CONVERGENCE FOR STABILISATION OF DEGENERATELY CONVEX MINIMISATION PROBLEMS

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Abstract. Degenerate variational problems often result from a relaxation technique in effective numerical simulation of nonconvex minimisation problems. The relaxed energy density is the convex envelope of the original one and so convex but not strictly convex. Hence strong convergence of straightforward finite element approximations cannot be expected but is relevant in many applications. This paper establishes a modified discretization by stabilisation and proves its convergence in strong norms.

1. Motivation and Introduction

The relaxation procedure in the calculus of variations allows the direct macroscopic simulation of models with finer and finer oscillations [L1, L2]. For the discrete problem that means that the non-convex energy density is removed and replaced by some quasiconvex envelope or —in some applications— even its convex envelope; we refer to the Example 1.1 for an illustration. The resulting discrete problem is then degenerated in the sense that it is convex but *not* strictly convex and so the Newton solver faces situations where the Hessian matrix for the tangential stiffness matrix is not positive definite and may be singular. Standard numerical regularisations are analysed in this paper as stabilisation techniques. Example 1.1 illustrates that the stabilisation allows less Newton-iterations than the original relaxed problem. We prove for relevant examples that proper stabilisation maintains the convergence rates of the discrete problem, and, which came much to a surprise, yields even strong convergence of the strain variables in certain circumstances.

Key words and phrases. degenerate variational problems, convexification, stabilisation, strong convergence, Euler-Lagrange equations, calculus of variations.

Example 1.1 (3-Well Problem). Given $\Omega = (0,1)^2$ and boundary data $u_D(x) = v_0(x_1) + v_0(x_2)$ for $x = (x_1, x_2) \in \Omega$ and

$$v_0(t) = \begin{cases} (t-1/4)^3/6 + (t-1/4)/8 & \text{for } t \le 1/4, \\ -(t-1/4)^5/40 - (t-1/4)^3/8 & \text{for } t \ge 1/4, \end{cases}$$

the relaxation W^{**} (i.e. the lower convex envelope) of the 3-well energy density

$$W(F) = \min\{|F|^2, |F-(1,0)|^2, |F-(0,1)|^2\}$$

leads to the energy minimisation problem

$$\min_{u\in\mathcal{A}} E(u) \quad \text{for} \quad \mathcal{A} = \{v \in W^{1,2}(\Omega) : v = u_D \text{ on } \partial\Omega\} \quad \text{and}$$

$$E(u) = \int_\Omega W^{**}(Du)\,dx + \int_\Omega |u_D - u|^2\,dx + \int_\Omega fv\,dx$$

with $f = \operatorname{div} DW^{**}(Du_D)$. The exact solution of the relaxed minimisation problem reads $u(x) = u_D(x)$ for $x \in \Omega$. Its finite element approximation is computed on a sequence of uniform triangulations $\mathcal{T}$ of Ω with mesh-size $h = 1/2, 1/4, ..., 1/32$ and degrees of freedom $N = 1, 9, 49, 225, 961$ into triangles which are translated copies of $\operatorname{conv}\{(0,0),(0,h),(h,h)\}$ and $\operatorname{conv}\{(0,0),(h,h),(h,0)\}$. Notice that W^{**} vanishes identically in $\operatorname{conv}(0,0),(1,0),(0,1) \subset \mathbb{R}^2$ and hence stabilisation is in order. The resulting discrete problem reads

$$\min_{u_h\in\mathcal{A}_h} E_h(u_h) \quad \text{for} \quad E_h(u_h) = E(u_h) + h^{\gamma-1}\int_\Omega |Du_h|^2\,dx$$

and $\mathcal{A}_h = \{v_h \in \mathcal{S}^1(\mathcal{T}) : v_h = u_{D,h} \text{ on } \partial\Omega\}$ where $\mathcal{S}^1(\mathcal{T}) \subseteq W^{1,2}(\Omega)$ is the lowest order finite element space related to $\mathcal{T}$ and $u_{D,h}(z) = u_D(z)$ for all nodes z on $\partial\Omega$.

For the exponents $\gamma = 0, 1/2, 1, 2$ and $\gamma = \infty$ ($\gamma = \infty$ means $E_h = E$, i.e. no stabilisation) we run a nested Newton-Raphson scheme. The termination criterion was an ℓ^2 norm of the residual less than 10^{-9}. Table 1 displays the history of iteration numbers K as a function of γ and h. This experimental result supports our interpretation from general observation that stabilisation improves the efficiency of the discrete problem solve.

The paper is concerned with the convergence behaviour of the perturbed discrete solutions. The class of problems analysed in this paper reads as follows. A natural finite element discretization of the Euler-Lagrange equations of a degenerately convex minimisation problem

$$(P) \quad \text{Seek } u \in \mathcal{A} \text{ with } \int_\Omega S(Du) : Dv\,dx + J(u;v) = 0 \text{ for all } v \in \mathcal{A}_D$$

h	1/2	1/4	1/8	1/16	1/32
$\gamma = 0$	4	4	5	7	8
$\gamma = 1/2$	4	4	5	10	9
$\gamma = 1$	4	4	5	13	16
$\gamma = 2$	4	6	10	29	-
$\gamma = \infty$	4	10	98	-	-

TABLE 1. Iteration numbers K required in the relaxed 3-well problem of Example 1.1 as a function of uniform mesh-size h and parameter γ. A minus sign means no convergence within 250 iteration steps.

(colon denotes the scalar product in $\mathbb{R}^{m\times n}$) with discrete spaces $\mathcal{A}_h = u_{D,h} + \mathcal{A}_{D,h}$ and $\mathcal{A}_{D,h} \subseteq \mathcal{A}_D$ reads

$$(P_h) \quad \text{Seek } u_h \in \mathcal{A}_h \text{ with } \int_\Omega S(Du_h) : Dv_h \, dx + J_h(u_h; v_h) = 0 \quad \text{for all } v_h \in \mathcal{A}_{D,h}.$$

Typically, the nonlinear stress-strain function $S : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is the derivative $S = D\varphi$ of an energy density function φ that is (quasi-)convex but not strictly (quasi-)convex. Lacking uniform convexity of φ and so lacking uniform monotonicity of S we *cannot* generally expect strong convergence of the error $e := u - u_h$, namely

$$\lim_{h\to 0} \|De\|_{L^p(\Omega)} = 0, \tag{1.1}$$

if an underlying mesh $\mathcal{T}_h$ becomes finer and finer such that the maximal meshsize tends to zero as $h \to 0$. Instead of (1.1), one may merely expect weak convergence $Du_h \rightharpoonup Du$ in $L^p(\Omega)$ or convergence in weaker norms, e.g. $\lim_{h\to 0} \|u - u_h\|_{L^r(\Omega)} = 0$. It turns out that the continuous lower order term $J : W^{1,p}(\Omega;\mathbb{R}^m) \to W^{1,p}(\Omega;\mathbb{R}^m)^*$ as well as boundary conditions in $\mathcal{A} := \{v \in W^{1,p}(\Omega;\mathbb{R}^m) : v = u_D \text{ on } \Gamma_D\}$ from some part Γ_D of the boundary $\partial\Omega$ of the domain Ω determine whether solutions u or u_h are unique or not; we refer to Section 2 for detailed assumptions. A typical time-step in evolution of phase transitions leads to (P) with an L^2-uniformly convex low-order term J (see, e.g. [CP3]) and requires strong convergence of gradients.

It is the aim of this paper to introduce stabilisation strategies to guarantee (1.1). For a mesh-dependent bilinear form $a_h : X_h \times Y_h \to \mathbb{R}$ such that $\mathcal{A}_h \subseteq X_h$ and $\mathcal{A}_{D,h} \subseteq Y_h$ we set $J_h(u_h, v_h) := J(u_h, v_h) +$

$a_h(u_h, v_h)$. For relaxed nonconvex minimisation problems the additional term $a_h(u_h, v_h)$ allows a physical interpretation of a discrete surface energy. Provided that (P) involves sufficient convexity, e.g. if J is uniformly monotone with respect to an L^p norm (on low-order terms) and φ is convex, there exists a unique solution u of (P). Then, if $u \in H^{3/2+\varepsilon}(\Omega; \mathbb{R}^m)$ for some $\varepsilon > 0$ we prove (1.1) for the unique discrete solution u_h of (P_h).

In order to illustrate some of the arguments in the proof of (1.1) we avoid in this introduction any technicality through the (unrealistic) assumption $\mathcal{A}_h, \mathcal{A}_{D,h} \subseteq H^2(\Omega; \mathbb{R}^m)$ and consider only one stabilisation term

$$J_h(u_h; v_h) := J(u_h; v_h) + h^2 \int_\Omega \Delta u_h \cdot \Delta v_h \, dx \tag{1.2}$$

(dot denotes the scalar product in $\mathbb{R}^m$). Suppose furthermore that the low-order term J is uniformly monotone such that standard arguments with the Galerkin orthogonality yield

$$h^2 \|\Delta e\|^2_{L^2(\Omega)} + \|e\|^2_{L^2(\Omega)} \le C h^2 \tag{1.3}$$

for $u \in H^2(\Omega; \mathbb{R}^m) \cap \mathcal{A}$. Then, an integration by parts and $e = 0$ on $\partial\Omega$ lead to

$$\|De\|^2_{L^2(\Omega)} = \int_\Omega De : De \, dx = -\int_\Omega e \cdot \Delta e \, dx.$$

Cauchy's inequality, Young's inequality in the resulting upper bound, and (1.3) in the final step prove

$$\|De\|^2_{L^2(\Omega)} \le \|e\|_{L^2(\Omega)} \|\Delta e\|_{L^2(\Omega)} \le \frac{h}{2} \|\Delta e\|^2_{L^2(\Omega)} + \frac{h^{-1}}{2} \|e\|^2_{L^2(\Omega)} \le C h.$$

Hence there holds strong convergence of gradients (1.1) for $p = 2$ if $u \in H^2(\Omega; \mathbb{R}^m)$. Since this argumentation requires C^1 conforming finite elements the practical use of stabilisation (1.2) is limited. Therefore, this paper establishes three discrete stabilisations which lead to (1.1) in case that $\mathcal{A}_h$, $\mathcal{A}_{D,h}$ are lowest order finite element spaces.

It should be stressed that stabilisation is in fact equivalent to [NW] stated for $m = n = 1$ and for a numerical modification that replaces J by a lumped version J_h. The proof of (1.1) in [NW] employs particular one-dimensional arguments for a particular model example. In contrast to this, stabilisation as introduced in this paper, appears to be a robust and flexible tool for a large class of degenerately convex minimisation problems. Convergence rates for the gradient error, however, requires strong regularity conditions of the exact solution along with its uniqueness.

The remaining part of this paper is organised as follows. The general setting and the main results are presented in Section 2. A list of examples for S and J that meet the abstract framework in (P) are given in Section 3. In Section 4 we prove the main result. Notation and basic results related to finite element discretizations are introduced and recalled in Section 5. Sections 6-8 are devoted to three different stabilisations that define (P_h) and lead to (1.1) via the abstract result of Section 2. Section 9 discusses strong convergence for a 2-well problem which results from a model for phase transitions in crystalline solids. Numerical examples are reported on in [Ba].

2. General Setting and Main Result

This section is devoted to a general framework that allows several particular choices of the stabilisation term a_h for a large class of examples indicated below. For this section, J_h is quite general and could model a numerical quadrature for J as well.

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with polygonal (for $n = 2$) or polyhedral (for $n = 3$) boundary $\partial\Omega$ and $1 < q \leq 2 \leq p < \infty$, $1/p + 1/q = 1$. Given $u_D \in W^{1,p}(\Omega; \mathbb{R}^m)$ set

$$\mathcal{A}_D := W_0^{1,p}(\Omega; \mathbb{R}^m) \quad \text{and} \quad \mathcal{A} := u_D + \mathcal{A}_D,$$

with $W_0^{1,p}(\Omega; \mathbb{R}^m) = \{v \in W^{1,p}(\Omega; \mathbb{R}^m) : v|_{\partial\Omega} = 0\}$; $|\cdot|_{W^{1,p}(\Omega)}$ abbreviates the seminorm $|v|_{W^{1,p}(\Omega)} := \|Dv\|_{L^p(\Omega)}$ of $v \in W^{1,p}(\Omega; \mathbb{R}^m)$. For a discrete space $\mathcal{A}_{D,h} \subseteq W_0^{1,p}(\Omega; \mathbb{R}^m)$, spaces X_h and Y_h, and an approximation $u_{D,h}$ of u_D we merely suppose

$$\mathcal{A}_h = u_{D,h} + \mathcal{A}_{D,h} \subseteq X_h \quad \text{and} \quad \mathcal{A}_{D,h} \subseteq Y_h.$$

The stress function $S : \mathbb{M}^{m \times n} \to \mathbb{R}$, the low-order terms

$$J : W^{1,p}(\Omega; \mathbb{R}^m) \to (W^{1,p}(\Omega; \mathbb{R}^m))^* \quad \text{and} \quad J_h : X_h \to Y_h^*$$

of the continuous and discrete level, respectively, are supposed to satisfy the following hypotheses (H1)-(H3) for the exact and the discrete solution $u \in \mathcal{A}$ and $u_h \in \mathcal{A}_h$, respectively. A list of examples follows below in Section 3.

(H1). There exist positive constants α, r, s with $1 < r \leq 2$, $0 \leq s < \infty$, and a function $S : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ such that, for all $A, B \in \mathbb{M}^{m\times n}$,

$$|S(A) - S(B)|^r \leq \alpha(1 + |A|^s + |B|^s)(S(A) - S(B)) : (A - B).$$

Here, $\mathbb{M}^{m\times n}$ denotes the real $m \times n$ matrices, $|\cdot|$ the Frobenius norm related to the scalar product

$$A : B = \sum_{k=1}^{m}\sum_{\ell=1}^{n} A_{k\,\ell}B_{k\,\ell} \quad \text{for } A, B \text{ in } \mathbb{M}^{m\times n}.$$

(H2). There exist solutions u and u_h of (P) and (P_h), respectively. [Their uniqueness is not assumed explicitly, at this stage, any choice will do it. However, the uniqueness of u is later an implication of our strong regularity assumption.] That is suppose that $u \in \mathcal{A}$ with $\sigma := S(Du)$ and $u_h \in \mathcal{A}_h$ with $\sigma_h := S(Du_h)$ satisfy

$$\int_\Omega \sigma : Dv\,dx + J(u; v) = 0 \quad \text{for all } v \in \mathcal{A}_D,$$

$$\int_\Omega \sigma_h : Dv_h\,dx + J_h(u_h; v_h) = 0 \quad \text{for all } v_h \in \mathcal{A}_{D,h}.$$

Throughout this paper, set

$$e := u - u_h \quad \text{and} \quad \delta := \sigma - \sigma_h.$$

(H3). There exist a constant $B > 0$, a strictly convex function $\beta : [0,\infty) \to [0,\infty)$ with $\beta(0) = 0$, and seminorms $\|\cdot\|_{X_h}$ and $\|\cdot\|_{Y_h}$ on the function spaces X_h and Y_h with $\mathcal{A}_h \subseteq X_h \subseteq W^{1,p}(\Omega;\mathbb{R}^m)$ and $\mathcal{A}_{D,h} \subseteq Y_h \subseteq W^{1,p}(\Omega;\mathbb{R}^m)$ such that $e \in X_h$, $e - \mathcal{A}_{D,h} \subseteq Y_h$, and

$$\beta\big(\|e\|_{X_h}\big) \leq J_h(u; e) - J_h(u_h; e),$$
$$J_h(u, v) - J_h(u_h; v) \leq B\|e\|_{X_h}\|v\|_{Y_h}$$

for the exact and discrete solution u and u_h with the error $e = u - u_h$ from (H2), and $v \in e - \mathcal{A}_{D,h}$.

Theorem 2.1. *Suppose (H1)-(H3) and let β^* denote the dual functional to β, i.e. $\beta^*(t) = \sup\{st - \beta(s) : s \geq 0\}$. Then, for all $e_h \in \mathcal{A}_{D,h}$, there holds*

$$(1 - 1/r)\int_\Omega \delta : De\,dx + (1/c_1)\,\|\delta\|^r_{L^q(\Omega)} + \beta\big(\|e\|_{X_h}\big)$$
$$\leq c_2|e - e_h|^{r/(r-1)}_{W^{1,p}(\Omega)} + \beta^*\big(2B\|e - e_h\|_{Y_h}\big) + 2\big(J_h(u; e_h) - J(u; e_h)\big).$$

The constants c_1 and c_2 depend on α, p, r, s, and upper bounds for $|u|_{W^{1,p}(\Omega)}$ and $|u_h|_{W^{1,p}(\Omega)}$.

Remark 2.1. It follows from (H1) that $0 \leq \delta : De$ almost everywhere on Ω; hence all the terms on the left-hand side in the estimate of the theorem are non-negative.

Remark 2.2. It is known that $\beta^* : [0,\infty) \to [0,\infty)$ is a convex function with $\beta^*(0) = 0$. In particular, for $\beta(t) = t^2/2$ one finds $\beta^*(t) = t^2/2$.

Remark 2.3. The bounds of $|u|_{W^{1,p}(\Omega)}$ and $|u_h|_{W^{1,p}(\Omega)}$ may follow from further natural growth conditions on S, J, and J_h which we have not stated here.

Throughout this paper we consider $J_h = J|_{X_h \times Y_h} + a_h$ for a continuous bilinear form $a_h : X_h \times Y_h \to \mathbb{R}$. Then we can replace (H3) by the following hypothesis.

(H4). Let $0 < m \leq M < \infty$ satisfy

$$m\|e\|^2_{L^2(\Omega)} \leq J(u;e) - J(u_h;e),$$
$$J(u;v) - J(u_h;v) \leq M\|e\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$

for all $v \in W^{1,p}(\Omega;\mathbb{R}^m)$.

Proposition 2.2. *Suppose (H4), $\mathcal{A}_h \subseteq X_h \subseteq W^{1,p}(\Omega;\mathbb{R}^m)$ and $\mathcal{A}_{D,h} \subseteq Y_h \subseteq W^{1,p}(\Omega;\mathbb{R}^m)$ are such that $e \in X_h$ and $e - \mathcal{A}_{D,h} \subseteq Y_h$. Assume that $a_h : X_h \times Y_h \to \mathbb{R}$ is a continuous bilinear form, $\|\cdot\|^2_{X_h} = \|\cdot\|^2_{Y_h} = \|\cdot\|^2_{L^2(\Omega)} + a_h(\cdot,\cdot)$, and $J_h = J|_{X_h\times Y_h} + a_h$. Then, there holds (H3) with $\beta(t) = \min\{1,m\}\, t^2$ and $B := \max\{1,M\}$.*

Proof. This follows directly from the definitions of J_h, $\|\cdot\|_{X_h}$, $\|\cdot\|_{Y_h}$. ☐

3. Examples

Example 3.1 (p-Laplacian). An energy minimisation of $|Du|^p/p$ leads to the p-Laplacian problem with operator $S(F) = |F|^{p-2}F$ and $2 \leq p < \infty$. Since (e.g. by a combination of Lemma 2.1-2.3 in [CK]) for any distinct $A, B \in \mathbb{R}^n$ and $\alpha = 1 + \max\{1, p-2\}^2$ there holds

$$\frac{|S(A) - S(B)|^2}{(S(A) - S(B)) : (A - B)} \leq \alpha(|A|^{p-2} + |B|^{p-2})$$

and (H1) is valid with $r = 2$, $s = p - 2$. See [CM, LB] for further results.

Example 3.2 (Optimal Design). The relaxed model for an optimal design problem derived in [GKR] leads to a minimisation problem with energy density $\varphi(F) = \psi(|F|)$ and $S(F) = D\varphi(F)$. Given positive parameters $0 < t_1 < t_2$ and $0 < \mu_2 < \mu_1$ with $t_1\mu_1 = t_2\mu_2$, the C^1 function $\psi : [0,\infty) \to [0,\infty)$ is defined by $\psi(0) = 0$ and

$$\psi'(t) = \begin{cases} \mu_1 t & \text{if } 0 \le t \le t_1, \\ t_1\mu_1 = t_2\mu_2 & \text{if } t_1 \le t \le t_2, \\ \mu_2 t & \text{if } t_2 \le t. \end{cases}$$

The function $S(F)$ satisfies (H1) with $r = 2$, $s = 0$, and $\alpha = \mu_1$ [CP1]; cf. also [F].

Example 3.3 (Scalar 2-Well Problem). Given distinct wells $F_1, F_2 \in \mathbb{R}^n$, $F_1 \neq F_2$, the relaxed scalar 2-well problem leads to a convexified minimisation problem with energy density

$$\varphi(F) = \max\{|F - B|^2 - |A|^2, 0\}^2 + 4(|A|^2\,|F - B|^2 - [A^T\,(F - B)]^2) \tag{3.1}$$

where $A = (F_2 - F_1)/2$ and $B = (F_1 + F_2)/2$. and satisfies (H1) with $r = 2$, $s = 2$, and $\alpha = 4\max\{2, |F_1 - F_2|^2\}$ [CP1, F]. This scalar problem can be deduced from the Ericksen-James energy density in an anti-plane shear model; the version for $n = 1$, due to O. Bolza [Bo], serves as a master example in non-convex minimisation [Y].

Example 3.4 (Compatible Vectorial 2-Well Problem). Given two symmetric matrices $E_1, E_2 \in \mathbb{M}^{n\times n}_{sym}$, real numbers $W_1^0, W_2^0 \in \mathbb{R}$, and a positive definite fourth order tensor $\mathbb{C} : \mathbb{M}^{n\times n}_{sym} \to \mathbb{M}^{n\times n}_{sym}$, let

$$W_j(E) = \frac{1}{2}(E - E_j) : \mathbb{C}(E - E_j) + W_j^0$$

for $E \in \mathbb{M}^{n\times n}_{sym}$ and $j = 1, 2$. Then, if $E_1 = E_2 + (a\otimes b + b\otimes a)/2$ for $a, b \in \mathbb{R}^n$ the quasiconvex hull of $W : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$, $E \mapsto \min\{W_1(E), W_2(E)\}$, is convex and given by [K]

$$\varphi(E) = \begin{cases} W_1(E) & \text{for } W_2(E) + \gamma \le W_1(E), \\ \frac{1}{2}(W_2(E) + W_1(E)) - \frac{1}{4\gamma}(W_2(E) - W_1(E))^2 - \frac{4}{\gamma} & \\ & \text{for } |W_1(E) - W_2(E)| \le \gamma, \\ W_2(E) & \text{for } W_1(E) + \gamma \le W_2(E), \end{cases}$$

for $E \in \mathbb{M}^{n\times n}_{sym}$ and $\gamma = \frac{1}{2}(E_1 - E_2) : \mathbb{C}(E_1 - E_2)$. There holds (H1) for $S(A) = D\varphi((A + A^T)/2)$, $A \in \mathbb{M}^{n\times n}$, with $r = 2$, $s = 0$, and a constant $0 < \alpha$ that depends on $\mathbb{C}$ [CP2].

More physical examples in the context of non-convex minimization are included in [L1, L2, R].

Example 3.5 (Linear Right-Hand Side). Given functions $f \in L^q(\Omega;\mathbb{R}^m)$ and $g \in L^q(\Gamma_N;\mathbb{R}^m)$ a typical linear right-hand side reads, for $u, v \in W^{1,p}(\Omega;\mathbb{R}^m)$,

$$J(u;v) = \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds$$

where Γ_N is a (possibly empty) part of $\partial\Omega$. Note that J is independent of u and hence does not satisfy (H4).

Example 3.6 (linear Low-Order Terms). The derivative $J = D\Psi$ of a strictly convex low-order term Ψ in a model situation of [CP1] reads, for $u, v \in W^{1,p}(\Omega;\mathbb{R}^m)$,

$$J(u;v) = \int_\Omega u \cdot v \, ds$$

and satisfies (H4) for $m = M = 1$.

4. Proof of Theorem 2.1

The proof of Theorem 2.1 extends a technique from [CP1]. From there we quote the first lemma.

Lemma 4.1. *Suppose (H1)-(H2) and* $|\Omega|^{s/p} + |u|^s_{W^{1,p}(\Omega)} + |u_h|^s_{W^{1,p}(\Omega)} \leq c_1\alpha$. *Then*

$$\|\delta\|^r_{L^q(\Omega)} \leq c_1 \int_\Omega \delta : De \, dx.$$

Proof. The proof follows (in different notation) the arguments that lead to formula (3.7) in [CP1] and is hence omitted. □

Direct algebra and (H3) imply the following result.

Lemma 4.2. *Suppose (H2)-(H3) and* $e_h \in \mathcal{A}_{D,h}$. *Then*

$$2\int_\Omega \delta : De \, dx + \beta\big(\|e\|_{X_h}\big) \leq 2\int_\Omega \delta : D(e - e_h) \, dx + \beta^*\big(2B\|e - e_h\|_{Y_h}\big) + 2\big(J_h(u;e_h) - J(u;e_h)\big).$$

Proof. The two identities in (H2) with $v = e_h = v_h \in \mathcal{A}_{D,h} \subseteq \mathcal{A}_D$ yield

$$\int_\Omega \delta : De \, dx + J(u;e) - J_h(u_h;e) = \int_\Omega \delta : D(e - e_h) \, dx + J(u;e - e_h) - J_h(u_h;e - e_h).$$

The differences on the left- and right-hand side are estimated with the first and second inequality of (H3) after inserting $J_h(u;e)$ and $J_h(u;e-e_h)$, respectively, where $v = e - e_h$. Hence,

$$\int_\Omega \delta : De\,dx + \beta\big(\|e\|_{X_h}\big) + \big(J(u;e) - J_h(u;e)\big) \le \int_\Omega \delta : D(e-e_h)\,dx \\ + B\|e\|_{X_h}\|e-e_h\|_{Y_h} + \big(J(u;e-e_h) - J_h(u;e-e_h)\big).$$

The definition of β^* shows $st \le \beta(s) + \beta^*(t)$ which, for $s = \|e\|_{X_h}$ and $t = 2B\|e-e_h\|_{Y_h}$, results in

$$2B\|e\|_{X_h}\|e-e_h\|_{Y_h} \le \beta\big(\|e\|_{X_h}\big) + \beta^*\big(2B\|e-e_h\|_{Y_h}\big).$$

The combination of the last two estimates proves the lemma. □

Lemma 4.3. *Suppose (H1)-(H2) and let* $c_2 := 2^{r'}c_1^{r'-1}/r'$. *Then*

$$2\int_\Omega \delta : D(e-e_h)\,dx \le (1/r)\int_\Omega \delta : De\,dx + c_2|e-e_h|^{r'}_{W^{1,p}(\Omega)}.$$

Proof. Hölder's and Young's inequality show

$$2\int_\Omega \delta : D(e-e_h)\,dx \le \|\delta\|^r_{L^q(\Omega)}/(rc_1) + 2^{r'}c_1^{r'/r}|e-e_h|^{r'}_{W^{1,p}(\Omega)}/r'.$$

The assertion then follows from Lemma 4.1. □

Proof of Theorem 2.1. This follows from Lemma 4.1, 4.2, and 4.3. □

5. Finite Element Discretization

Let $\mathcal{T}$ be a regular triangulation of Ω into triangles ($n=2$) or tetrahedra ($n=3$) in the sense of [BS], i.e. no hanging nodes, the domain is matched exactly, $\overline{\Omega} = \cup_{T\in\mathcal{T}}T$, and $\mathcal{T}$ satisfies the maximum angle condition. The extremal points of $T \in \mathcal{T}$ are called nodes and $\mathcal{N}$ denotes the set of all such nodes; $\mathcal{K} := \mathcal{N}\backslash\partial\Omega$ is the subset of free nodes. The set of edges ($n=2$) or faces ($n=3$) $E = \operatorname{conv}\{z_1,...,z_n\} \subseteq \partial T$ for pairwise distinct $z_1,...,z_n \in \mathcal{N}$ and $T \in \mathcal{T}$ is denoted as $\mathcal{E}$. By $\mathcal{E}_\Omega$ we denote the set of interior edges or faces, $\mathcal{E}_\Omega = \{E \in \mathcal{E} : \exists T_1, T_2 \in \mathcal{T}, E = T_1 \cap T_2\}$. We assume that $\partial\Omega$ is matched exactly by edges on $\partial\Omega$ which implies $\partial\Omega = \cup_{E\in\mathcal{E}_D}E$ for the set of boundary edges $\mathcal{E}_D := \{E \in \mathcal{E} : E \subseteq \partial\Omega\}$. Let $P_k(\omega)$ denote the set of algebraic polynomials of (total) degree $\le k$ regarded as scalar functions on ω. The set

$$\mathcal{P}_k(\mathcal{T}) := \{v_h \in L^\infty(\Omega) : \forall T \in \mathcal{T}, v_h|_T \in P_k(T)\}$$

consists of all (possibly discontinuous) $\mathcal{T}$–elementwise polynomials of degree at most k. We define

$$\mathcal{S}^1(\mathcal{T}) := \mathcal{P}_1(\mathcal{T}) \cap C(\overline{\Omega}) \text{ and } \mathcal{A}_{D,h} = \mathcal{S}^1_0(\mathcal{T})^m := \mathcal{S}^1(\mathcal{T})^m \cap W^{1,2}_0(\Omega;\mathbb{R}^m).$$

Supposing that u_D is continuous on $\partial\Omega$ we choose $u_{D,h} \in \mathcal{S}^1(\mathcal{T})^m$ with $u_{D,h}(z) = u_D(z)$ for all $z \in \mathcal{N} \cap \partial\Omega$ and set

$$\mathcal{A}_h := u_{D,h} + \mathcal{S}^1_0(\mathcal{T})^m.$$

Let $(\varphi_z : z \in \mathcal{N})$ be the nodal basis of $\mathcal{S}^1(\mathcal{T})$, i.e. $\varphi_z \in \mathcal{S}^1(\mathcal{T})$ satisfies $\varphi_z(x) = 0$ if $x \in \mathcal{N} \setminus \{z\}$ and $\varphi_z(z) = 1$. We set $h_T := \operatorname{diam}(T)$ for all $T \in \mathcal{T}$ and $h_E := \operatorname{diam}(E)$ for all $E \in \mathcal{E}$ and define a function $h_{\mathcal{T}} \in \mathcal{L}^0(\mathcal{T})$ by $h_{\mathcal{T}}|_T := h_T$ for $T \in \mathcal{T}$. Abbreviate $h := \|h_{\mathcal{T}}\|_{L^\infty(\Omega)}$. We will frequently assume that $\mathcal{T}$ is quasiuniform which implies that $h \approx \|h_{\mathcal{T}}^{-1}\|^{-1}_{L^\infty(\Omega)}$.

We write $H^s(U;\mathbb{R}^m)$ for $W^{s,2}(U;\mathbb{R}^m)$ for an open set $U \subseteq \mathbb{R}^n$ and

$$H^s(\mathcal{T};\mathbb{R}^m) = \big\{ v \in L^2(\Omega;\mathbb{R}^m) : \forall T \in \mathcal{T}, v|_T \in H^s\big(\operatorname{int}(T);\mathbb{R}^m\big)\big\}.$$

The elementwise application of the differential operators D^2 (the matrix of all second order derivatives) and Δ (the Laplace operator) to a function $v \in H^2(\mathcal{T};\mathbb{R}^m)$ is denoted by $D^2_{\mathcal{T}} v$ and $\Delta_{\mathcal{T}} v$, respectively.

For each edge $E \in \mathcal{E}_\Omega$ we choose a vector $\nu_E \in \mathbb{R}^n$ (with selected and then fixed orientation) with $|\nu_E| = 1$ orthogonal to E.

Assume $v \in H^1(\Omega;\mathbb{R}^m) \cap H^2(\mathcal{T};\mathbb{R}^m)$, let $E \in \mathcal{E}_\Omega$ be such that $E = T_+ \cap T_-$ for $T_+, T_- \in \mathcal{T}$ and suppose ν_E points from T_+ to T_-. Then, define $[Dv] \in L^2(E;\mathbb{M}^{m\times n})$ by

$$[Dv] := (Dv|_{T_+})|_E - (Dv|_{T_-})|_E.$$

For a function $\phi \in C(\partial\Omega;\mathbb{R}^m)$ such that $\phi|_E \in H^2(E;\mathbb{R}^m)$ for all $E \in \mathcal{E}_D$, $\partial^2_{\mathcal{E}}\phi/\partial s^2$ is the edgewise second derivative of ϕ along $\partial\Omega$; $H^2(\mathcal{E}_D;\mathbb{R}^m)$ denotes the set of all such functions ϕ.

Throughout this paper we abbreviate inequalities $A \leq C\,B$ with an h-independent constant $C > 0$ by $A \lesssim B$ and $A \approx B$ replaces $A \lesssim B \lesssim A$. The constant C may well depend on the shape of the elements; e.g. $h_E \approx h_T$ for $E \in \mathcal{E}$ and $T \in \mathcal{T}$ with $E \subseteq \partial T$. For instance, the well-established trace inequality reads

$$\|\phi\|^2_{L^2(\partial T)} \lesssim h_T^{-1}\|\phi\|^2_{L^2(T)} + h_T\|D\phi\|^2_{L^2(T)} \tag{5.1}$$

for any $T \in \mathcal{T}$ and $\phi \in H^1(T;\mathbb{R}^m)$.

6. Stabilisation via jumps of gradients

This section is devoted to the discrete problem (P_h) with $J_h := J + a_h$ for the bilinear form

$$a_h : X_h \times Y_h \to \mathbb{R}, \quad (v, w) \mapsto \sum_{E \in \mathcal{E}_\Omega} h_E^\gamma \int_E [Dv] : [Dw] \, ds. \tag{6.1}$$

Therein, the spaces X_h and Y_h are arbitrary with

$$X_h = Y_h \subseteq W^{1,p}(\Omega; \mathbb{R}^m) \cap H^{3/2+\varepsilon}(\mathcal{T}; \mathbb{R}^m) \tag{6.2}$$

for some ε with $0 < \varepsilon \leq 1/2$. Then the traces of Dv and Dw on $\cup \mathcal{E}_\Omega$ for $v, w \in X_h = Y_h$ belong to $L^2(\cup \mathcal{E}_\Omega)$. Notice that $\mathcal{S}^1(\mathcal{T})^m \subseteq X_h$ but $e \in X_h$ is some additional (and strong) hypothesis on u and that we will even suppose $u \in H^2(\Omega; \mathbb{R}^m)$.

Theorem 6.1. *Suppose (H1), (H2), and (H4) and $u_D \in H^2(\mathcal{E}_D; \mathbb{R}^m)$. Moreover, assume that $u \in H^2(\Omega; \mathbb{R}^m) \cap W^{1,p}(\Omega; \mathbb{R}^m)$ and $\mathcal{T}$ is quasi-uniform. Then, there holds*

$$\lim_{h \to 0} \|Du - Du_h\|_{L^2(\Omega)} = 0 \quad \text{for } -1 < \gamma < 3,$$

$$\|u - u_h\|_{W^{1,2}(\Omega)} \leq c_3 h^{1/2} \quad \text{for } \gamma = 1.$$

The constant $c_3 > 0$ depends on c_1, c_2, and upper bounds for $\|u\|_{H^2(\Omega)}$, $|u_h|_{W^{1,p}(\Omega)}$, $|u|_{W^{1,p}(\Omega)}$, and $\|\partial_\mathcal{E}^2 u_D / \partial s^2\|_{L^2(\partial\Omega)}$.

Remark 6.1. Provided $u \in (H^2(\mathcal{T}; \mathbb{R}^m) \cap W^{1,p}(\Omega; \mathbb{R}^m)) \setminus H^2(\Omega; \mathbb{R}^m)$ there holds $a_h(u, \cdot) \not\equiv 0$. Then, for $\gamma = 5/2$, the proof of Theorem 6.1 below can be modified to obtain the estimate

$$\|u - u_h\|_{W^{1,2}(\Omega)} \lesssim h^{1/8}.$$

The proof of the theorem follows from the abstract estimate of Theorem 2.1 and the following lemmas. Throughout this section, abbreviate

$$|v|_h := \|h_\mathcal{E}^{\gamma/2} [Dv]\|_{L^2(\cup \mathcal{E}_\Omega)} \quad \text{and} \quad \|v\|_{X_h}^2 = \|v\|_{Y_h}^2 := \|v\|_{L^2(\Omega)}^2 + |v|_h^2$$

for $v \in H^{3/2+\varepsilon}(\mathcal{T}; \mathbb{R}^m)$.

Lemma 6.1. *If e_h is the nodal interpolant of $e \in C(\overline{\Omega}; \mathbb{R}^m)$ then*

$$\|e - e_h\|_{Y_h} \lesssim \|h_\mathcal{T}^{(1+\gamma)/2} D_\mathcal{T}^2 e\|_{L^2(\Omega)}.$$

Proof. This is an immediate consequence of the trace inequality (5.1) and standard error estimates of nodal interpolation. □

Proposition 2.2 and Lemma 6.1 allow for the application of Theorem 2.1. The strong convergence, however, is obtained by a combination with the following argument.

Lemma 6.2. *There holds*

$$
\begin{aligned}
|e|^2_{W^{1,2}(\Omega)} \lesssim \|e\|_{L^2(\Omega)}\|\Delta_{\mathcal{T}} e\|_{L^2(\Omega)} \\
+ |e|_h\big(\|h_{\mathcal{T}}^{(1-\gamma)/2} De\|_{L^2(\Omega)} + \|h_{\mathcal{T}}^{-(1+\gamma)/2} e\|_{L^2(\Omega)}\big) \\
+ \|h_{\mathcal{T}}^2 \partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\partial\Omega)}\big(\|u\|_{H^2(\Omega)} + \|h_{\mathcal{T}}^{-1/2} Du_h\|_{L^2(\Omega)}\big).
\end{aligned}
$$

Proof. We perform an integration by parts on each $T \in \mathcal{T}$, use the estimates $\|Du\cdot\nu\|_{L^2(\partial\Omega)} \lesssim \|u\|_{H^2(\Omega)}$ and $\|Du_h\cdot\nu\|_{L^2(\partial\Omega)} \lesssim \|h_{\mathcal{T}}^{-1/2} Du_h\|_{L^2(\Omega)}$, and employ Cauchy inequalities to verify

$$
\begin{aligned}
\|De\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}} \int_{\partial T} (De\cdot\nu)\cdot e\, ds - \sum_{T\in\mathcal{T}} \int_T (\Delta e)\cdot e\, dx \\
&= \sum_{E\in\mathcal{E}_\Omega} \int_E \big([De]\cdot\nu_E\big)\cdot e\, ds - \int_\Omega (\Delta_{\mathcal{T}} e)\cdot e\, dx + \int_{\partial\Omega} (De\cdot\nu)\cdot e\, ds \\
&\lesssim \Big(\sum_{E\in\mathcal{E}_\Omega} h_E^{\gamma}\|[De]\|^2_{L^2(E)}\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_\Omega} h_E^{-\gamma}\|e\|^2_{L^2(E)}\Big)^{1/2} \\
&\quad + \|\Delta_{\mathcal{T}} e\|_{L^2(\Omega)}\|e\|_{L^2(\Omega)} + \big(\|u\|_{H^2(\Omega)} + \|h_{\mathcal{T}}^{-1/2} Du_h\|_{L^2(\Omega)}\big)\|e\|_{L^2(\partial\Omega)} \\
&= |e|_h\Big(\sum_{E\in\mathcal{E}_\Omega} h_E^{-\gamma}\|e\|^2_{L^2(E)}\Big)^{1/2} + \|e\|_{L^2(\Omega)}\|\Delta_{\mathcal{T}} e\|_{L^2(\Omega)} \\
&\quad + \big(\|u\|_{H^2(\Omega)} + \|h_{\mathcal{T}}^{-1/2} Du_h\|_{L^2(\Omega)}\big)\|e\|_{L^2(\partial\Omega)}.
\end{aligned}
$$

The trace inequality (5.1) yields

$$
\sum_{E\in\mathcal{E}_\Omega} h_E^{-\gamma}\|e\|^2_{L^2(E)} \lesssim \|h_{\mathcal{T}}^{-(1+\gamma)/2}\, e\|^2_{L^2(\Omega)} + \|h_{\mathcal{T}}^{(1-\gamma)/2}\, De\|^2_{L^2(\Omega)}.
$$

Nodal interpolation estimates on each $E \in \mathcal{E}_D$ show

$$
\|e\|_{L^2(\partial\Omega)} \lesssim \|h_{\mathcal{T}}^2 \partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\partial\Omega)}.
$$

The combination of the last three estimates concludes the proof. ☐

Proof of Theorem 6.1. Notice that $[Du]|_E = 0$ for all $E \in \mathcal{E}_\Omega$ so that $a_h(u, e_h) = 0$. It follows from Theorem 2.1 and Lemma 6.1 that

$$
\begin{aligned}
\|e\|^2_{L^2(\Omega)} + |e|^2_h &\lesssim |e - e_h|^{r/(r-1)}_{W^{1,p}(\Omega)} + \beta^*\big(2B\|e - e_h\|_{Y_h}\big) + 2a_h(u, e_h) \\
&\lesssim \|h_{\mathcal{T}} D^2_{\mathcal{T}} e\|^{r/(r-1)}_{L^p(\Omega)} + \|h_{\mathcal{T}}^{(\gamma+1)/2} D^2_{\mathcal{T}} e\|^2_{L^2(\Omega)} \\
&\lesssim h^{r/(r-1)} + h^{\gamma+1} =: \mathrm{RHS}^2.
\end{aligned}
$$

The combination of this with Lemma 6.2 and $\|\Delta_{\mathcal{T}} e\|_{L^2(\Omega)}$, $\|u\|_{H^2(\Omega)}$, $\|\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\partial\Omega)}$, $\|Du_h\|_{L^2(\Omega)} \lesssim 1$ yields

$$\begin{aligned}|e|^2_{W^{1,2}(\Omega)} \lesssim{} & \mathrm{RHS} + \|h^2_{\mathcal{T}}\|_{L^\infty(\Omega)} \|h_{\mathcal{T}}^{-1/2}\|_{L^\infty(\Omega)} \\ & + \mathrm{RHS}\big(\|h_{\mathcal{T}}^{-(1+\gamma)/2}\|_{L^\infty(\Omega)}\mathrm{RHS} + \|h_{\mathcal{T}}^{(1-\gamma)/2}\|_{L^\infty(\Omega)}\|De\|_{L^2(\Omega)}\big).\end{aligned}$$

Young's inequality allows us to absorb $\|De\|_{L^2(\Omega)} = |e|_{W^{1,2}(\Omega)}$ on the right-hand side and hence shows

$$\begin{aligned}|e|^2_{W^{1,2}(\Omega)} \lesssim{} & \mathrm{RHS} + \mathrm{RHS}^2 \|h_{\mathcal{T}}^{-(1+\gamma)/2}\|_{L^\infty(\Omega)} \\ & + \mathrm{RHS}^2\|h_{\mathcal{T}}^{(1-\gamma)/2}\|^2_{L^\infty(\Omega)} + \|h^2_{\mathcal{T}}\|_{L^\infty(\Omega)}\|h_{\mathcal{T}}^{-1/2}\|_{L^\infty(\Omega)}.\end{aligned}$$

Since $r \le 2$, $h^{r/(r-1)} \lesssim h^2$. With $\|h_{\mathcal{T}}^{-1}\|_{L^\infty(\Omega)} \approx \|h_{\mathcal{T}}\|^{-1}_{L^\infty(\Omega)}$ we deduce

$$|e|^2_{W^{1,2}(\Omega)} \lesssim h + h^{(\gamma+1)/2} + h^{(3-\gamma)/2} + h^{(\gamma+1)/2} + h^{3-\gamma} + h^2 + h^{3/2}.$$

This and $\|e\|_{L^2(\Omega)} \lesssim h^2 + h^{\gamma+1}$ prove Theorem 6.1. □

Remark 6.2. If boundary conditions are imposed only on some part Γ_D of $\partial\Omega$ (and not on the entire boundary $\partial\Omega$) one obtains an additional term

$$\int_{\partial\Omega\setminus\Gamma_D} (De\cdot\nu)\cdot e\, ds$$

which we failed to control.

7. Stabilisation via distances to averages of gradients

This section is devoted to a stabilisation $J_h = J + a_h$ with distances to averages of gradients, i.e.

$$a_h(v,w) := \int_\Omega h_{\mathcal{T}}^{\gamma-1}(Dv - ADv) : (Dw - ADw)\, dx \tag{7.1}$$

for $\gamma \in \mathbb{R}$, $v, w \in W^{1,p}(\Omega;\mathbb{R}^m)$, and for the averaging operator

$$A : L^2(\Omega;\mathbb{M}^{m\times n}) \to \mathcal{S}^1(\mathcal{T})^{m\times n}, \quad p \mapsto Ap := \sum_{z\in\mathcal{N}} |\omega_z|^{-1}\int_{\omega_z} p\, dx\, \varphi_z.$$

Here, for each node $z \in \mathcal{N}$, $\omega_z = \{x \in \Omega : \varphi_z(x) > 0\}$ denotes its patch of area or volume $|\omega_z|$. Let $X_h = Y_h$ be as in Section 6. For $v \in X_h$ we abbreviate

$$|||v|||^2_h = a_h(v,v) = \|h_{\mathcal{T}}^{(\gamma-1)/2}(Dv - ADv)\|^2_{L^2(\Omega)}$$

and define $\|\cdot\|_{X_h} = \|\cdot\|_{Y_h}$ by

$$\|v\|^2_{X_h} = \|v\|^2_{Y_h} = \|v\|^2_{L^2(\Omega)} + |||v|||^2_h.$$

Theorem 7.1. *Under the hypotheses of Theorem 6.1 there holds*

$$\lim_{h\to 0}\|Du-Du_h\|_{L^2(\Omega)}=0 \quad \textit{for } -1<\gamma<3,$$
$$\|u-u_h\|_{W^{1,2}(\Omega)}\le c_4 h^{1/2} \quad \textit{for } \gamma=1.$$

Remark 7.1. Provided $u\in (H^2(\mathcal{T};\mathbb{R}^m)\cap W^{1,p}(\Omega;\mathbb{R}^m))\setminus H^2(\Omega;\mathbb{R}^m)$ and $\gamma=5/2$, one can prove

$$\|u-u_h\|_{W^{1,2}(\Omega)}\lesssim h^{1/8}.$$

The following lemma shows that the stabilisation defined by (7.1) is equivalent to the one discussed in the previous section and will be used to reduce the proof of Theorem 7.1 to the one of Theorem 6.1. The semi-norm $|\cdot|_h$ is defined as in the previous section.

Lemma 7.1 ([C])**.** *For $v_h\in\mathcal{S}^1(\mathcal{T})^m$ there holds $|v_h|_h\approx |||v_h|||_h$.* □

Proof of Theorem 7.1. Let e_h denote the nodal interpolant of $e\in C(\overline{\Omega};\mathbb{R}^m)$. Theorem 2.1 and Proposition 2.2 show

$$\|e\|^2_{L^2(\Omega)}+|||e|||_h^2\lesssim |e-e_h|^{r/(r-1)}_{W^{1,p}(\Omega)} + \|e-e_h\|^2_{L^2(\Omega)}+|||e-e_h|||_h^2+2a_h(u,e_h).$$

Lemma 7.1 shows

$$\begin{aligned}|e|_h&\le |e_h|_h+|e-e_h|_h\lesssim |||e_h|||_h+|e-e_h|_h\\ &\lesssim |||e|||_h+|||e-e_h|||_h+|e-e_h|_h.\end{aligned}$$

Nodal interpolation estimates and continuity of A then imply

$$|e|_h^2\lesssim |||e_h|||_h^2+h^{\gamma+1}.$$

We employ Hölder's inequality, Young's inequality, and nodal interpolation estimates to verify for $\varrho>0$

$$\begin{aligned}a_h(u,e_h)&\lesssim |||u|||_h^2+\varrho|||e_h|||_h^2\lesssim |||u|||_h^2+\varrho|||e|||_h^2+\varrho|||e-e_h|||_h^2\\ &\lesssim |||u|||_h^2+\varrho|||e|||_h^2+h^{\gamma+1}.\end{aligned}$$

Using $\sum_{z\in\mathcal{N}}\varphi_z=1$, we deduce

$$\begin{aligned}|||u|||_h^2&=\sum_{z\in\mathcal{N}}\int_\Omega h_{\mathcal{T}}^{\gamma-1}\varphi_z(Du-p_z)(Du-ADu)\,dx\\ &\le\Big(\sum_{z\in\mathcal{N}}\|h_{\mathcal{T}}^{(\gamma-1)/2}\varphi_z^{1/2}(Du-p_z)\|^2_{L^2(\Omega)}\Big)^{1/2}\|h_{\mathcal{T}}^{(\gamma-1)/2}(Du-ADu)\|_{L^2(\Omega)},\end{aligned}$$

where $p_z = |\omega_z|^{-1} \int_{\omega_z} Du\, dx$ for all $z \in \mathcal{N}$. Poincaré's inequality and $|\varphi_z| \le 1$ show

$$\sum_{z\in\mathcal{N}} \|h_{\mathcal{T}}^{(\gamma-1)/2} \varphi_z^{1/2}(Du - p_z)\|_{L^2(\Omega)}^2 \lesssim \|h_{\mathcal{T}}^{(\gamma+1)/2} D^2 u\|_{L^2(\Omega)}^2.$$

The combination of the preceding three estimates proves

$$\|e\|_{L^2(\Omega)}^2 + |e|_h^2 \lesssim h^2 + h^{\gamma+1}.$$

The assertions of the theorem then follow with Lemma 6.2 and the arguments of the proof of Theorem 6.1. □

8. Stabilisation via gradients

This section is devoted to a stabilisation $J_h = J + a_h$ with gradients, i.e. with some $\gamma > 0$ and

$$a_h(v, w) = h^\gamma \int_\Omega Dv : Dw\, dx \tag{8.1}$$

for all $v, w \in X_h = Y_h = W^{1,p}(\Omega; \mathbb{R}^m)$. For $v \in X_h$ we define

$$\|v\|_{X_h}^2 = \|v\|_{Y_h}^2 := \|v\|_{L^2(\Omega)}^2 + h^\gamma \|Dv\|_{L^2(\Omega)}^2.$$

Theorem 8.1. *Suppose (H1), (H2), and (H4). Assume that $\mathcal{T}$ is quasiuniform and $u \in W^{1,p}(\Omega; \mathbb{R}^m) \cap H^{1+s}(\Omega; \mathbb{R}^m)$ for some $s \in (1/2, 1]$. Then, there holds*

$$\lim_{h\to 0} \|Du - Du_h\|_{L^2(\Omega)} = 0 \quad \text{for } \gamma \in (2(1-s), 2s),$$
$$\|u - u_h\|_{W^{1,2}(\Omega)} \le c_5 h^{s-1/2} \quad \text{for } \gamma = 1.$$

The constant $c_5 > 0$ depends on c_1, c_2, and upper bounds for $\|u\|_{H^{1+s}(\Omega)}$, $|u_h|_{W^{1,p}(\Omega)}$, $|u|_{W^{1,p}(\Omega)}$.

Proof. Proposition 2.2 and Theorem 2.1 prove

$$\begin{aligned} \|e\|_{L^2(\Omega)}^2 + h^\gamma \|De\|_{L^2(\Omega)}^2 \lesssim{}& |e - e_h|_{W^{1,p}(\Omega)}^{r/(r-1)} + \|e - e_h\|_{L^2(\Omega)}^2 \\ &+ h^\gamma \|D(e - e_h)\|_{L^2(\Omega)}^2 + a_h(u, e_h) \end{aligned} \tag{8.2}$$

for the nodal interpolant $e_h \in \mathcal{S}_0^1(\mathcal{T})^m$ of $e \in C(\overline{\Omega}; \mathbb{R}^m)$. Standard estimates on nodal interpolation in $H^{1+s}(\Omega)$ and $r/(r-1) \ge 2$ imply

$$|e - e_h|_{W^{1,p}(\Omega)}^{r/(r-1)} + \|e - e_h\|_{L^2(\Omega)}^2 + h^\gamma \|D(e - e_h)\|_{L^2(\Omega)}^2 \lesssim h^{2s} + h^{2+2s} + h^{\gamma+2s}.$$

If $u \in H^2(\Omega; \mathbb{R}^m)$ then integration by parts and $e_h = 0$ on $\partial\Omega$ show

$$h^\gamma \int_\Omega Du : De_h \, dx \leq h^\gamma \|u\|_{H^2(\Omega)} \|e_h\|_{L^2(\Omega)}.$$

Hölder's inequality and an elementwise inverse estimate verify

$$h^\gamma \int_\Omega Du : De_h \, dx \lesssim h^{\gamma-1} \|u\|_{H^1(\Omega)} \|e_h\|_{L^2(\Omega)}.$$

Interpolation of the last two estimates yields

$$a_h(u, e_h) = h^\gamma \int_\Omega Du : De_h \, dx \lesssim h^{\gamma-(1-s)} \|u\|_{H^{1+s}(\Omega)} \|e_h\|_{L^2(\Omega)}.$$

We further estimate

$$\begin{aligned} a_h(u, e_h) &\lesssim h^{\gamma-(1-s)} \|u\|_{H^{1+s}(\Omega)} \|e_h\|_{L^2(\Omega)} \\ &\leq h^{\gamma-(1-s)} \|u\|_{H^{1+s}(\Omega)} \|e - e_h\|_{L^2(\Omega)} + h^{\gamma-(1-s)} \|u\|_{H^{1+s}(\Omega)} \|e\|_{L^2(\Omega)}. \end{aligned}$$

Nodal interpolation estimates and Young's inequality imply for $\varrho > 0$

$$a_h(u, e_h) \lesssim h^{\gamma-(1-s)+1+s} + h^{2\gamma-2(1-s)} + \varrho \|e\|^2_{L^2(\Omega)}.$$

The combination with (8.2) shows, after absorbing $\|e\|_{L^2(\Omega)}$ on the right-hand side,

$$\|e\|^2_{L^2(\Omega)} + h^\gamma \|De\|^2_{L^2(\Omega)} \lesssim h^{2s} + h^{2\gamma-2(1-s)}. \quad \square$$

The following theorem states that the stabilisation scheme (8.1) is in fact the scheme of [NW] in 1D (up to a lumped integration of the right hand side f).

Theorem 8.2. *Let $n = m = 1$, $\Omega := (0,1)$, $\mathcal{A} = \mathcal{A}_D := W_0^{1,p}(0,1)$,*

$$J(u; v) := \int_0^1 uv \, dx \quad \text{and} \quad J_h(u_h; v_h) := \frac{1}{2} \sum_{z \in \mathcal{K}} h_z u_h(z) v_h(z)$$

for $u, v \in W_0^{1,p}(0,1)$ and $u_h, v_h \in \mathcal{A}_h = \mathcal{A}_{D,h} := \mathcal{S}_0^1(\mathcal{T})$. Then, for all $u_h, v_h \in \mathcal{A}_h$, there holds

$$J_h(u_h; v_h) = J(u_h; v_h) + \frac{1}{6} \int_0^1 h_\mathcal{T}^2 Du_h Dv_h \, dx.$$

Proof. Let $0 = z_0 < z_1 < ... < z_{m+1} = 1$ be such that $\mathcal{N} = \{z_0, z_1, ..., z_{m+1}\}$ and set $h_j := z_j - z_{j-1}$ for $j = 1, ..., m+1$ so that

$h_{z_j} = h_j + h_{j+1}$ for $j = 1, ..., m$. Elementary calculations with $v_h(z_0) = v_h(z_{m+1}) = 0$ show

$$\begin{aligned} J(u_h; v_h) &= \frac{1}{6}\sum_{j=1}^{m+1} h_j\big(2u_h(z_{j-1})v_h(z_{j-1}) + u_h(z_{j-1})v_h(z_j) \\ &\qquad + u_h(z_j)v_h(z_{j-1}) + 2u_h(z_j)v_h(z_j)\big), \\ J_h(u_h; v_h) &= \frac{1}{2}\sum_{j=1}^{m+1} h_j\big(u_h(z_{j-1})v_h(z_{j-1}) + u_h(z_j)v_h(z_j)\big). \end{aligned}$$

Hence, there holds

$$\begin{aligned} &J_h(u_h; v_h) - J(u_h; v_h) \\ &\quad = \frac{1}{6}\sum_{j=1}^{m+1} h_j\big(u_h(z_{j-1})v_h(z_{j-1}) - u_h(z_{j-1})v_h(z_j) \\ &\qquad\qquad - u_h(z_j)v_h(z_{j-1}) + u_h(z_j)v_h(z_j)\big) \\ &\quad = \frac{1}{6}\sum_{j=1}^{m+1} h_j\big(u_h(z_j) - u_h(z_{j-1})\big)\big(v_h(z_j) - v_h(z_{j-1})\big) \\ &\quad = \frac{1}{6}\int_0^1 h_{\mathcal{T}}^2 Du_h Dv_h\, dx. \quad \square \end{aligned}$$

The parameter $\gamma = 2$ is critical in Theorem 8.1 and excluded in our analysis. In fact, the arguments in [NW] are quite different and restricted to a model scenario in 1D.

9. Strong convergence in the scalar 2-well problem

In case of the 2-well energy from Example 3.3 and $n \geq 2$, $m = 1$ we can weaken (H4), i.e. the uniform monotonicity of J can in fact be replaced by monotonicity.

(H5). Suppose that there exists $B \geq 0$ such that, for $v \in W^{1,p}(\Omega)$,

$$\begin{aligned} 0 &\leq J(u; e) - J(u_h; e), \\ J(u; v) - J(u_h; v) &\leq B\|e\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}. \end{aligned}$$

We suppose that $J_h := J + a_h$ with a_h as in (6.1), (7.1), or (8.1).

Theorem 9.1. *Suppose $n \geq 2$ and $m = 1$. Let $S = D\varphi$ with φ as in Example 3.3. Suppose (H5) and $u_D \in H^2(\mathcal{E}_D; \mathbb{R})$. Assume that $\mathcal{T}$ is quasiuniform and $u \in H^2(\Omega) \cap W^{1,p}(\Omega)$. Then, there holds*

$$\|u - u_h\|_{W^{1,2}(\Omega)} \leq c_6 h^{1/2} \quad \textit{for } \gamma = 1.$$

The constant $c_6 > 0$ *depends on* c_1, c_2, *and upper bounds for* $\|u\|_{H^2(\Omega)}$, $|u_h|_{W^{1,p}(\Omega)}$, $|u|_{W^{1,p}(\Omega)}$, *and* $\|\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\partial\Omega)}$.

The proof of the theorem follows from the following lemma and the estimates of the previous sections.

Lemma 9.1. *Let* $n \geq 2$ *and let* φ *be as in Example 3.3 and* $S = D\varphi$. *Suppose* $e_h \in H^1(\Omega)$ *satisfies* $e_h = 0$ *on* $\partial\Omega$. *Then, there holds*

$$\|e\|^2_{L^2(\Omega)} \lesssim \int_\Omega \delta : De\,dx + \|e - e_h\|^2_{L^2(\Omega)} + \|D(e - e_h)\|^2_{L^2(\Omega)}.$$

Proof. Proposition 3 in [CP1] ensures the existence of some $a \in \mathbb{R}^n$ with $|a| = 1$ such that

$$\|a \cdot De\|^2_{L^2(\Omega)} \lesssim \int_\Omega \delta \cdot De\,dx.$$

A fine version of Friedrichs' inequality (which follows from the one dimensional Friedrichs inequality) proves

$$\|e_h\|_{L^2(\Omega)} \lesssim \|a \cdot De_h\|_{L^2(\Omega)}.$$

Two applications of the triangle inequality and the last two estimates prove the lemma. □

Proof of Theorem 9.1. Proposition 2 and Theorem 2 in [CP1] prove (H1)-(H2). Setting $\|v\|^2_{X_h} = \|v\|^2_{Y_h} := a_h(v, v)$ we observe that the first estimate in (H3) is satisfied. Instead of the second estimate in (H3) we have

$$J_h(u; v) - J_h(u_h; v) \leq \|e\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|e\|_{X_h}\|v\|_{Y_h}$$

for all $v \in Y_h$. This and Lemma 9.1 imply the estimate of Theorem 2.1. Hence,

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} + a_h(e, e) \lesssim |e - e_h|^{r/(r-1)}_{W^{1,p}(\Omega)} + \|e - e_h\|^2_{L^2(\Omega)} \\ + \|D(e - e_h)\|^2_{L^2(\Omega)} + a_h(e - e_h, e - e_h) + 2a_h(u, e_h)\end{aligned}$$

for $e_h \in \mathcal{S}^1_0(\mathcal{T})$. The estimate of the theorem then follows as in the proofs of Theorem 6.1, 7.1, and 8.1 for a_h defined by (6.1), (7.1), and (8.1), respectively. □

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References

[Ba] Bartels, S. (2001), Numerical analysis of some non-convex variational problems. PhD thesis, Christian Albrechts-University, Kiel. Available at `http://e-diss.uni-kiel.de/math-nat.html`

[BS] Brenner, S.C., Scott, L.R. (1994), *The mathematical theory of finite element methods.* Texts in Applied Mathematics. 15. Springer-Verlag.

[Bo] Bolza, O. (1906), A fifth necessary condition for a strong extremum of the integral $\int_{x_0}^{x_1} f(x,y,y')\,dx$. *Trans. Amer. Math. Soc.*, **7**(2), 314-324.

[C] Carstensen, C. (2002), All first order averaging techniques yield reliable and efficient a posteriori finite element error control. *Math. Comp.* (to appear).

[CK] Carstensen, C., Klose, R. (2002), Guaranteed a posteriori finite element error control for the p-Laplace problem. Preprint.

[CM] Carstensen, C., Müller, S. (2002), Local Stress Regularity in Scalar Non-Convex Variational Problem. *SIAM J. Math. Anal.* **34**, 495–509.

[CP1] Carstensen, C., Plecháč, P. (1997), Numerical solution of the scalar double-well problem allowing microstructure. *Math. Comp.* **66**, 997-1026.

[CP2] Carstensen, C., Plecháč, P. (2000), Numerical analysis of compatible phase transitions in elastic solids. *SIAM J. Numer. Anal.* **37**, 2061-2081.

[CP3] Carstensen, C., Plecháč, P. (2001), Numerical analysis of a relaxed variational model of hysteresis in two-phase solids. *M2AN* **35**, 865-878.

[F] French, D.A. (1990), On the convergenc of finite element approximations of a relaxed variational problem. *SIAM J. Numer. Anal.* **27**, 419-436.

[GKR] Goodman, J., Kohn, R., Reyna, L. (1986), A numerical study of a relaxed variational problem from optimal design. *Comput. Methods Appl. Engrg.*, **57**(1), 107-127.

[K] Kohn, R. V. (1991), The relaxation of a double-well energy. *Contin. Mech. Thermodyn.* **3**, No.3, 193-236.

[LB] Lui, W.B., Barrett, J.W. (1993), A remark on the regularity of the solution of the p-Laplacian and its application to their finite element approximation. *J. Math. Anal. Appl.* **178**, 470-487.

[L1] Luskin, M. (1996), On the computation of crystalline microstructure. *Acta Numerica* **5**, 191-257.

[L2] Luskin, M. (1996), Approximation of a laminated microstructure for a rotational invariant double-well energy density. *Numer. Math.* **75**, 205-221.

[NW] Nicolaides, R. A., Walkington, N. J. (1995), Strong convergence of numerical solutions to degenerate variational problems. *Math. Comp.* **64**, 117-127.

[R] Roubíček, T. (1997), *Relaxation in optimization theory and variational calculus.* De Gruyter Series in Nonlinear Analysis and Applications 4, New York.

[Y] Young, L.C. (1937), Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *Comptes Rendues de la Société des Sciences et des Lettres de Varsovie, classe III* **30**, 212-234.

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