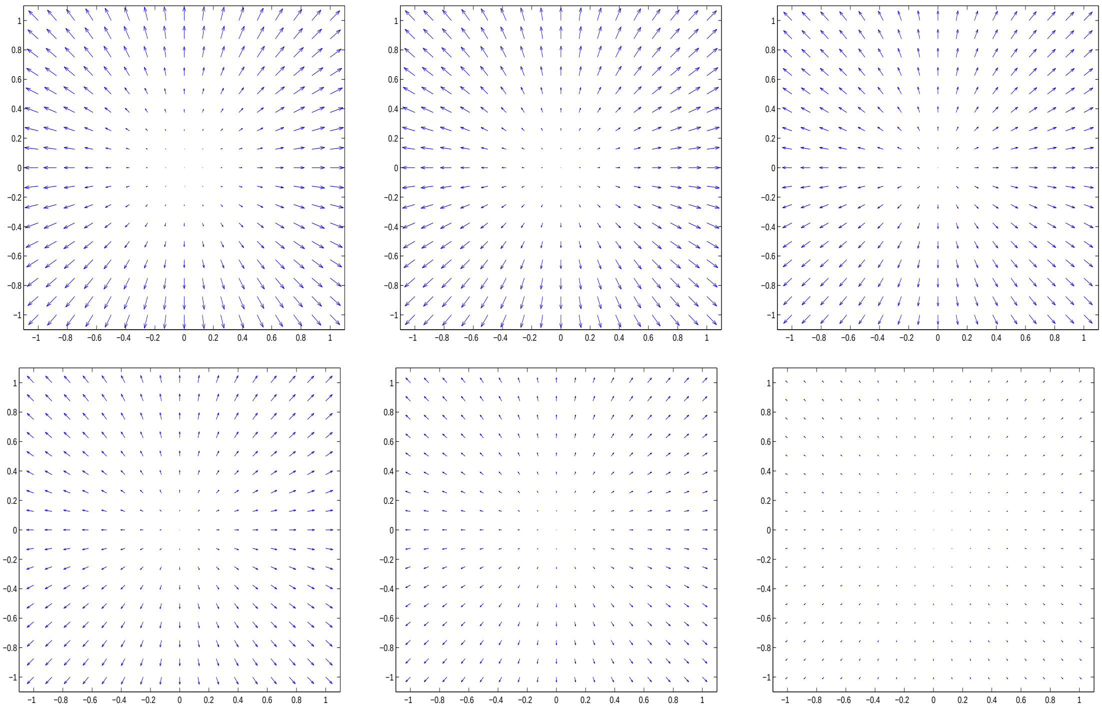

FIG. 5.1. $\mathbf{U}(t,\cdot)$ *in Example 5.1 (ai) for* $t = 0$, 0.0195, 0.1758, 0.3320, 0.5078, 0.9180.

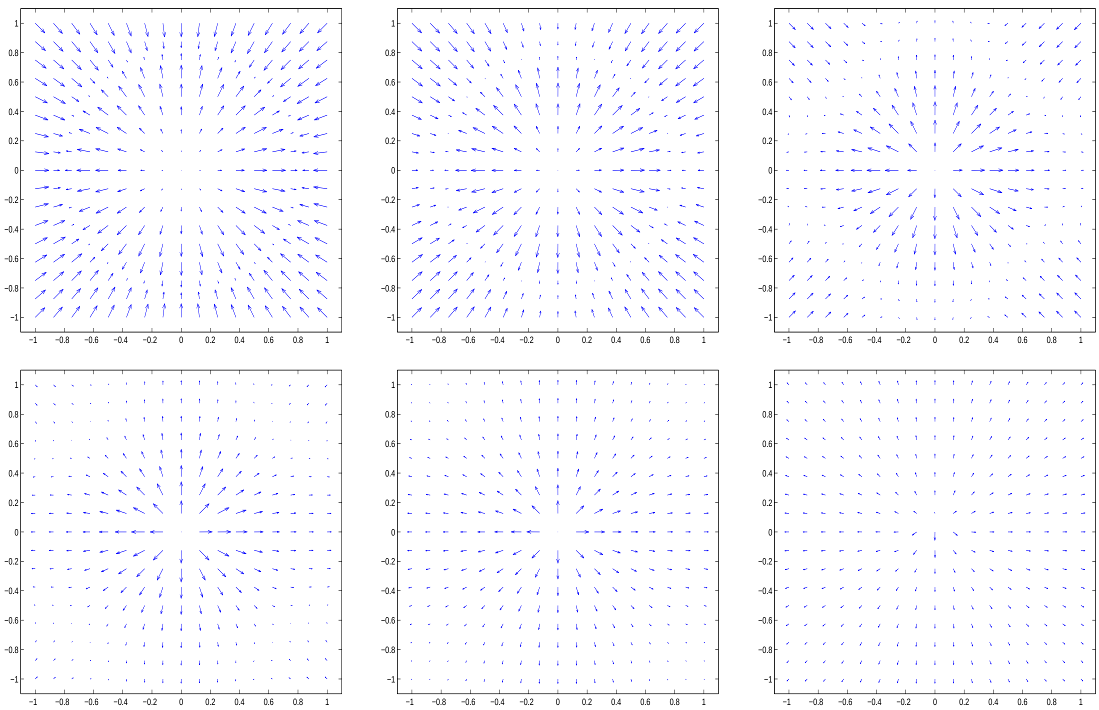

FIG. 5.2. $\mathbf{U}(t,\cdot)$ *in Example 5.1 (aii) for* $t = 0$, 0.0195, 0.0977, 0.1758, 0.2539, 0.3516.

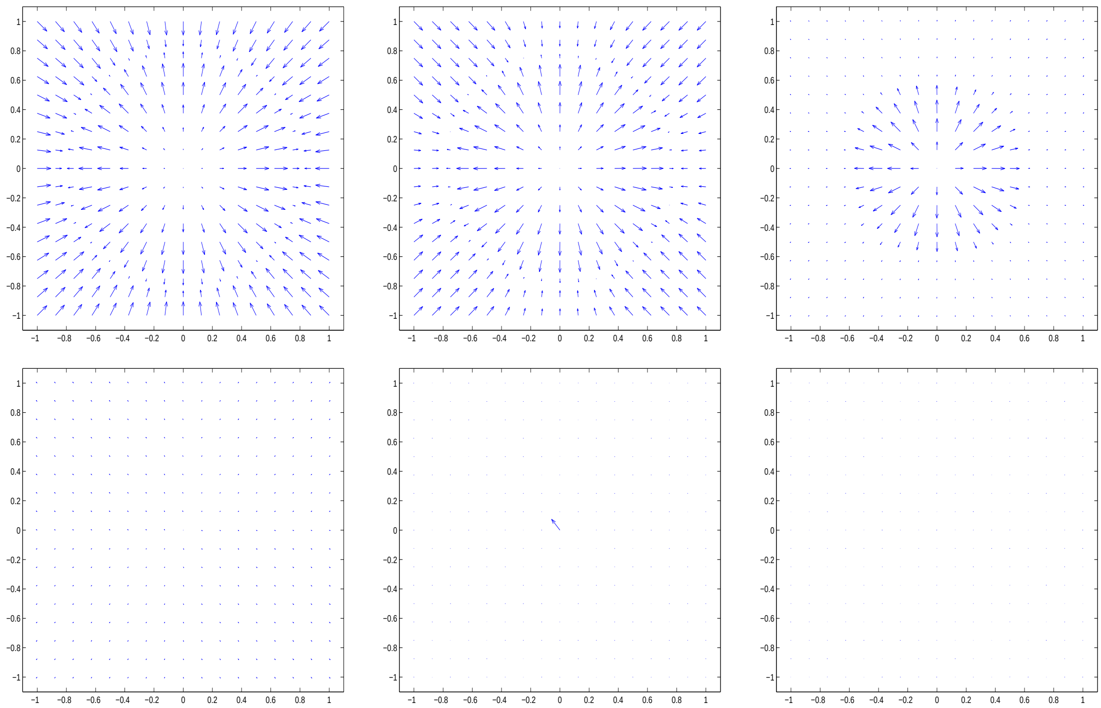

FIG. 5.3. $\mathbf{U}(t,\cdot)$ *in Example 5.1 (bii) for* $t = 0$, 0.1198, 0.4790, 0.8383, 1.3173, 1.6765.

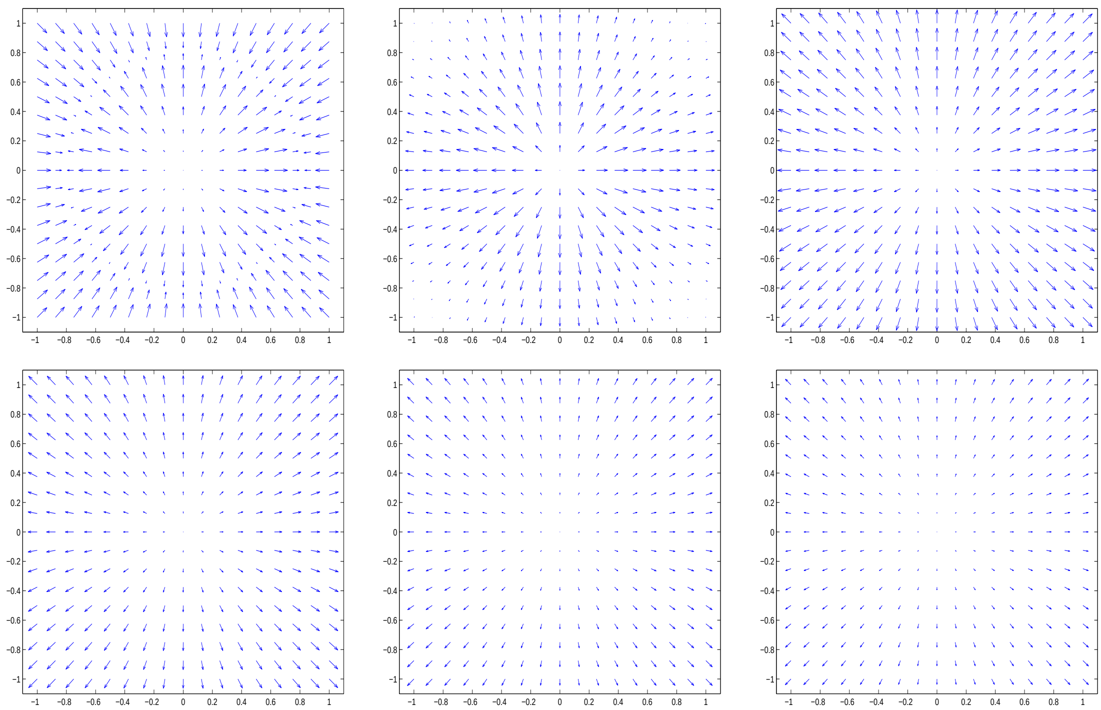

FIG. 5.4. $\mathbf{U}(t,\cdot)$ *in Example 5.1 (cii) for* $t = 0$, 0.1000, 0.2999, 0.4999, 0.6998, 0.7998.

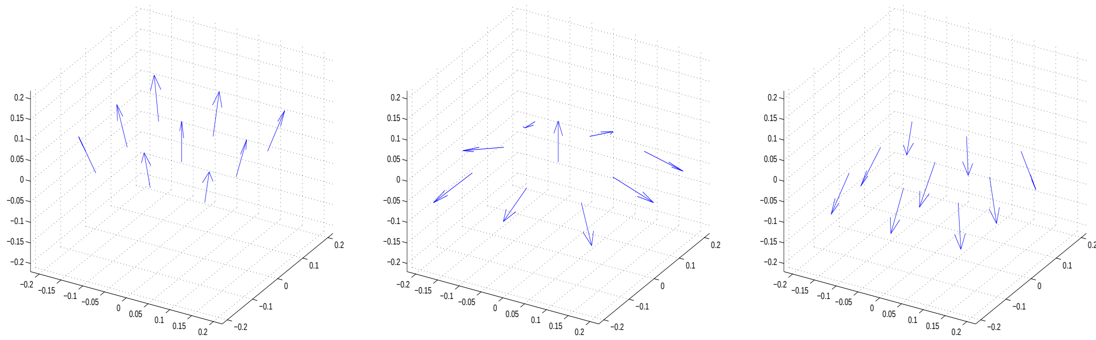

FIG. 5.5. *Nodal values* $\mathbf{U}(t,\mathbf{q}_i)$ *for nodes* $\mathbf{q}_i$ *close to the origin in Example 5.1 (aii) for* $t = 0.0195,\ 0.2539,\ 0.3516$.

In Figure 5.2 we plotted the numerical solution in Example 5.1 (aii) at various times. Blow-up occurs at $t \approx 0.3516$ when the vector at the origin changes its direction from $(0,0,1)$ to $-(0,0,1)$. A zoom at the values of the nodes in a neighborhood of the origin at some times is displayed in Figure 5.5 and magnifies the change of direction at the origin.

The blow-up happens differently for (bii). Some snapshots of its dynamics are displayed in Figure 5.3. In the time interval $1 \le t \le 1.7$ all vectors apart from the one at $\mathbf{x} = \mathbf{0}$ approximately point out of the plane. Then, at time $t \approx 1.78$ the vector at the origin changes direction so that a uniform state is achieved.

The behaviour in (cii) is different from that in (aii) and (bii). No blow-up occurs, cf. Figure 5.6. The vector field $\mathbf{U}$ obtained in (cii) is shown for various times in Figure 5.4.

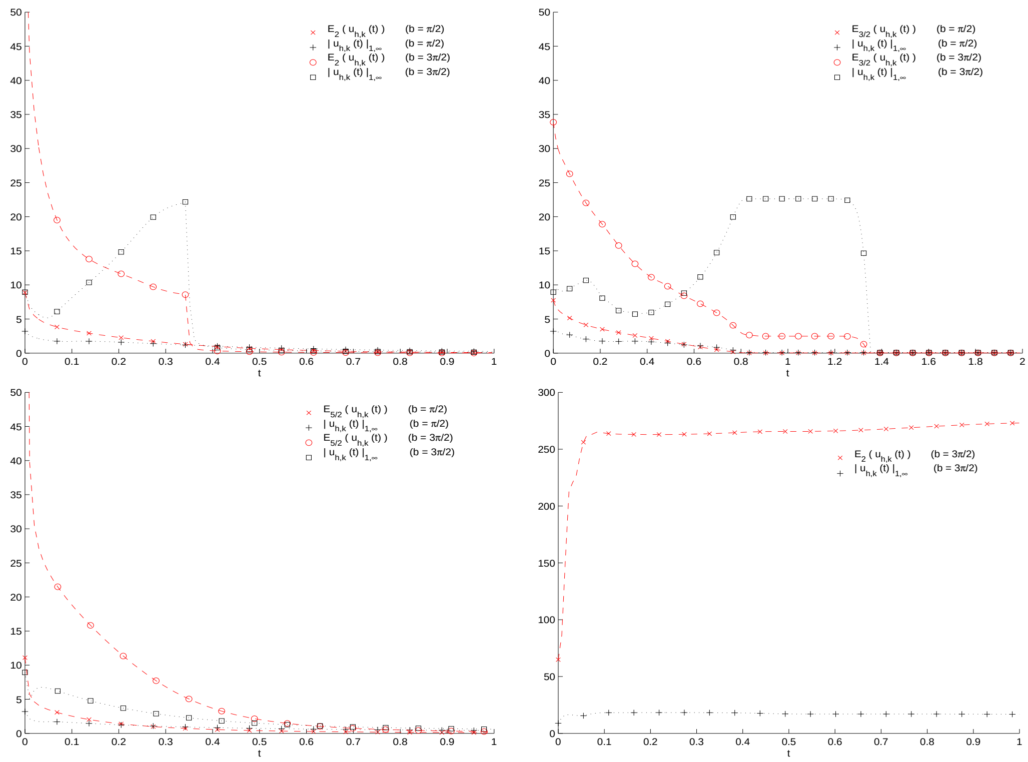

FIG. 5.6. *Energy decay and* $W^{1,\infty}$ *seminorm in Example 5.1 (a), (b), and (c) and instability in Example 5.1 (ai) for* $k = h^2$.

The lower right plot in Figure 5.6 displays the energy $E_2\big(\mathbf{U}(t,\cdot)\big)$ in Example 5.1 (ai) obtained with $k = h^2$. The results clearly indicate that $k = h^2$ is not small enough for $p = 2$ and $n = 2$ in this experiment.

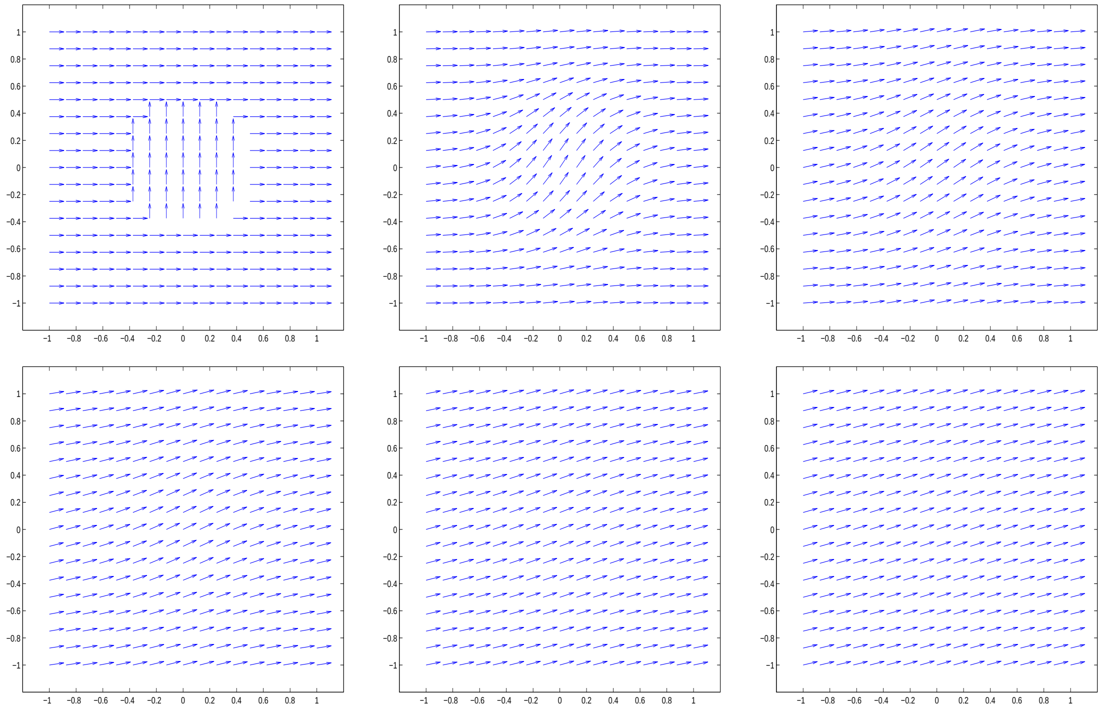

FIG. 5.7. $\mathbf{U}(t,\cdot)$ *in Example 5.2 (i) for* $t = 0$, 0.0600, 0.1200, 0.1800, 0.2400, 0.3000.

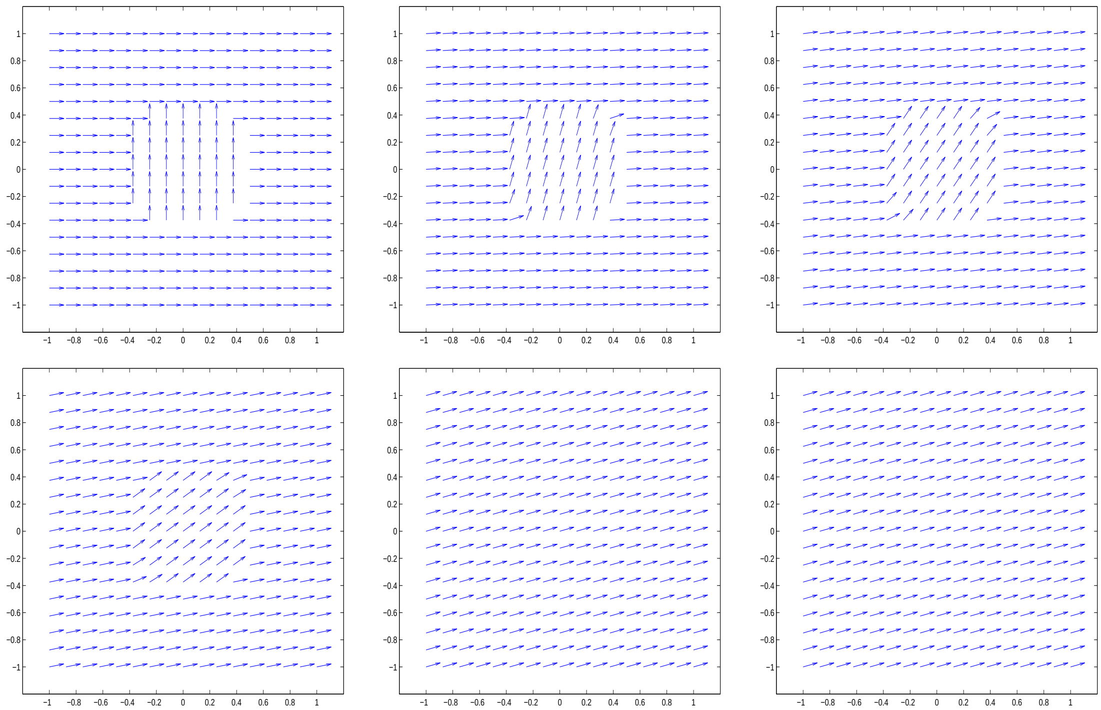

FIG. 5.8. $\mathbf{U}(t,\cdot)$ *in Example 5.2 (ii) for* $t = 0$, 0.0799, 0.1599, 0.2399, 0.3199, 0.3999.

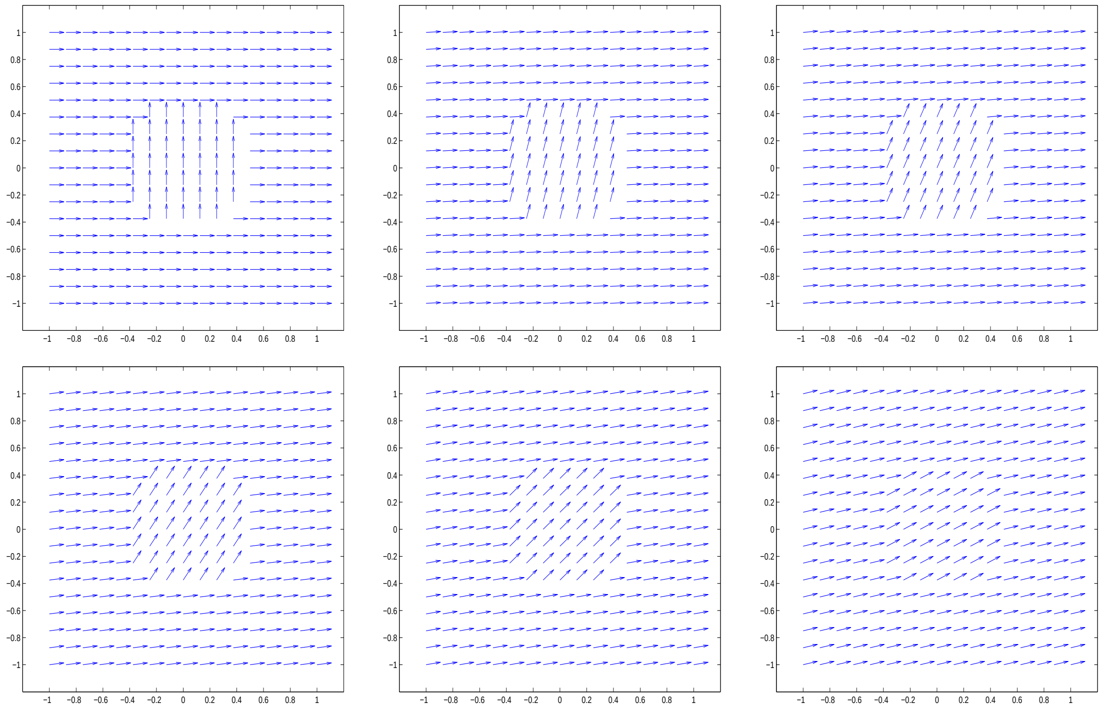

FIG. 5.9. $\mathbf{U}(t,\cdot)$ *in Example 5.2 (iii) for* $t = 0$, 0.2400, 0.3600, 0.4800, 0.6000, 0.7200.

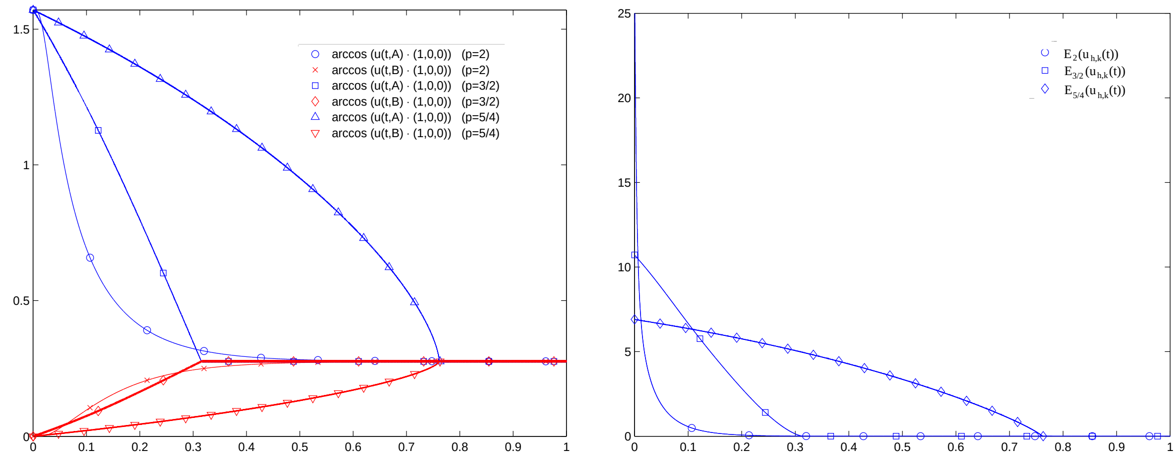

FIG. 5.10. *Angles and energy decay in Example 5.2.*

Analytical studies [3] of the scalar-valued total variation (TV) flow ($p = 1$) $-u_t \in \partial J(u)$, $u(0) = u_0 \in L^2(\Omega)$, for $J(u) = |\,Du\,|(\Omega)$ show interesting characterizations of the strong solution in the sense of semigroup theory: i) finite extinction time ($n = 2$), ii) $u(t,\cdot) \in L^\infty(\Omega)$, $t > 0$, if $u_0 \in L^n(\Omega)$, and no $L^1 - L^2$-regularizing effect for $L^1(\Omega)$-initial data, in general, iii) $C^{1,\alpha}$-regularity of level sets $\partial^*\big[u(t) > \lambda\big]$ for $u_0 \in L^n(\Omega)$ of decreasing size, i.e., $\frac{d}{dt}\mathcal{H}^{n-1}\big(\partial^*\big[u(t) > \lambda\big]\big) \leq 0$, and iv) invariance of supports, provided e.g. the curvature of the smooth boundary of the simply connected convex staring support is not too large; cf. [20] for a convergence analysis of a regularized, fully discrete scheme, and corresponding computational studies. We next discuss the latter issue in the present vectorial case.

Example 5.2. *We define* $\mathbf{u}_0 : \overline{\Omega} \to S^2$ *by*

$$\mathbf{u}_0(\mathbf{x}) := \begin{cases} (1,0,0) & \textit{for } |\mathbf{x}| < 0.5, \\ (0,1,0) & \textit{for } |\mathbf{x}| \geq 0.5; \end{cases}$$

and set (i) $p = 2$*, (ii)* $p = 3/2$*, and (iii)* $p = 5/4$*.*

Figure 5.7, 5.8, and 5.9 display snapshots of the numerical solutions in Example 5.2 (i), (ii), and (iii), respectively. For $p = 2$ in (i) we observe that the solution is rather smooth for positive times and that at $t \approx 0.24$ a uniform (constant) state is obtained. As opposed to the results in (i) for $p = 2$, the discontinuity along the circle $|\mathbf{x}| = 0.5$ is preserved for $p = 3/2$ in (ii) until $t \approx 0.31992$ when a constant state is achieved. For $p = 5/4$ the discontinuity is preserved for a significantly longer time, cf. Figure 5.9. In the left plot of Figure 5.10 we displayed the angle between the vectors $\mathbf{U}(t, \mathbf{x})$ and $(1, 0, 0)$ for $t \in (0, 1)$ and $\mathbf{x} \in \{\mathbf{A}, \mathbf{B}\}$, where $\mathbf{A} = (0, 0)$ and $\mathbf{B} = (3/4, 3/4)$, and for $p = 2$, $p = 3/2$, and $p = 5/4$. We observe that the angle at the origin changes almost linearly in case $p = 3/2$. In the right plot of Figure 5.10 we displayed the energies $E_2(\mathbf{U}(t, \cdot))$, $E_{3/2}(\mathbf{U}(t, \cdot))$, and $E_{5/4}(\mathbf{U}(t, \cdot))$ as a function of t for the solutions in Example 5.2 (i), (ii), and (iii), respectively. Of course, even though $\mathbf{u}_0$ is discontinuous, $\mathbf{U}^0 \equiv \mathcal{I}_h \mathbf{u}_0 \in W^{1,p}(\Omega, \mathbb{R}^3)$ with a mesh dependent norm, and so we still expect energy decay. We observe that this energy decay is slower for smaller exponents p.

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REFERENCES

[1] F. Alouges, *A new algorithm for computing liquid crystal stable configurations: the harmonic mapping case*, SIAM J. Numer. Anal. **34**, pp. 1708–1726 (1997).
[2] F. Alouges, P. Jaisson *Convergence of a finite elements discretization for the Landau Lifshitz equations*, downloadable at `mahery.math.u-psud.fr/alouges/` (2004).
[3] F. Andreu-Vaillo, V. Caselles, J.M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals*, Birkhäuser (2004).
[4] J. W. Barrett, X. Feng, A. Prohl, *p-harmonic map heat flows for* $1 \leq p < \infty$ *and their finite element approximations with applications to image processing* (manuscript) (2005).
[5] J. W. Barrett, X. Feng, A. Prohl, *Convergence of a fully discrete finite element approximation of an Ericksen-Leslie model for the flow of liquid crystals* (manuscript) (2005).
[6] J. W. Barrett, R. Nürnberg, *Finite element approximation of a Stefan problem with degenerate Joule heating*, M^2AN **38**, pp. 633–652 (2004).
[7] J. W. Barrett, W. B. Liu, *Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow*, Numer. Math. **68**, pp. 437–456 (1994).
[8] S. Bartels, *Stability and convergence of finite element approximation schemes for harmonic maps*, SIAM J. Numer. Anal. (accepted). Downloadable at `www.math.umd.edu/ sba/` (2004).
[9] F. Bethuel and X. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal. **80**, pp. 60–75 (1988).
[10] P. V. Blomgren, *Total variation methods for restoration of vector valued images*, Ph.D. Thesis, University of California, Los Angeles (1998).
[11] W. F. Brown, *Micromagnetics*, Interscience (1963).
[12] K.-C. Chang, W.-Y. Ding, R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Diff. Geom. **36**, pp. 507–515 (1992).
[13] Y. Chen, *The weak solutions to the evolution problems of harmonic maps*, Math. Z. **201**, pp. 69-74 (1989).
[14] Y. Chen, M.-C. Hong, N. Hungerbühler, *Heat flow of p-harmonic maps with values into spheres*, Math. Z. **215**, pp. 25–35 (1994).
[15] Y. Chen, M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. **201**, pp. 83–103 (1999).
[16] R. Cohen, S.-Y. Lin, M. Luskin, *Relaxation and gradient methods for molecular orientation in liquid crystals*, Comput. Phys. Comm. **53**, pp. 455–465 (1989).
[17] R. Cohen, R. Hardt, D. Kinderlehrer, S.-Y. Lin, M. Luskin, *Minimum energy configurations for liquid crystals: computational results*, in: Theory and Applications of Liquid Crystals, IMA Vol. Math. Appl **5**, pp. 99–122, Springer (1987).
[18] P. Courilleau, F. Demengel, *Heat flow for p-harmonic maps with values in the circle*, Nonl. Analysis **41**, pp. 689–700 (2000).

[19] A. Fardoun, R. Regbaoui, *Heat flow for p-harmonic map with small initial data*, to appear in Calc. Var. PDE's (2004).

[20] X. Feng, M.v. Oehsen, A. Prohl, *Rate of convergence of regularization procedures and finite element approximations for the total variation flow*, to appear in Num. Math. (2005).

[21] Y. Giga, Y. Kashima, N. Yamazaki, *Local solvability of a constrained gradient system of total variation*, Abstr. Appl. Anal. **8**, pp. 651–682 (2004).

[22] R. Hardt, D. Kinderlehrer, M. Luskin, *Remarks about the mathematical theory of liquid crystals*, in: Calc. Var. PDE's, LNM **1340**, Springer, New York, pp. 123–138 (1988).

[23] N. Hungerbühler, *Heat flow into spheres for a class of energies*, Progr. Nonlinear Diff. Eqns. Appl. **59**, pp. 45–65 (2004).

[24] N. Hungerbühler, *Non-uniqueness for the p-harmonic flow*, Canad. Math. Bull. **40**, pp. 174–182 (1997).

[25] N. Hungerbühler, *p-harmonic flow*. Diss. Math. Wiss. ETH Zürich, Nr. 10740 (1994).

[26] N. Hungerbühler, *Global weak solutions to the p-harmonic flow into homogeneous spaces*, Indiana Univ. Math. J. **45**, pp. 275–288 (1996).

[27] M. Kružík, A. Prohl, *Recent developments in modeling, analysis and numerics of ferromagnetism*, FIM-Preprint-Series (ETH Zürich), downloadable at: `www.fim.math.ethz.ch/preprints/` (2005).

[28] S. Y. Lin, M. Luskin, *Relaxation methods for liquid crystal problems*, SIAM J. Numer. Anal. **26**, pp. 1310–1324 (1989).

[29] M. Misawa, *Approximation of p-harmonic maps by the penalized equation*, Nonl. Analysis **47**, pp. 1069–1080 (2001).

[30] M. Misawa, *On the p-harmonic flow into spheres in the singular case*, Nonl. Analysis **50**, pp. 485–494 (2002).

[31] M. Struwe, *Geometric evolution problems*, IAS/Park City Math. Series, vol. **2**, pp. 259–339 (1996).

[32] B. Tang, G. Sapiro, V. Caselles, *Diffusion of generated data on non-flat manifolds via harmonic maps theory: the direction diffusion case*, Int. J. Comput. Vision **36**, pp. 149–161 (2000).

[33] B. Tang, G. Sapiro, V. Caselles, *Color image enhancement via chromaticity diffusion*, IEEE Trans. Image Proc. **10**, pp. 701–707 (2001).

[34] L. A. Vese, S. J. Osher, *Numerical methods for p-harmonic flows and applications to image processing*, SIAM J. Numer. Anal. **40**, pp. 2085–2104 (2002).

[35] E. Zeidler, *Nonlinear functional analysis and its applications,* **vol. II/B***: Nonlinear monotone operators*, Springer (1990).