

ALTERNATING DIRECTION METHOD OF MULTIPLIERS WITH VARIABLE STEP SIZES

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ABSTRACT. The alternating direction method of multipliers (ADMM) is a flexible method to solve a large class of convex minimization problems. Particular features are its unconditional convergence with respect to the involved step size and its direct applicability. This article deals with the ADMM with variable step sizes and devises an adjustment rule for the step size relying on the monotonicity of the residual and discusses proper stopping criteria. The numerical experiments show significant improvements over established variants of the ADMM.

1. INTRODUCTION

The development of iterative schemes for convex minimization problems is a fundamental and challenging task in applied mathematics with a long history reflected in a number of articles, e.g., in [3, 4, 7, 8, 13, 16, 17, 21, 31, 33, 34, 35, 38]. These include gradient descent methods, semi-smooth Newton methods, (accelerated) primal-dual-methods, dual methods, Brègman iteration and operator splitting methods. Here, we aim at developing a strategy for an automated adjustment of the step size of the alternating direction method of multipliers (ADMM), which is an operator splitting method, motivated by the fact that its performance is known to critically depend on the involved step size.

We consider convex variational problems of the form

$$\inf_{u \in X} F(Bu) + G(u)$$

that arise in various applications from partial differential equations, mechanics, imaging and economics, e.g., the p -Laplace equation, the ROF model for image denoising, obstacle problems and convex programming. We assume that possible natural constraints are encoded in the objective functionals via indicator functionals. One way to solve this minimization problem is to introduce an auxiliary variable $p = Bu$ which leads to the constrained minimization problem

$$\inf_{(u,p) \in X \times Y} F(p) + G(u) \quad \text{subject to} \quad p = Bu.$$

With a Lagrange multiplier λ and a positive step size $\tau > 0$ the corresponding augmented Lagrangian reads

$$\mathcal{L}_\tau(u, p; \lambda) = F(p) + G(u) + (\lambda, Bu - p)_Y + \frac{\tau}{2} \|Bu - p\|_Y^2,$$

which has been introduced by Hestenes in [30] and Powell in [36]. The augmented Lagrangian method (ALM) minimizes \mathcal{L}_τ with respect to (u, p) jointly and then updates the Lagrange multiplier λ . Since the joint minimization with respect to u and p is almost as hard as the original problem, the idea of the alternating direction method of multipliers (ADMM) is to decouple the minimization and to minimize \mathcal{L}_τ with respect to u and p successively and then update the Lagrange multiplier. In this way one can benefit from the particular features of the objective functionals F and G in the sense that the separate minimization problems can often be solved directly.

The ADMM was first introduced by Glowinski and Marroco in [18] and Gabay and Mercier in [15]. For a comprehensive discussion on ALM and ADMM and applications in partial differential equations, consider, for instance, [16], and for the connection of ADMM to other splitting methods, particularly the Douglas-Rachford splitting method (DRSM) [11], see, e.g., [14, 12]. In the recent literature versions of ADMM for convex minimization problems with more than two primal and/or auxiliary linearly constrained variables have also been analyzed, see, e.g., [9, 23].

He and Yuan establish in [28] an $\mathcal{O}(1/J)$ convergence rate in an ergodic sense of a quantity related to an optimality condition which is based on a variational inequality reformulation of the constrained minimization problem. In [29], they prove an $\mathcal{O}(1/J)$ convergence rate for the residual of the ADMM. Shen and Xu prove in [40] an $\mathcal{O}(1/J)$ convergence rate in an ergodic sense for a modified ADMM proposed by Ye and Yuan in [41] which augments the original ADMM with fixed step size by an additional extrapolation of the primal and dual variable. In [20], motivated by the work of Nesterov [34], Goldstein et al. consider an accelerated version of ADMM (Fast-ADMM) and prove an $\mathcal{O}(1/J^2)$ convergence rate for the objective value of the dual problem of the constrained minimization problem under the assumption that both objective functionals are strongly convex and G is quadratic. Furthermore, they prove an $\mathcal{O}(1/J^2)$ convergence rate for the residuals if in addition B has full row rank and also propose a Fast-ADMM with restart for the case of F and G being only convex for which, however, a convergence rate $\mathcal{O}(1/J^2)$ could not be proven. Recently, Deng and Yin established in [10] the linear convergence of a generalized ADMM under a variety of different assumptions on the objective functionals and the operator B . Particularly, they derive an explicit upper bound for the linear convergence rate of the ADMM and, by optimizing the rate with respect to the step size, obtain an optimized step size. However, experiments reveal that the optimized step size often leads to a pessimistic convergence

rate. Furthermore, the computation of the optimized step size requires the knowledge of the strong convexity constant of G and the Lipschitz constant of ∇G as well as the computation of the minimal eigenvalue of the operator $B'B$. In many of the cited results, the convergence rate of the ADMM critically depends on the dimension of X via the operator norm of B and an inf-sup-condition associated to B .

Since the convergence rate of the ADMM depends on the step size τ it seems reasonable to consider a variation of the step size. He and Yang have already studied the ADMM with variable step sizes in [27] under the assumption that F and G are continuously differentiable. They prove termination of the algorithm for monotonically decreasing or monotonically increasing step sizes and prove convergence for fixed step sizes. In [32], Kontogiorgis and Meyer also consider the ADMM with variable step sizes and prove convergence provided that the step sizes decrease except for finitely many times. However, the suggested strategy in varying the step sizes is tailored to the problems discussed in their work. In [26, 25] He et al. consider the ADMM with self-adaptive step sizes and prove convergence provided that the step size is uniformly bounded from above and below and that the sequence of difference quotients of the step size is summable. Their proposed adjustment rule for the step size aims at balancing the two components of the residual of the ADMM. In [24], He et al. consider an inexact ADMM where certain proximal terms are added to the augmented Lagrangian functional \mathcal{L}_τ and the subproblems are only solved approximately. They also allow for variable step sizes and proximal parameters and prove convergence under a summability condition of the difference quotients of the step sizes and proximal parameters and suggest to adjust the step size in such a way that the two contributions of the residual of the ADMM balance out.

In this paper we aim at devising a general automatic step size adjustment strategy. The adjustment of the step size is based on the observation that the residual of the ADMM decreases as long as the sequence of step sizes is non-increasing. Particularly, we prove that the residual reduces by a factor of at least $\gamma < 1$ as long as the step size decreases and under the assumption that ∇G is Lipschitz continuous and B' is injective with bounded left-inverse. More precisely, we propose the following strategy:

- (1) Choose a feasible initialization (u^0, λ^0) , a large step size τ_0 and a contraction factor $\gamma \in (0, 1)$.
- (2) Minimize \mathcal{L}_τ with respect to p , then minimize \mathcal{L}_τ with respect to u and finally update λ .
- (3) Check if the residual is decreased by the factor γ . If this is not the case decrease the step size.
- (4) If the current step size is smaller than a chosen lower bound, restart the algorithm with a larger $\gamma \in (0, 1)$. Otherwise continue with (2).

We furthermore address the choice of accurate stopping criteria and propose a stopping criterion that controls the distance between the primal iterates and the exact solution if strong coercivity is given.

The paper is organized as follows. In Section 2 we state some basic notation that we use and briefly introduce the finite element spaces we use when applying the general framework to convex model problems. Section 3 is devoted to the analysis of ADMM with variable step sizes. In Subsection 3.1 we present the minimization problem and the corresponding saddle-point formulation on which the ADMM is based. The ADMM with variable step sizes is then recalled in Subsection 3.2 and a convergence proof, which is similar to that of Kontogiorgis and Meyer in [32], is given. Additionally, we show that the residual controls the distance between iterates and a saddle-point if strong coercivity is given. The monotonicity of the residual of the ADMM with variable step sizes, which is proven in Subsection 3.3, is a crucial property of the method to ensure the termination of the scheme even if the step size is being decreased. Furthermore, it directly implies a sub-linear convergence rate of the method without any additional conditions on the functionals. In Subsection 3.4 we deal with the linear convergence of the ADMM with variable step sizes which motivates to adjust the step size according to the contraction properties of the residual. Subsequently, we make our approach more precise and present the Variable-ADMM in Subsection 3.5. In Section 4 we apply our algorithm to the obstacle problem and the TV - L^2 minimization (or ROF) problem and compare its performance to the classical ADMM with fixed step size and the Fast-ADMM proposed by Goldstein et al. in [20], which we specify in the appendix. Here, we also discuss stopping criteria used in the literature that may fail to lead to an accurate approximation. Finally, we give a conclusion in Section 5.

2. PRELIMINARIES

2.1. Notation. We consider two Hilbert spaces X, Y equipped with inner products $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$, respectively, and identify their duals, denoted by X' and Y' , with X and Y , respectively. For a linear operator $B : X \rightarrow Y$ we denote by $B' : Y' \rightarrow X'$ its adjoint. We furthermore let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polygonal Lipschitz domain. The L^2 -norm on Ω is denoted by $\|\cdot\|$ and is induced by the scalar product

$$(v, w) := \int_{\Omega} v \cdot w \, dx$$

for scalar functions or vector fields $v, w \in L^2(\Omega; \mathbb{R}^r)$, $r \in \{1, d\}$, and we write $|\cdot|$ for the Euclidean norm. We use for arbitrary sequences $(a^j)_{j \in \mathbb{N}}$ and step sizes $\tau_j > 0$ the backward difference quotient

$$d_t a^{j+1} := \frac{a^{j+1} - a^j}{\tau_{j+1}}.$$

Using this definition we will also work with

$$d_t^2 a^{j+1} = \frac{d_t a^{j+1} - d_t a^j}{\tau_{j+1}} = \frac{\frac{a^{j+1} - a^j}{\tau_{j+1}} - \frac{a^j - a^{j-1}}{\tau_j}}{\tau_{j+1}}.$$

Note that we have the discrete product rules

$$(1a) \quad 2d_t a^{j+1} \cdot a^{j+1} = d_t |a^{j+1}|^2 + \tau_{j+1} |d_t a^{j+1}|^2,$$

$$(1b) \quad d_t (\tau_{j+1}^2 |a^{j+1}|^2) = (d_t \tau_{j+1}^2) |a^j|^2 + \tau_{j+1}^2 d_t |a^{j+1}|^2.$$

Finally, throughout the paper c will denote a generic, positive and mesh-independent constant.

2.2. Finite element spaces. We let $(\mathcal{T}_h)_{h>0}$ be a family of regular triangulations of Ω with mesh sizes $h = \max_{T \in \mathcal{T}_h} h_T$ with h_T being the diameter of the simplex T . We further denote $h_{\min} = \min_{T \in \mathcal{T}_h} h_T$. For a given triangulation \mathcal{T}_h the set \mathcal{N}_h contains the corresponding nodes and we consider the finite element spaces of continuous, piecewise affine functions

$$\mathcal{S}^1(\mathcal{T}_h) := \{v_h \in C(\bar{\Omega}) : v_h|_T \text{ affine for all } T \in \mathcal{T}_h\}$$

and of elementwise constant functions ($r = 1$) or vector fields ($r = d$)

$$\mathcal{L}^0(\mathcal{T}_h)^r := \{q_h \in L^\infty(\Omega; \mathbb{R}^r) : q_h|_T \text{ constant for all } T \in \mathcal{T}_h\}.$$

Correspondingly, we denote by \mathcal{T}_ℓ , $\ell \in \mathbb{N}$, a triangulation of Ω generated from an initial triangulation \mathcal{T}_0 by ℓ uniform refinements. The refinement level ℓ will be related to the mesh size h by $h \sim 2^{-\ell}$. The set of nodes \mathcal{N}_ℓ is then defined as before. In our experiments we will use the discrete norm $\|\cdot\|_h$ induced by the discrete scalar product

$$(v, w)_h := \sum_{z \in \mathcal{N}_h} \beta_z v(z) w(z)$$

for $v, w \in \mathcal{S}^1(\mathcal{T}_h)$, where $\beta_z = \int_\Omega \varphi_z dx$ and $\varphi_z \in \mathcal{S}^1(\mathcal{T}_h)$ is the nodal basis function associated with the node $z \in \mathcal{N}_h$. This mass lumping will allow for the nodewise solution of certain nonlinearities. We have the relation

$$\|v_h\| \leq \|v_h\|_h \leq (d+2)^{1/2} \|v_h\|$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$, cf. [1, Lemma 3.9]. On $\mathcal{L}^0(\mathcal{T}_h)^r$ we will also consider the weighted L^2 -inner product

$$(\cdot, \cdot)_w := h^d(\cdot, \cdot)$$

which has the property $\|q_h\|_w \leq c \|q_h\|_{L^1(\Omega)}$ due to an inverse estimate, cf. [5].

3. ALTERNATING DIRECTION METHOD OF MULTIPLIERS

3.1. Minimization problem and saddle-point formulation. We are given convex, proper, and lower-semicontinuous functionals $F : Y \rightarrow \mathbb{R} \cup \{+\infty\}$, $G : X \rightarrow \mathbb{R} \cup \{+\infty\}$, and a bounded linear operator $B : X \rightarrow Y$ such that the functional $I(\cdot) = F(B\cdot) + G(\cdot)$ is proper and coercive. We consider the minimization problem

$$\inf_{u \in X} I(u) = \inf_{u \in X} F(Bu) + G(u).$$

Upon introducing $p = Bu$ and choosing $\tau > 0$ we obtain the equivalent, consistently stabilized saddle-point problem defined by

$$\inf_{(u,p) \in X \times Y} \sup_{\lambda \in Y} \mathcal{L}_\tau(u, p; \lambda) = F(p) + G(u) + (\lambda, Bu - p)_Y + \frac{\tau}{2} \|Bu - p\|_Y^2.$$

Remark 3.1. *If there exists a saddle-point $(u, p; \lambda)$ for \mathcal{L}_τ , then u is a minimizer of I and $p = Bu$, cf. [16, Chapter VI, Thm. 2.1]. On the other hand, if X and Y are finite-dimensional and if $u \in X$ is a minimizer of I , the existence of a saddle-point $(u, p; \lambda)$ for \mathcal{L}_τ can be proven by taking $p = Bu$, using [37, Thm. 23.8] and [37, Thm. 23.9], and incorporating the fact that $(v, q) \mapsto \mathcal{L}_\tau(v, q; \lambda)$ for fixed λ is convex, proper, coercive and lower-semicontinuous and therefore admits a minimizer. The characterizing optimality conditions for such a minimizer are satisfied by the pair (u, p) if λ is chosen properly and one deduces that $(u, p; \lambda)$ is a saddle-point for \mathcal{L}_τ (see also [16, 37]).*

In this paper we make the following assumption.

Assumption 3.2. *There exists a saddle-point $(u, p; \lambda)$ for \mathcal{L}_τ .*

Possible strong convexity of F or G is characterized by nonnegative functionals $\varrho_F : Y \times Y \rightarrow \mathbb{R}$ and $\varrho_G : X \times X \rightarrow \mathbb{R}$ in the following lemma.

Lemma 3.3 (Optimality conditions). *A triple (u, p, λ) is a saddle point for \mathcal{L}_τ if and only if $Bu = p$ and*

$$\begin{aligned} (\lambda, q - p)_Y + F(p) + \varrho_F(q, p) &\leq F(q), \\ -(\lambda, B(v - u))_Y + G(u) + \varrho_G(v, u) &\leq G(v), \end{aligned}$$

for all $(v, q) \in X \times Y$.

Proof. The variational inequalities characterize stationarity with respect to u and p , respectively, i.e., that, e.g., $0 \in \partial_u \mathcal{L}_\tau(u, p; \lambda)$. \square

For ease of presentation we introduce the symmetrized coercivity functionals

$$\widehat{\varrho}_G(u, u') = \varrho_G(u, u') + \varrho_G(u', u), \quad \widehat{\varrho}_F(p, p') = \varrho_F(p, p') + \varrho_F(p', p).$$

3.2. Algorithm and convergence. We approximate a saddle-point using the following iterative scheme which has been introduced in [18, 15, 16] with fixed step sizes.

Algorithm 3.4 (Generalized ADMM). *Choose $(u^0, \lambda^0) \in X \times Y$ such that $G(u^0) < \infty$. Choose $\bar{\tau} \geq \underline{\tau} > 0$ and $\bar{R} \gg 0$ and set $j = 1$.*

(1) *Set $\tau_1 = \bar{\tau}$ and $R_0 = \bar{R}$.*

(2) *Compute a minimizer $p^j \in Y$ of the mapping*

$$p \mapsto \mathcal{L}_{\tau_j}(u^{j-1}, p; \lambda^{j-1}).$$

(3) *Compute a minimizer $u^j \in X$ of the mapping*

$$u \mapsto \mathcal{L}_{\tau_j}(u, p^j; \lambda^{j-1}).$$

(4) *Update $\lambda^j = \lambda^{j-1} + \tau_j(Bu^j - p^j)$.*

(5) *Define*

$$R_j = (\|\lambda^j - \lambda^{j-1}\|_Y^2 + \tau_j^2 \|B(u^j - u^{j-1})\|_Y^2)^{1/2}.$$

(6) *Stop if R_j is sufficiently small.*

(7) *Choose step size $\tau_{j+1} \in [\underline{\tau}, \bar{\tau}]$.*

(8) *Set $j \rightarrow j + 1$ and continue with (2).*

Remarks 3.5. (1) *A variant of the ADMM with variable step sizes has been proposed and analyzed in [32]. Therein, a more general scheme with symmetric positive definite matrices H_j is presented. We will give a more compact proof of boundedness of the iterates and termination of the algorithm related to [32, Lem. 2.5, Lem. 2.6] with $H_j = \tau_j I$.*

(2) *We call Algorithm 3.4 with fixed step sizes, i.e., $\bar{\tau} = \underline{\tau}$, simply “ADMM”.*

(3) *In Subsection 3.5 we present a strategy for the adjustment of τ_j based on contraction properties and introduce the “Variable-ADMM”.*

The iterates of Algorithm 3.4 satisfy the following optimality conditions.

Lemma 3.6 (Decoupled optimality). *With $\tilde{\lambda}^j := \lambda^{j-1} + \tau_j(Bu^{j-1} - p^j)$ the iterates $(u^j, p^j, \lambda^j)_{j=0,1,\dots}$ satisfy for $j \geq 1$ the variational inequalities*

$$\begin{aligned} (\tilde{\lambda}^j, q - p^j)_Y + F(p^j) + \varrho_F(q, p^j) &\leq F(q), \\ -(\lambda^j, B(v - u^j))_Y + G(u^j) + \varrho_G(v, u^j) &\leq G(v), \end{aligned}$$

for all $(v, q) \in X \times Y$. In particular, $(u^j, p^j; \lambda^j)$ is a saddle-point of \mathcal{L}_τ if and only if $\lambda^j - \lambda^{j-1} = 0$ and $B(u^j - u^{j-1}) = 0$.

Proof. By step (1) in Algorithm 3.4 we have $0 \in \partial_p \mathcal{L}_{\tau_j}(u^{j-1}, p^j; \lambda^{j-1})$ which is equivalent to the first variational inequality using the definition of $\tilde{\lambda}^j$. Step (2) implies $0 \in \partial_u \mathcal{L}_{\tau_j}(u^j, p^j; \lambda^{j-1})$ which is equivalent to the second variational inequality using the definition of λ^j in step (3). \square

We set $\tau_0 := \tau_1$ for $d_t \tau_1 = 0$ to be well-defined in the convergence proof.

Theorem 3.7 (Convergence). *Let $(u, p; \lambda)$ be a saddle point for \mathcal{L}_τ . Suppose that the step sizes satisfy the monotonicity property*

$$0 < \tau \leq \tau_{j+1} \leq \tau_j$$

for $j \geq 1$. For the iterates $(u^j, p^j; \lambda^j)$, $j \geq 0$, of Algorithm 3.4, the corresponding differences $\delta_\lambda^j = \lambda - \lambda^j$, $\delta_p^j = p - p^j$ and $\delta_u^j = u - u^j$, and the distance

$$D_j^2 = \|\delta_\lambda^j\|_Y^2 + \tau_j^2 \|B\delta_u^j\|_Y^2,$$

we have that for every $J \geq 1$ it holds

$$\frac{1}{2}D_J^2 + \sum_{j=1}^J \left(\tau_j (\widehat{\varrho}_G(u, u^j) + \widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u^{j-1}, u^j)) + \frac{1}{2}R_j^2 \right) \leq \frac{1}{2}D_0^2.$$

In particular, $R_j \rightarrow 0$ as $j \rightarrow \infty$ and Algorithm 3.4 terminates.

Proof. Choosing $(v, q) = (u^j, p^j)$ in Lemma 3.3 and $(v, q) = (u, p)$ in Lemma 3.6 and adding corresponding inequalities we obtain

$$\begin{aligned} (\tilde{\lambda}^j - \lambda, p - p^j)_Y + \widehat{\varrho}_F(p, p^j) &\leq 0, \\ (\lambda - \lambda^j, B(u - u^j))_Y + \widehat{\varrho}_G(u, u^j) &\leq 0. \end{aligned}$$

Adding the inequalities, inserting λ^j , and using that $Bu = p$ we obtain

$$\widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u, u^j) \leq (\lambda - \lambda^j, Bu^j - p^j)_Y + (\lambda^j - \tilde{\lambda}^j, p - p^j)_Y.$$

With $\lambda^j - \tilde{\lambda}^j = \tau_j B(u^j - u^{j-1}) = -\tau_j^2 d_t B\delta_u^j$ and $Bu^j - p^j = d_t \lambda^j = -d_t \delta_\lambda^j$ we get

$$(2) \quad \widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u, u^j) \leq -(\delta_\lambda^j, d_t \delta_\lambda^j)_Y - \tau_j^2 (d_t B\delta_u^j, \delta_p^j)_Y.$$

Testing the optimality conditions of u^j and u^{j-1} with $v = u^{j-1}$ and $v = u^j$, respectively, and adding the corresponding inequalities gives

$$\widehat{\varrho}_G(u^{j-1}, u^j) \leq -\tau_j^2 (d_t \lambda^j, d_t Bu^j)_Y.$$

Using $d_t \lambda^j = Bu^j - p^j$ and inserting $p = Bu$ on the right-hand side yields

$$(3) \quad \begin{aligned} \widehat{\varrho}_G(u^{j-1}, u^j) &\leq -\tau_j (B(u^j - u) + p - p^j, B(u^j - u^{j-1}))_Y \\ &= -\tau_j^2 (B\delta_u^j, d_t B\delta_u^j)_Y + \tau_j^2 (\delta_p^j, d_t B\delta_u^j)_Y. \end{aligned}$$

Adding (2) and (3) and using the discrete product rules (1a) and (1b) gives

$$(4) \quad \begin{aligned} &\widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u, u^j) + \widehat{\varrho}_G(u^{j-1}, u^j) + \frac{\tau_j}{2} \|d_t \lambda^j\|_Y^2 + \frac{\tau_j^3}{2} \|d_t B\delta_u^j\|_Y^2 \\ &\leq -\frac{d_t}{2} \|\delta_\lambda^j\|_Y^2 - \tau_j^2 \frac{d_t}{2} \|B\delta_u^j\|_Y^2 \\ &= -\frac{d_t}{2} \|\delta_\lambda^j\|_Y^2 - d_t \left(\frac{\tau_j^2}{2} \|B\delta_u^j\|_Y^2 \right) + \left(d_t \frac{\tau_j^2}{2} \right) \|B\delta_u^{j-1}\|_Y^2. \end{aligned}$$

Multiplication by τ_j , summation over $j = 1, \dots, J$, and noting that $R_j^2 = \tau_j^2 \|d_t \lambda^j\|_Y^2 + \tau_j^4 \|d_t B u^j\|_Y^2$ yields

$$\begin{aligned} & \frac{1}{2} \left(\|\delta_\lambda^J\|_Y^2 + \tau_J^2 \|B \delta_u^J\|_Y^2 \right) \\ & + \sum_{j=1}^J \tau_j \left(\widehat{\varrho}_G(u, u^j) + \widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u^{j-1}, u^j) + \frac{1}{2\tau_j} R_j^2 \right) \\ & \leq \frac{1}{2} \left(\|\delta_\lambda^0\|_Y^2 + \tau_0^2 \|B \delta_u^0\|_Y^2 + \sum_{j=1}^J \tau_j (d_t \tau_j^2) \|B \delta_u^{j-1}\|_Y^2 \right). \end{aligned}$$

The fact that $d_t \tau_j^2 \leq 0$ proves the assertion. \square

Remarks 3.8. (1) Note that Algorithm 3.4 is convergent independently of the choice $\tau_0 > 0$.

(2) The estimate shows that a large step size τ_0 may affect the convergence behavior. However, experiments indicate that the algorithm is slow if the step size is chosen too small. This motivates to consider a variable step size that is adjusted to the performance of the algorithm during the iteration.

(3) If we change the order of minimization in Algorithm 3.4 we obtain the estimate with δ_p^J , $d_t p^j$, $\widehat{\varrho}_F(p^{j-1}, p^j)$, δ_p^0 and δ_p^{j-1} instead of $B \delta_u^J$, $d_t B u^j$, $\widehat{\varrho}_G(u^{j-1}, u^j)$, $B \delta_u^0$ and $B \delta_u^{j-1}$, respectively. The second minimization should thus be carried out with respect to the variable for which we have strong convexity to have control over the distance between two consecutive iterates.

(4) If X and Y are finite element spaces related to a triangulation with maximal mesh size $h > 0$ and if we have u^0 with the approximation property $\|B(u - u^0)\| \leq ch^\alpha$ we may choose the initial step size τ_0 as $\tau_0 = h^{-\alpha}$. In general, we may initialize the algorithm with a sufficiently large step size τ_0 and gradually decrease the step size, e.g., whenever the algorithm computes iterates which do not lead to a considerable decrease in the residual.

(5) Note that the convergence proof allows for finitely many reinitializations of the step size. If $u^j := u^0$ and $\lambda^j := \lambda^0$ whenever the step size is reinitialized, this resembles a restart of the algorithm. To be more precise, if J_1, \dots, J_L denote the iterations after which the algorithm is reinitialized, i.e., we set $u^{J_k} := u^0$ and $\lambda^{J_k} := \lambda^0$ and $\tau_{J_k+1} = \tau_0$, $k = 1, \dots, L$, we obtain for any $1 \leq k \leq L$ and any $J_k \leq J < J_{k+1}$

$$\frac{1}{2} D_J^2 + \sum_{j=J_k}^J \left(\tau_j (\widehat{\varrho}_G(u, u^j) + \widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u^{j-1}, u^j)) + \frac{1}{2} R_j^2 \right) \leq \frac{1}{2} D_0^2,$$

where we used that $\delta_\lambda^{J_k} = \delta_\lambda^0$ and $B \delta_u^{J_k} = B \delta_u^0$. Summation over $k = 1, \dots, L$ then gives for any $J \geq 1$

$$\frac{L}{2} D_J^2 + \sum_{j=1}^J \left(\tau_j (\widehat{\varrho}_G(u, u^j) + \widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u^{j-1}, u^j)) + \frac{1}{2} R_j^2 \right) \leq \frac{L}{2} D_0^2.$$

(6) The conditions on the step size are the same as in [32].

The residuals R_j control the distance between iterates and a saddle-point $(u, p; \lambda)$ provided that strong coercivity applies.

Corollary 3.9. *For all $j \geq 1$ we have*

$$\widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u, u^j) + \widehat{\varrho}_G(u^{j-1}, u^j) \leq 2C_0 R_j$$

with $C_0 = \max\{\frac{1}{\tau_j}\|\lambda\|_Y + \|Bu\|_Y, \frac{1}{\tau_j}\|\lambda^j\|_Y + \|Bu^j\|_Y\}$.

Proof. Adding (2) and (3) gives

$$\begin{aligned} & \widehat{\varrho}_F(p, p^j) + \widehat{\varrho}_G(u, u^j) + \widehat{\varrho}_G(u^{j-1}, u^j) \\ & \leq -(\delta_\lambda^j, d_t \delta_\lambda^j)_Y - \tau_j^2 (B\delta_u^j, d_t B\delta_u^j)_Y \\ & \leq \frac{1}{\tau_j} \|\lambda - \lambda^j\|_Y \|\lambda^j - \lambda^{j-1}\|_Y + \tau_j \|B(u - u^j)\|_Y \|B(u^j - u^{j-1})\|_Y \\ & \leq \left(\frac{1}{\tau_j} \|\lambda - \lambda^j\|_Y + \|B(u - u^j)\|_Y \right) R_j, \end{aligned}$$

which implies the estimate. \square

Remarks 3.10. (1) If G is strongly coercive there exists a coercivity constant $\alpha_G > 0$ such that $\varrho_G(v, w) = \alpha_G \|v - w\|_X^2$. Particularly, we then have $\widehat{\varrho}_G(v, w) = 2\alpha_G \|v - w\|_X^2$.

(2) Corollary 3.9 motivates to use the stopping criterion $R_j \leq \varepsilon_{stop}/C_0$ for a prescribed accuracy $\varepsilon_{stop} > 0$.

3.3. Monotonicity and convergence rate. In [29] a sublinear $\mathcal{O}(1/J)$ convergence rate for the ADMM is shown with a contraction-type analysis. For this, the authors prove (4) with constant step sizes and a monotonicity property of the residual. The residual of Algorithm 3.4 also enjoys a monotonicity property which is stated in the following proposition which is a generalization of [29, Thm. 5.1].

Proposition 3.11 (Monotonicity of residual). *For all $j \geq 1$ we have*

$$2\tau_{j+1}(\widehat{\varrho}_F(p^j, p^{j+1}) + \widehat{\varrho}_G(u^j, u^{j+1})) + R_{j+1}^2 \leq R_j^2$$

if $\tau_{j+1} \leq \tau_j$. Particularly, the residual is non-increasing.

Proof. Testing the decoupled optimality conditions of (p^{j+1}, u^{j+1}) and (p^j, u^j) in Lemma 3.6 with $(q, v) = (p^j, u^j)$ and $(q, v) = (p^{j+1}, u^{j+1})$, respectively, and adding the inequalities yields

$$\begin{aligned} & \widehat{\varrho}_F(p^j, p^{j+1}) + \widehat{\varrho}_G(u^j, u^{j+1}) \\ & \leq -(\widetilde{\lambda}^{j+1} - \widetilde{\lambda}^j, p^j - p^{j+1})_Y - (\lambda^j - \lambda^{j+1}, B(u^j - u^{j+1}))_Y \\ & = \tau_{j+1}^2 (d_t \widetilde{\lambda}^{j+1}, d_t p^{j+1})_Y - \tau_{j+1}^2 (d_t \lambda^{j+1}, d_t B u^{j+1})_Y. \end{aligned}$$

Using $d_t \tilde{\lambda}^{j+1} = d_t \lambda^{j+1} - d_t(\tau_{j+1}^2 d_t B u^{j+1})$ and $d_t p^{j+1} = -d_t^2 \lambda^{j+1} + d_t B u^{j+1}$ in the first term on the right-hand side, using the discrete product rules (1a), (1b) and Young's inequality $ab \leq \varepsilon a^2/2 + b^2/(2\varepsilon)$ with $\varepsilon = \tau_{j+1}$ we obtain

$$\begin{aligned}
& \widehat{\varrho}_F(p^j, p^{j+1}) + \widehat{\varrho}_G(u^j, u^{j+1}) \\
& \leq -\tau_{j+1}^2 (d_t \lambda^{j+1}, d_t^2 \lambda^{j+1})_Y - (d_t(\tau_{j+1}^2 d_t B u^{j+1}), \tau_{j+1}^2 d_t B u^{j+1})_Y \\
& \quad + \tau_{j+1}^2 (d_t(\tau_{j+1}^2 d_t B u^{j+1}), d_t^2 \lambda^{j+1})_Y \\
& = -\tau_{j+1}^2 \frac{d_t}{2} \|d_t \lambda^{j+1}\|_Y^2 - \frac{\tau_{j+1}^3}{2} \|d_t^2 \lambda^{j+1}\|_Y^2 \\
& \quad - \frac{d_t}{2} \|\tau_{j+1}^2 d_t B u^{j+1}\|_Y^2 - \frac{\tau_{j+1}}{2} \|d_t(\tau_{j+1}^2 d_t B u^{j+1})\|_Y^2 \\
& \quad + \tau_{j+1}^2 (d_t(\tau_{j+1}^2 d_t B u^{j+1}), d_t^2 \lambda^{j+1})_Y \\
& \leq -\frac{d_t}{2} \|\tau_{j+1}^2 d_t B u^{j+1}\|_Y^2 - d_t \left(\frac{\tau_{j+1}^2}{2} \|d_t \lambda^{j+1}\|_Y^2 \right) + \left(d_t \frac{\tau_{j+1}^2}{2} \right) \|d_t \lambda^j\|_Y^2,
\end{aligned}$$

which implies the assertion. \square

We can now deduce a convergence rate for the residual. This generalizes the result in [29, Thm. 6.1] for the ADMM with monotonically decreasing step sizes.

Corollary 3.12. *Suppose that the step sizes satisfy the monotonicity property $\tau_{j+1} \leq \tau_j$ for $j \geq 1$. Then we have*

$$R_J^2 = \mathcal{O}\left(\frac{1}{J}\right).$$

Proof. Proposition 3.11 guarantees $R_J \leq R_j$ for $1 \leq j \leq J$ which implies

$$JR_J^2 \leq \sum_{j=1}^J R_j^2 \leq D_0^2$$

for any $J \geq 1$ where the second inequality is due to Theorem 3.7. \square

3.4. Linear convergence. We extend the results in [10] concerning the linear convergence of the ADMM to the case of variable step sizes and prove additionally the linear convergence of the residual of the Variable-ADMM which serves as the basis for our adjustment rule for the step size. From now on we assume that G is Fréchet-differentiable with Lipschitz continuous derivative G' , i.e., there exists $L_G > 0$ such that for all $v, v' \in X$ we have

$$\|\nabla G(v) - \nabla G(v')\|_X \leq L_G \|v - v'\|_X,$$

and that G is strongly convex with coercivity constant α_G , i.e.,

$$\varrho_G(v, v') \geq \alpha_G \|v - v'\|_X^2.$$

Here, ∇G is the representation of G' with respect to the inner product $(\cdot, \cdot)_X$, i.e.,

$$(\nabla G(v), w)_X = G'(v)[w]$$

for all $w \in X$. We further assume that the adjoint of B is injective with bounded left-inverse, i.e., there exists a constant $\alpha_B > 0$ such that $\|B'\mu\|_X \geq \alpha_B \|\mu\|_Y$ for all $\mu \in Y$.

Lemma 3.13. *For all $v, w \in X$ and any $\theta \in [0, 1]$ we have*

$$\begin{aligned} & (\nabla G(v) - \nabla G(w), v - w)_X \\ & \geq (1 - \theta) \frac{1}{L_G} \|\nabla G(v) - \nabla G(w)\|_X^2 + \theta 2\alpha_G \|v - w\|_X^2. \end{aligned}$$

Proof. Due to the differentiability and strong convexity of G we have

$$(\nabla G(v), w - v)_X + G(v) + \alpha_G \|w - v\|_X^2 \leq G(w).$$

Exchanging the roles of v and w and adding the inequalities gives

$$(\nabla G(v) - \nabla G(w), v - w)_X \geq 2\alpha_G \|v - w\|_X^2.$$

By Lemma A.1 stated in Appendix A we also have

$$(\nabla G(v) - \nabla G(w), v - w)_X \geq \frac{1}{L_G} \|\nabla G(v) - \nabla G(w)\|_X^2,$$

which implies the estimate. \square

Theorem 3.14 (Linear convergence). *If $\underline{\tau} \leq \tau_{j+1} \leq \tau_j \leq \bar{\tau}$ for all $j \geq 0$ there exists a sequence $(\gamma_j)_{j \in \mathbb{N}} \subset (0, 1)$ with $\gamma_j \leq \gamma < 1$ for all $j \geq 1$ such that*

$$D_{j+1}^2 \leq \gamma_{j+1} D_j^2$$

with D_j as in Theorem 3.7.

Proof. From inequality (4) it follows that

$$(5) \quad 2\tau_{j+1} \widehat{\varrho}_G(u, u^{j+1}) + D_{j+1}^2 \leq D_j^2.$$

Here, the term $\widehat{\varrho}_G(u, u^{j+1})$ on the left-hand side results from the estimate

$$(6) \quad \widehat{\varrho}_G(u, u^{j+1}) \leq (\lambda^{j+1} - \lambda, B(u - u^{j+1}))_Y.$$

We aim at replacing this bound by a stronger bound using the differentiability of G and Lemma 3.13. With the differentiability of G the optimality conditions for u and u^{j+1} in Lemma 3.3 and Lemma 3.6, respectively, can be rewritten as

$$\begin{aligned} -(\lambda, Bv)_Y &= (\nabla G(u), v)_X, \\ -(\lambda^{j+1}, Bv)_Y &= (\nabla G(u^{j+1}), v)_X \end{aligned}$$

for all $v \in X$. Particularly, $\nabla G(u) = -B'\lambda$ and $\nabla G(u^{j+1}) = -B'\lambda^{j+1}$. Choosing $v = u - u^{j+1}$ and subtracting we obtain

$$(\lambda^{j+1} - \lambda, B(u - u^{j+1}))_Y = (\nabla G(u) - \nabla G(u^{j+1}), u - u^{j+1})_X.$$

Using Lemma 3.13 we find with the bounds $\|Bv\|_Y \leq c_B\|v\|_X$ and $\|B'\mu\|_X \geq \alpha_B\|\mu\|_Y$ that

$$\begin{aligned} & (\lambda^{j+1} - \lambda, B(u - u^{j+1}))_X \\ & \geq (1 - \theta) \frac{1}{L_G} \|B'(\lambda - \lambda^{j+1})\|_X^2 + \theta 2\alpha_G \|u - u^{j+1}\|_X^2 \\ & \geq (1 - \theta) \frac{\alpha_B^2}{L_G} \|\delta_\lambda^{j+1}\|_X^2 + \theta \frac{2\alpha_G}{c_B^2} \|B\delta_u^{j+1}\|_X^2 \\ & = (1 - \theta) s_1 \frac{1}{\tau_{j+1}} \|\delta_\lambda^{j+1}\|_X^2 + \theta s_2 \tau_{j+1} \|B\delta_u^{j+1}\|_X^2 \end{aligned}$$

where $s_1 = \alpha_B^2 \tau_{j+1} / L_G$ and $s_2 = 2\alpha_G / (c_B^2 \tau_{j+1})$. Choosing $\theta \in [0, 1]$ such that $(1 - \theta)s_1 = \theta s_2$, i.e.,

$$\theta = \left(1 + \frac{s_2}{s_1}\right)^{-1},$$

we have

$$\rho_{j+1} = (1 - \theta)s_1 = \theta s_2 = \frac{s_1 s_2}{s_1 + s_2} > 0.$$

This choice of θ yields

$$(\lambda^{j+1} - \lambda, B(u - u^{j+1}))_Y \geq \rho_{j+1} \left(\tau_{j+1} \|B\delta_u^{j+1}\|_X^2 + \frac{1}{\tau_{j+1}} \|\delta_\lambda^{j+1}\|_Y^2 \right).$$

Using this estimate instead of (6) in the proof of (5) we obtain

$$2\rho_{j+1} D_{j+1}^2 + D_{j+1}^2 \leq D_j^2.$$

The assertion follows with the choice $\gamma_{j+1} = (1 + 2\rho_{j+1})^{-1}$. \square

Proposition 3.15 (Linear convergence of residual). *If $\underline{\tau} \leq \tau_{j+1} \leq \tau_j \leq \bar{\tau}$ for all $j \geq 0$ there exists a sequence $(\gamma_j)_{j \in \mathbb{N}} \subset (0, 1)$ with $\gamma_j \leq \gamma < 1$ for all $j \geq 1$ such that*

$$R_{j+1}^2 \leq \gamma_{j+1} R_j^2.$$

Proof. We have by Proposition 3.11 that

$$2\tau_{j+1} \widehat{\rho}_G(u^j, u^{j+1}) + R_{j+1}^2 \leq R_j^2,$$

where the term $\widehat{\rho}_G(u^j, u^{j+1})$ on the left-hand side results from the estimate

$$\widehat{\rho}_G(u^j, u^{j+1}) \leq (\lambda^{j+1} - \lambda^j, B(u^j - u^{j+1}))_Y.$$

The optimality conditions for u^j and u^{j+1} in Lemma 3.6 and the differentiability of G imply

$$-B'\lambda^{j+1} = \nabla G(u^{j+1}), \quad -B'\lambda^j = \nabla G(u^j).$$

The assertion follows with the same rate γ_{j+1} as in Theorem 3.14 using the same arguments as in the proof of the previous theorem. \square

Remark 3.16. (1) Note that by the previous proposition and Corollary 3.12 we have

$$R_J^2 = \mathcal{O}(\min\{\gamma^J, 1/J\}).$$

(2) Minimizing γ_{j+1} with respect to τ_{j+1} yields the step size $\tau_{j+1} \equiv \tau$ with corresponding rate $\gamma_{j+1} \equiv \gamma$ given by

$$\tau = \left(\frac{2\alpha_G L_G}{\alpha_B^2 c_B^2} \right)^{1/2}, \quad \gamma = \left(1 + \left(\frac{2\alpha_G L_G \alpha_B^2}{c_B^2} \right)^{1/2} \right)^{-1}.$$

3.5. Step size adjustment. The previous discussion shows that the convergence rate critically depends on the step size. Moreover, the optimized step size may lead to a pessimistic contraction order as observed in [10]. This motivates to incorporate an automated adjustment of the step size. With regard to Proposition 3.15 the idea is to prescribe a contraction factor $\gamma \geq \underline{\gamma}$, start with a large step size $\tau = \bar{\tau}$ and decrease τ whenever the contraction property is violated. When a lower bound $\underline{\tau}$ is reached, the algorithm is restarted with a larger contraction factor γ . To account for cases which do not satisfy the conditions for linear convergence of the algorithm, one has to choose an upper bound $\bar{\gamma} \approx 1$ for the contraction factor to guarantee convergence of the algorithm. We make our procedure precise in the following algorithm which is identical to Algorithm 3.4 except for the initialization of additional parameters and the specification of step (7) of Algorithm 3.4.

Algorithm 3.17 (Variable-ADMM). Choose $(u^0, \lambda^0) \in X \times Y$ such that $G(u^0) < \infty$. Choose $\bar{\tau} > \underline{\tau} > 0$, $\delta \in (0, 1)$, $0 < \underline{\gamma} \leq \bar{\gamma} < 1$, $\bar{R} \gg 0$ and $\varepsilon_{stop} > 0$. Set $j = 1$.

(1) Set $\gamma_1 = \underline{\gamma}$, $\tau_1 = \bar{\tau}$ and $R_0 = \bar{R}$.

(2)-(6) As in Algorithm 3.4.

(7) Define $(\tau_{j+1}, \gamma_{j+1})$ as follows:

- If $R_j \leq \gamma_j R_{j-1}$ or if $\tau_j = \underline{\tau}$ and $\gamma_j = \bar{\gamma}$ set

$$\tau_{j+1} = \tau_j \quad \text{and} \quad \gamma_{j+1} = \gamma_j.$$

- If $R_j > \gamma_j R_{j-1}$ and $\tau_j > \underline{\tau}$ set

$$\tau_{j+1} = \max\{\delta \tau_j, \underline{\tau}\} \quad \text{and} \quad \gamma_{j+1} = \gamma_j.$$

- If $R_j > \gamma_j R_{j-1}$, $\tau_j = \underline{\tau}$ and $\gamma_j < \bar{\gamma}$ set

$$\tau_{j+1} = \bar{\tau}, \quad \gamma_{j+1} = \min\left\{ \frac{\gamma_j + 1}{2}, \bar{\gamma} \right\}, \quad u^j = u^0 \quad \text{and} \quad \lambda^j = \lambda^0.$$

(8) Set $j = j + 1$ and continue with (2).

Remark 3.18. The total number of restarts is bounded by $\lceil \log((\underline{\gamma} - 1)/(\bar{\gamma} - 1)) / \log(2) \rceil$. The minimal number of iterations between two restarts is given by $\lceil \log(\underline{\tau}/\bar{\tau}) / \log(\delta) \rceil$. Since the contraction factor is constant between two restarts, i.e., $\gamma_j \equiv \tilde{\gamma}$ for a $\tilde{\gamma} \in [\underline{\gamma}, \bar{\gamma})$, the maximal number of iterations between two restarts is bounded by $\lceil \log(\underline{\tau}/\bar{\tau}) / \log(\delta) \rceil + \lceil \log(\varepsilon_{stop}/R_1) / \log(\tilde{\gamma}) \rceil$.

4. NUMERICAL EXPERIMENTS

We tested the ADMM (Algorithm 3.4 with fixed step size), the Variable-ADMM (Algorithm 3.17) and the Fast-ADMM proposed in [20], which we present in Appendix B as Algorithm B.1, for some prototypical minimization problems which were discretized using low order finite elements. An overview of relevant parameters and abbreviations is given in the following and concern all three algorithms if not otherwise stated:

- *Abbreviations:* $N = \#$ of iterations for termination; u_h^{stop} = output; u_h^{ref} = reference solution computed with ADMM, $\tau = h^{-1}$, $\varepsilon_{stop} = 10^{-9}$ (Example 4.1) and $\tau = h^{-3/2}$, $\varepsilon_{stop} = 10^{-4}$ (Example 4.2)
- *Geometry:* $\Omega = (0, 1)^2$; coarse mesh $\mathcal{T}_0 = \{T_1, T_2\}$; \mathcal{T}_ℓ generated from \mathcal{T}_0 by ℓ uniform refinements
- *Mesh sizes:* $h = \sqrt{2}2^{-\ell}$, $\ell = 3, \dots, 9$
- *(Initial) step sizes:* $\bar{\tau} = h^{-m}$, $m = 0, \dots, 3$
- *Initialization:* $u_h^0 = 0$ and $\lambda_h^0 = 0$
- *Stopping criteria:* (a) $\widehat{\varrho}_G(u_h^{ref}, u_h^j)^{1/2} \leq \varepsilon_{stop}^{(1)}$
(b) $R_j \leq \varepsilon_{stop}^{(2)}/C_0$
- *Stopping tolerance:* $\varepsilon_{stop}^{(1)} = 10^{-3}$, $\varepsilon_{stop}^{(2)} = h^2$ (Example 4.1);
 $\varepsilon_{stop}^{(1)} = 10^{-2}$, $\varepsilon_{stop}^{(2)} = h$ (Example 4.2)
- *Error:* $E_h = \widehat{\varrho}_G(u_h^{ref}, u_h^{stop})^{1/2}$
- *Termination:* A hyphen (-) abbreviates $N \geq 10^3$ (Example 4.1)
 $N \geq 10^4$ (Example 4.2)
- *Fast-ADMM:* $\gamma = 0.999$; $N_{re} = \#$ of restarts (see step (7))
- *Variable-ADMM:* $\gamma = 0.5$, $\delta = 0.5$, $\underline{\tau} = 1$, $\bar{\gamma} = 0.999$;
 $\bar{N}_\tau = \#$ of τ -adjustments after last restart;
 $N_\gamma = \#$ of γ -adjustments

The number N_γ represents also the total number of restarts of the Variable-ADMM when $\bar{\tau} > \underline{\tau}$. Moreover, the integer N also regards the discarded iterations due to restart in the Fast-ADMM and the Variable-ADMM.

4.1. Application to obstacle problem. We consider

$$G(u_h) = \frac{1}{2} \int_{\Omega} |\nabla u_h|^2 dx - \int_{\Omega} f u_h dx, \quad F(Bu_h) = I_K(u_h)$$

with $X_\ell = (\mathcal{S}_0^1(\mathcal{T}_\ell), (\nabla \cdot, \nabla \cdot))$, $Y_\ell = (\mathcal{S}_0^1(\mathcal{T}_\ell), (\cdot, \cdot)_h)$, $B = \text{id} : X \rightarrow Y$ and K being the convex set $K = \{v_h \in \mathcal{S}^1(\mathcal{T}_\ell) : v_h \geq \chi\}$ for some obstacle function $\chi \in \mathcal{S}^1(\mathcal{T}_\ell)$.

4.1.1. Example and stopping criterion. For our experiments we use the following specifications.

Example 4.1. We let $f \equiv -5$, $\chi \equiv -1/4$, $\Gamma_D = \partial\Omega$, $u = 0$ on Γ_D .

		ADMM (obstacle; $\varepsilon_{stop}^{(1)} = 10^{-3}, \tau_j \equiv \bar{\tau}$)							
		$\bar{\tau} = 1$		$\bar{\tau} = h^{-1}$		$\bar{\tau} = h^{-2}$		$\bar{\tau} = h^{-3}$	
ℓ		N		N		N		N	
3		880		159		31		30	
4		—		221		28		217	
5		—		241		76		—	
6		—		151		294		—	
7		—		103		—		—	
8		—		59		—		—	
9		—		60		—		—	
		Fast-ADMM ($\gamma = 0.999$)							
ℓ		N	(N_{re})	N	(N_{re})	N	(N_{re})	N	(N_{re})
3		123	(3)	45	(3)	17	(1)	19	(2)
4		241	(6)	59	(2)	22	(1)	51	(2)
5		474	(17)	67	(4)	34	(2)	157	(2)
6		—	(—)	47	(2)	58	(2)	—	(—)
7		—	(—)	62	(6)	123	(2)	—	(—)
8		—	(—)	37	(4)	320	(32)	—	(—)
9		—	(—)	32	(2)	—	(—)	—	(—)
		Variable-ADMM ($\tau = 1, \gamma = 0.5, \bar{\gamma} = 0.999, \delta = 0.5$)							
ℓ		N	(N_τ, N_γ)	N	(N_τ, N_γ)	N	(N_τ, N_γ)	N	(N_τ, N_γ)
3		880	(0, 8)	236	(0, 5)	83	(0, 3)	59	(1, 2)
4		—	(—, —)	406	(0, 6)	60	(0, 2)	116	(3, 3)
5		—	(—, —)	547	(0, 7)	123	(1, 3)	213	(5, 4)
6		—	(—, —)	442	(0, 7)	134	(3, 3)	233	(8, 4)
7		—	(—, —)	410	(0, 7)	151	(8, 3)	247	(11, 4)
8		—	(—, —)	237	(0, 5)	254	(6, 4)	263	(14, 4)
9		—	(—, —)	185	(0, 4)	264	(8, 4)	277	(17, 4)

TABLE 1. Iteration numbers for Example 4.1 using ADMM, Fast-ADMM and Variable-ADMM with stopping criterion $\|\nabla(u_h^{ref} - u_h^j)\| \leq 10^{-3}$. A hyphen (—) means that the algorithm did not terminate within 10^3 iterations. In parenthesis: total number of restarts (Fast-ADMM) and total number of adjustments of τ_j and γ_j (Variable-ADMM).

Note that G is strongly convex with coercivity constant $\alpha_G = 1/2$, i.e., we have $\widehat{\varrho}_G(v, w) = \|v - w\|_X^2 = \|\nabla(v - w)\|^2$. We have that the unique solution of the infinite-dimensional obstacle problem satisfies $u \in H^2(\Omega)$, cf. [6, 1]. Hence, Corollary 3.9 implies that the error tolerance should be chosen as $\varepsilon_{stop} = h^2$ so that the convergence rate $\|\nabla(u - u_h)\| = \mathcal{O}(h)$ is not violated. Particularly, G is differentiable and its gradient is Lipschitz

ADMM (obstacle; $\varepsilon_{stop}^{(2)} = h^2, \tau_j \equiv \bar{\tau}$)								
	$\bar{\tau} = 1$		$\bar{\tau} = h^{-1}$		$\bar{\tau} = h^{-2}$		$\bar{\tau} = h^{-3}$	
ℓ	N	E_h/h	N	E_h/h	N	E_h/h	N	E_h/h
3	4	0.8658	14	0.1659	9	0.0523	26	0.0160
4	62	0.5479	35	0.1401	25	0.0160	218	0.0106
5	221	0.3439	57	0.0819	87	0.0050	—	—
6	697	0.2868	197	0.0357	382	0.0026	—	—
7	—	—	538	0.0221	—	—	—	—
8	—	—	—	—	—	—	—	—
9	—	—	—	—	—	—	—	—
Fast-ADMM ($\gamma = 0.999$)								
3	4	0.8455	10	0.1723	9	0.0490	15	0.0081
4	17	0.5550	18	0.1395	19	0.0161	51	0.0089
5	104	0.3442	39	0.0799	36	0.0044	184	0.0043
6	543	0.2870	56	0.0368	71	0.0014	—	—
7	—	—	185	0.0220	168	0.0010	—	—
8	—	—	462	0.0124	466	0.0004	—	—
9	—	—	—	—	—	—	—	—
Variable-ADMM ($\underline{\tau} = 1, \gamma = 0.5, \bar{\gamma} = 0.999, \delta = 0.5$)								
3	4	0.8658	7	0.3332	8	0.1586	10	0.1474
4	62	0.5479	36	0.3808	29	0.1734	37	0.1228
5	221	0.3439	68	0.2260	36	0.2848	47	0.1877
6	697	0.2868	85	0.2750	78	0.1189	96	0.1727
7	—	—	145	0.2175	154	0.0898	168	0.1433
8	—	—	251	0.1708	166	0.1914	290	0.1090
9	—	—	327	0.1557	304	0.1359	500	0.0775

TABLE 2. Iteration numbers and ratio E_h/h for Example 4.1 using ADMM, Fast-ADMM and Variable-ADMM with $R_j \leq h^2/\tilde{C}_0$. A hyphen (—) means that the algorithm did not terminate within 10^3 iterations.

continuous and B' is invertible, i.e., the conditions for linear convergence of the ADMM are satisfied.

With the chosen inner-products on X and Y we obtain the constants

$$\alpha_G = \frac{1}{2}, \quad L_G = 1, \quad \alpha_B \approx \frac{h}{2}, \quad c_B = 2c_P$$

with c_P denoting the Poincaré constant associated to the domain Ω , which can in turn be bounded by $c_P \leq \sqrt{2}/\pi$. Using these constants in Remark 3.16 leads to $\tau_{opt} \approx \pi/(\sqrt{2}h)$ and $\gamma_{opt} \approx (1 + \pi h/\sqrt{32})^{-1}$.

With regard to Corollary 3.9 we have to provide a computable upper bound \tilde{C}_0 for

$$C_0 = \max \left\{ \frac{1}{\tau_j} \|\lambda_h\|_h + \|u_h\|_h, \frac{1}{\tau_j} \|\lambda_h^j\|_h + \|u_h^j\|_h \right\}.$$

We have $\|u_h\|_h \leq 2\|u_h\| \leq c(\|\chi\| + \|f\|)$. Furthermore, the optimality condition $0 \in G'(u_h) + \partial F(u_h)$ implies the existence of a Lagrange multiplier $\lambda_h \in \partial F(u_h)$ with

$$-(\lambda_h, v_h)_h = (\nabla u_h, \nabla v_h) - (f, v_h) \quad \text{for all } v_h \in \mathcal{S}_0^1(\mathcal{T}_\ell).$$

Particularly, Assumption 3.2 is satisfied for Example 4.1. Inserting $(\nabla u, \nabla v_h)$ on the right-hand side, using standard interpolation estimates, an inverse estimate, the fact that $u \in H^2(\Omega)$ and integration by parts gives

$$\begin{aligned} |(\lambda_h, v_h)_h| &\leq \|\nabla(u - u_h)\| \|\nabla v_h\| + \|f\| \|v_h\| + \|\Delta u\| \|v_h\| \\ &\leq c(1 + \|f\| + \|\Delta u\|) \|v_h\| \end{aligned}$$

which means that $\|\lambda_h\|_h$ is uniformly bounded in h . Therefore, since we only consider $\tau_j \geq \underline{\tau} = 1$ in our numerical experiments we set

$$\tilde{C}_0 = \max \left\{ 1, \frac{1}{\tau_j} \|\lambda_h^j\|_h + \|u_h^j\|_h \right\},$$

which is, up to constants, an upper bound for C_0 .

4.1.2. Results. We report the iteration numbers for the ADMM, the Fast-ADMM and the Variable-ADMM applied to Example 4.1 with stopping criterion $\|\nabla(u_h^{ref} - u^j)\| \leq 10^{-3}$ in Table 1. Note that $\varepsilon_{stop}^{(1)} = 10^{-3}$ is a lower bound for the minimal mesh size we are considering in our experiments, i.e., the outputs of the algorithms do not affect the order of convergence $\|\nabla(u - u_h)\| = \mathcal{O}(h)$. We infer that, for large initial step sizes, the iteration numbers of the Variable-ADMM are considerably smaller than those of the ADMM and also smaller or at least comparable to those of the Fast-ADMM. Particularly, one can observe a mesh-independent convergence behavior for Variable-ADMM. Note that $\bar{\tau} = h^{-1}$ happens to be approximately the optimal step size τ_{opt} which explains the lower iteration numbers for ADMM and Fast-ADMM in the case $\bar{\tau} = h^{-1}$ since Variable-ADMM had to restart several times to recognize the actual contraction order.

In Table 2 the iteration numbers and the ratio E_h/h , which identifies the quality of the stopping criterion, for the three algorithms with stopping criterion $R_j \leq h^2/C_0$ are displayed which also reflect a considerable improvement of the Variable-ADMM over the ADMM and Fast-ADMM especially for large initial step sizes $\bar{\tau} = h^{-2}, h^{-3}$. The ADMM and Variable-ADMM do not differ for $\bar{\tau} = 1$ since we have set $\underline{\tau} = 1$. A remarkable feature of the Variable-ADMM is that it performs robustly with respect to the choice of the initial step size $\bar{\tau}$ in contrast to the ADMM and Fast-ADMM. Let us finally remark that the ratio E_h/h remains bounded as $h \rightarrow 0$ which underlines that $R_j \leq \varepsilon_{stop}/C_0$ is a reliable stopping criterion and optimal for Variable-ADMM.

4.2. Application to TV- L^2 minimization. In this subsection we apply the algorithms to a prototypical total variation minimization problem, the so called *ROF problem*, cf. [39]. We set

$$G(u_h) = \frac{\alpha}{2} \|u_h - g\|^2, \quad F(Bu_h) = \int_{\Omega} |\nabla u_h| \, dx$$

with $X_{\ell} = (\mathcal{S}^1(\mathcal{T}_{\ell}), (\cdot, \cdot))$, $Y_{\ell} = (\mathcal{L}^0(\mathcal{T}_{\ell})^d, (\cdot, \cdot)_w)$, and $B = \nabla : X_{\ell} \rightarrow Y_{\ell}$.

4.2.1. Example and stopping criterion. We consider the following specification of the minimization problem.

Example 4.2. We let $\alpha = 20$ and $g = \tilde{g} + \xi \in \mathcal{S}^1(\mathcal{T}_{\ell})$ where $\tilde{g} \in \mathcal{S}^1(\mathcal{T}_{\ell})$ is the piecewise linear approximation of the characteristic function $\chi_{B_{1/5}(x_{\Omega})}$ of the circle with radius $r = 1/5$ around the center x_{Ω} of Ω and $\xi \in \mathcal{S}^1(\mathcal{T}_{\ell})$ is a perturbation function whose coefficients are samples of a uniformly distributed random variable in the interval $[-1/10, 1/10]$.

Note that G is strongly convex with coercivity constant $\alpha_G = \alpha/2$, i.e., we have $\hat{\varrho}_G(v, w) = \alpha \|v - w\|^2$. Let $u \in BV(\Omega) \cap L^2(\Omega)$ denote the continuous minimizer. Since $g \in L^{\infty}(\Omega)$ one can show that $u \in L^{\infty}(\Omega)$. Then by Corollary 3.9 the error tolerance has to be chosen as $\varepsilon_{stop} = h$ to match the optimal convergence rate $\|u - u_h\| = \mathcal{O}(h^{1/2})$, cf. [2, Rmk. 7.2] and [1, Rmk. 10.9 (ii)].

The optimality condition $0 \in G'(u_h) + \partial F(\nabla u_h)$ implies the existence of a Lagrange multiplier $\lambda_h \in \partial F(\nabla u_h)$ with $\operatorname{div} \lambda_h = \alpha(u_h - g)$ (cf. [37, Thm. 23.9]) where the operator $\operatorname{div} : \mathcal{L}^0(\mathcal{T}_{\ell})^d \rightarrow \mathcal{S}^1(\mathcal{T}_{\ell})$ is defined via $-(\operatorname{div} \mu_h, v_h) = (\mu_h, \nabla v_h)_w$ for all $\mu_h \in \mathcal{L}^0(\mathcal{T}_{\ell})^d$ and $v_h \in \mathcal{S}^1(\mathcal{T}_{\ell})$. Hence, Assumption 3.2 is satisfied. In this setting the constant C_0 from Corollary 3.9 is given by

$$C_0 = \max \left\{ \frac{1}{\tau_j} \|\lambda_h\|_w + \|\nabla u_h\|_w, \frac{1}{\tau_j} \|\lambda_h^j\|_w + \|\nabla u_h^j\|_w \right\}.$$

The specific choice of the norm ensures that by an inverse estimate it holds $\|\nabla u_h\|_w \leq c \|\nabla u_h\|_{L^1(\Omega)}$. The optimality condition $\lambda_h \in \partial F(p_h)$ implies that

$$\lambda_h = \begin{cases} h^{-d} \frac{p_h}{|p_h|}, & \text{if } p_h \neq 0, \\ h^{-d} \xi, & \text{if } p_h = 0, \end{cases}$$

with $\xi \in B_1(0)$. Therefore we have $\|\lambda_h\|_w \leq ch^{-d/2}$. This scaling of the Lagrange multiplier has to be taken into account in the tolerance for the residual to obtain meaningful outputs, i.e., we set

$$\tilde{C}_0 = \max \left\{ \frac{1}{h^{d/2} \tau_j}, \frac{1}{\tau_j} \|\lambda_h^j\|_w + \|\nabla u_h^j\|_w \right\}.$$

ADMM (ROF; $\varepsilon_{stop}^{(1)} = 10^{-2}$, $\tau_j \equiv \bar{\tau}$)									
	$\bar{\tau} = 1$		$\bar{\tau} = h^{-1}$		$\bar{\tau} = h^{-2}$		$\bar{\tau} = h^{-3}$		
ℓ	N		N		N		N		
3	195		36		21		127		
4	647		58		56		623		
5	2443		109		178		3994		
6	9827		218		578		—		
7	—		427		810		—		
8	—		842		1430		—		
9	—		1670		3666		—		
Fast-ADMM ($\gamma = 0.999$)									
ℓ	N	(N_{re})	N	(N_{re})	N	(N_{re})	N	(N_{re})	
3	60	(2)	19	(1)	9	(0)	77	(8)	
4	174	(9)	31	(2)	25	(2)	386	(42)	
5	1596	(560)	50	(4)	88	(7)	6410	(2907)	
6	—	(—)	113	(9)	289	(24)	—	(—)	
7	—	(—)	226	(19)	410	(33)	—	(—)	
8	—	(—)	485	(46)	790	(72)	—	(—)	
9	—	(—)	1042	(111)	5093	(2158)	—	(—)	
Variable-ADMM ($\tau = 1$, $\gamma = 0.5$, $\bar{\gamma} = 0.999$, $\delta = 0.5$)									
ℓ	N	(N_τ, N_γ)	N	(N_τ, N_γ)	N	(N_τ, N_γ)	N	(N_τ, N_γ)	
3	195	(0, 6)	111	(0, 4)	21	(2, 1)	25	(5, 1)	
4	647	(0, 8)	183	(0, 5)	50	(3, 2)	62	(6, 2)	
5	2443	(0, 9)	319	(0, 6)	62	(5, 2)	82	(10, 2)	
6	9827	(0, 9)	594	(0, 7)	129	(5, 3)	166	(10, 3)	
7	—	(—, —)	1136	(0, 8)	140	(7, 3)	180	(14, 3)	
8	—	(—, —)	2194	(0, 9)	156	(8, 3)	197	(14, 3)	
9	—	(—, —)	—	(—, —)	326	(7, 4)	401	(16, 4)	

TABLE 3. Iteration numbers for Example 4.2 using ADMM, Fast-ADMM and Variable-ADMM with stopping criterion $\sqrt{\alpha}\|u_h^{ref} - u_h^j\| \leq 10^{-2}$. A hyphen (—) means that the algorithm did not converge within 10^4 iterations. In parenthesis: total number of restarts (Fast-ADMM) and total number of adjustments of τ_j and γ_j (Variable-ADMM).

4.2.2. *Results.* The iteration numbers for the ADMM, the Fast-ADMM and the Variable-ADMM applied to Example 4.2 with stopping criterion $\sqrt{\alpha}\|u_h^{ref} - u_h^j\| \leq 10^{-2}$ are displayed in Table 3. Note that $\varepsilon_{stop}^{(1)} = 10^{-2}$ is a lower bound for $h^{1/2}$, i.e., the optimal convergence rate $\|u - u_h\| = \mathcal{O}(h^{1/2})$ is not affected by the computed outputs. The results again underline that

	ADMM (ROF; $\varepsilon_{stop}^{(2)} = h, \tau_j \equiv \bar{\tau}$)							
	$\bar{\tau} = 1$		$\bar{\tau} = h^{-1}$		$\bar{\tau} = h^{-2}$		$\bar{\tau} = h^{-3}$	
ℓ	N	E_h/\sqrt{h}	N	E_h/\sqrt{h}	N	E_h/\sqrt{h}	N	E_h/\sqrt{h}
3	27	0.2242	7	0.1531	8	0.1479	49	0.1836
4	137	0.1690	14	0.1455	47	0.0558	521	0.0569
5	691	0.1657	32	0.1614	217	0.0248	4878	0.0251
6	3882	0.1739	87	0.1729	740	0.0529	—	—
7	—	—	286	0.1569	2037	0.0169	—	—
8	—	—	920	0.1169	7598	0.0131	—	—
9	—	—	2585	0.0834	—	—	—	—
ℓ	Fast-ADMM ($\gamma = 0.999$)							
3	15	0.1188	6	0.1013	7	0.0757	26	0.1625
4	63	0.1644	13	0.1080	26	0.0342	326	0.0554
5	246	0.1662	28	0.0975	107	0.0217	8178	0.0251
6	2370	0.1746	49	0.1693	384	0.0526	—	—
7	—	—	153	0.1561	1145	0.0168	—	—
8	—	—	537	0.1154	—	—	—	—
9	—	—	1669	0.0829	—	—	—	—
ℓ	Variable-ADMM ($\underline{\tau} = 1, \gamma = 0.5, \bar{\gamma} = 0.999, \delta = 0.5$)							
3	27	0.2242	14	0.1531	6	0.1244	20	0.1310
4	137	0.1690	34	0.1455	26	0.0957	33	0.1006
5	691	0.1657	78	0.1614	60	0.0568	81	0.0529
6	3882	0.1739	331	0.1729	137	0.0426	181	0.0178
7	—	—	995	0.1569	319	0.0156	375	0.0162
8	—	—	2272	0.1169	365	0.0161	430	0.0202
9	—	—	—	—	834	0.0140	1763	0.0080

TABLE 4. Iteration numbers for Example 4.2 using ADMM, Fast-ADMM and Variable-ADMM with $R_j \leq h/\tilde{C}_0$. A hyphen (—) means that the algorithm did not converge within 10^4 iterations. In parenthesis: total number of restarts (Fast-ADMM) and total number of adjustments of τ_j and γ_j (Variable-ADMM).

Variable-ADMM can lead to a considerable improvement for large initial step sizes even though this example does not fit into the framework of linear convergence of ADMM we addressed in Subsection 3.4.

In Table 4 the iteration numbers with the stopping criterion $R_j \leq h/C_0$ are reported. Here, the advantage of using the Variable-ADMM is even more pronounced than in Example 4.1 since again the iteration numbers especially for the initial step sizes $\bar{\tau} = h^{-2}, h^{-3}$ are lower compared to those

of ADMM and Fast-ADMM. The reported ratio E_h/\sqrt{h} once more confirms the reliability of the chosen stopping criterion.

4.3. Failure of other stopping criteria. We next demonstrate that it is not sufficient to stop the ADMM and Fast-ADMM using as the stopping criterion either $\|\lambda_h^j - \lambda_h^{j-1}\|_Y \leq \varepsilon_{stop}$ or $\tau_j \|B(u_h^j - u_h^{j-1})\|_Y \leq \varepsilon_{stop}$ because these stopping criteria may not always lead to a suitable approximation of the exact minimizer u_h and that one needs to resort to the stronger stopping criterion $R_j \leq \varepsilon_{stop}/C_0$ due to the saddle-point structure. To see this, we consider Example 4.1 with the stopping criterion $\|\lambda_h^j - \lambda_h^{j-1}\|_h \leq h^2/C_0$ and Example 4.2 with stopping criterion $\tau_j \|\nabla(u_h^j - u_h^{j-1})\|_w \leq h/C_0$ and investigate the ratio $\|\nabla(u_h^{ref} - u_h^{stop})\|/h$ and $\sqrt{\alpha} \|u_h^{ref} - u_h^{stop}\|/\sqrt{h}$, respectively. In Table 5 the corresponding results are shown and we infer that the ratios do not remain bounded as $h \rightarrow 0$ indicating suboptimal approximations. Comparing the results with those reported in Tables 2 and 4 we conclude that in order to obtain an accurate approximation one has to control both the primal iterates Bu_h^j and the dual iterates λ_h^j .

Obstacle ($\varepsilon_{stop}^{(2)} = h^2, \bar{\tau} = h^{-3}$)						
	ADMM		Fast-ADMM		Variable-ADMM	
ℓ	N	E_h/h	N	E_h/h	N	E_h/h
3	23	0.04	14	0.0296	10	0.147
4	152	0.164	43	0.0321	37	0.123
5	1123	0.453	120	0.142	47	0.188
6	2785	15.1	164	9.46	96	0.173
7	8455	50.9	–	57.9	168	0.143
ROF ($\varepsilon_{stop}^{(2)} = h, \bar{\tau} = 1$)						
ℓ	N	E_h/\sqrt{h}	N	E_h/\sqrt{h}	N	E_h/\sqrt{h}
3	6	1.19	5	1.10	6	1.19
4	11	2.10	7	2.09	11	2.10
5	24	3.20	11	3.24	24	3.20
6	54	4.81	19	4.76	54	4.81
7	123	7.13	31	7.04	123	7.13

TABLE 5. Iteration numbers and ratios E_h/h and E_h/\sqrt{h} with stopping criterion $\|\lambda_h^j - \lambda_h^{j-1}\| \leq h^2/\tilde{C}_0$ and $\tau_j \|\nabla(u_h^j - u_h^{j-1})\| \leq h/\tilde{C}_0$ and step size $\bar{\tau} = h^{-3}$ and $\bar{\tau} = 1$ for Example 4.1 and Example 4.2, respectively, using ADMM, Fast-ADMM and Variable-ADMM.

5. CONCLUSION

From our numerical experiments we infer the following observations:

- The Variable-ADMM can considerably improve the performance for any initial step size. If not improving the Variable-ADMM at least yields results comparable to those of ADMM and Fast-ADMM which differ by a fixed factor.
- For large initial step sizes, i.e., $\bar{\tau} = h^{-2}$, the Variable-ADMM always yields lower iteration numbers than the other two schemes. This suggests to choose a large initial step size.
- The reinitialization of the Variable-ADMM did not considerably influence the total number of iterations.
- In order to obtain meaningful approximations one has to control both contributions to the residual, i.e., both Bw^j and λ^j have to be controlled which is accomplished by $R_j \leq \varepsilon_{stop}/C_0$.

APPENDIX A. CO-COERCIVITY OF ∇G

For completeness we include a short proof of the co-coercivity estimate needed in Theorem 3.14, see, e.g., [22].

Lemma A.1. *Assume that G is convex and differentiable such that ∇G is Lipschitz continuous with Lipschitz constant L_G . For all $v, v' \in X$ it holds*

$$L_G(v - v', \nabla G(v) - \nabla G(v'))_X \geq \|\nabla G(v) - \nabla G(v')\|_X^2.$$

Proof. Let $v, v' \in X$. We define

$$G_v(w) = G(w) - (\nabla G(v), w)_X, \quad \text{and} \quad g_v(w) = \frac{L_G}{2} \|w\|_X^2 - G_v(w).$$

The functional G_v is convex since it is the sum of two convex functionals. Furthermore, one can check that for all w, w' we have

$$(\nabla g_v(w) - \nabla g_v(w'), w - w')_X \geq 0,$$

i.e., g_v is convex. Thus, we have for all $w, w' \in X$

$$\begin{aligned} g_v(w') &\geq (\nabla g_v(w), w' - w)_X + g_v(w) \\ \iff G_v(w') &\leq G_v(w) + (\nabla G_v(w), w' - w)_X + \frac{L_G}{2} \|w - w'\|_X^2. \end{aligned}$$

Note that v is a minimizer of G_v since $\nabla G_v(v) = \nabla G(v) - \nabla G(v) = 0$. Hence, we have for all $w, w' \in X$

$$(7) \quad G_v(v) \leq G_v(w) + (\nabla G_v(w), w' - w)_X + \frac{L_G}{2} \|w - w'\|_X^2.$$

By minimizing the right-hand side of (7) with respect to w' for fixed w we obtain the critical point $w^* = w - \frac{1}{L_G} \nabla G_v(w)$. Choosing $w' = w^*$ in (7) we obtain

$$\frac{1}{2L_G} \|\nabla G_v(w)\|_X^2 \leq G_v(w) - G_v(v).$$

This yields

$$\begin{aligned} G(v') - G(v) - (\nabla G(v), v' - v)_X &= G_v(v') - G_v(v) \\ &\geq \frac{1}{2L_G} \|\nabla G_v(v')\|_X^2 = \frac{1}{2L_G} \|\nabla G(v') - \nabla G(v)\|_X^2. \end{aligned}$$

Exchanging roles of v and v' and adding the inequalities yields the assertion. \square

APPENDIX B. FAST-ADMM

In [20] an accelerated version of ADMM with fixed step sizes is proposed. The work is inspired by an acceleration technique presented in [34] which has also been used in the context of forward-backward splitting in [3] and in [19] for the special case $B = I$. The technique consists in a certain extrapolation of the variables. The authors can prove a $\mathcal{O}(1/J^2)$ convergence rate for the objective value of the dual problem if F and G are strongly convex, G is quadratic and if the step size is chosen properly. The residual also enjoys this convergence rate if, in addition, B has full row rank. However, for problems with F or G being only convex they have to impose a restart condition to guarantee stability and convergence of the method. We will refer to this method with included restart condition as the Fast-ADMM. The details of the algorithm are given in the following.

Algorithm B.1 (Fast-ADMM). *Choose $(u^0, \lambda^0) \in X \times Y$ such that $G(u^0) < \infty$. Choose $\bar{\tau} > 0$, $\gamma \in (0, 1)$, $\bar{R} \gg 0$ and $\varepsilon_{stop} > 0$. Set $\hat{u}^0 = u^0$, $\hat{\lambda}^0 = \lambda^0$, $\theta_0 = 1$ and $j = 1$.*

(1) *Set $R_0 = \bar{R}$.*

(2)-(6) *As in Algorithm 3.4.*

(7) *If $R_j < \gamma R_{j-1}$ set*

$$\begin{aligned} \theta_j &= \frac{1 + \sqrt{1 + 4\theta_{j-1}^2}}{2}, \\ \hat{u}^j &= u^j + \frac{\theta_{j-1} - 1}{\theta_j} (u^j - u^{j-1}), \\ \hat{\lambda}^j &= \lambda^j \frac{\theta_{j-1} - 1}{\theta_j} (\lambda^j - \lambda^{j-1}). \end{aligned}$$

Otherwise, set $\theta_j = 1$, $\hat{u}^j = u^{j-1}$, $\hat{\lambda}^j = \lambda^{j-1}$ and $R_j = \gamma^{-1} R_{j-1}$.

(8) *Set $j \rightarrow j + 1$ and continue with (2).*

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REFERENCES

- [1] S. Bartels, *Numerical Methods for Nonlinear Partial Differential Equations*, Springer, Heidelberg, 2015.
- [2] S. Bartels, R. H. Nochetto and A. J. Salgado, Discrete total variation flows without regularization, *SIAM J. Numer. Anal.*, **52** (2014), 363–385.
- [3] A. Beck and M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, *SIAM Journal of Imaging Sciences*, **2** (2009), 183–202.
- [4] S. Boyd and L. Vandenberghe, *Convex Optimization*, University Press, Cambridge, 2004.
- [5] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd edition, Texts in Applied Mathematics, vol. 15, Springer, New York, 2008.
- [6] H. Brézis and G. Stampacchia, Sur la régularité de la solution d'inéquations elliptiques, *Bulletin de la S. M. F.*, **96** (1968), 153–180.
- [7] A. Chambolle, An algorithm for total variation minimization and applications, *J. Math. Imaging Vis.*, **20** (2004), 89–97.
- [8] A. Chambolle and T. Pock, A first-order primal-dual algorithm for convex problems with applications to imaging, *J. Math. Imaging Vis.*, **40** (2011), 120–145.
- [9] Y.-H. Dai, D. Han, X. Yuan and W. Zhang, A sequential updating scheme of the Lagrange multiplier for separable convex programming, *Math. Comp.*, **86** (2016), 315–343.
- [10] W. Deng and W. Yin, On the global and linear convergence of the generalized alternating direction method of multipliers, *J. Sci. Comput.*, **66** (2016), 889–916.
- [11] J. Douglas and H. Rachford, On the numerical solution of heat conduction problems in two and three space variables, *Trans. Am. Math. Soc.*, **82** (1998), 421–439.
- [12] J. Eckstein, *Splitting methods for monotone operators with applications to parallel optimization*, Ph.D thesis, Massachusetts Institute of Technology, 1989.
- [13] M. Fortin and R. Glowinski, *Augmented Lagrangian Methods*, 1st edition, North-Holland Publishing Co., Amsterdam, 1983.
- [14] D. Gabay, Applications of the method of multipliers to variational inequalities, in *Augmented Lagrangian Methods: Applications to the Solution of Boundary-Value Problems* (eds. M. Fortin and R. Glowinski), (1983), 299–331.
- [15] D. Gabay and B. Mercier, A dual algorithm for the solution of nonlinear variational problems via finite element approximation, *Comp. & Maths. with Appls.*, **2** (1976), 17–40.
- [16] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer, New York, 1984.
- [17] R. Glowinski and P. Le Tallec, *Augmented Lagrangians and Operator-Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, 1989.
- [18] R. Glowinski and A. Marroco, Sur l'approximation par éléments finis d'ordre un, et la résolution, par pénalisation-dualité d'une classe de problèmes de Dirichlet non linéaires, *Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique*, **9** (1975), 41–76.
- [19] D. Goldfarb, S. Ma and K. Scheinberg, Fast alternating linearization methods for minimizing the sum of two convex functions, *Math. Program.*, **141** (2013), 349–382.
- [20] T. Goldstein, B. O'Donoghue, S. Setzer and R. Baraniuk, Fast alternating direction optimization methods, *SIAM J. Imaging Sci.*, **7** (2014), 1588–1623.
- [21] T. Goldstein and S. Osher, The split Bregman method for L1 regularized problems, *SIAM J. Imaging Sci.*, **2** (2009), 323–343.
- [22] J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer, Berlin, Heidelberg, 2004.

- [23] D. Han, X. Yuan and W. Zhang, An augmented Lagrangian based parallel splitting method for separable convex minimization with applications to image processing, *Math. Comp.*, **83** (2014), 2263–2291.
- [24] B. He, L.-Z. Liao, D. Han and H. Yang, A new inexact alternating directions method for monotone variational inequalities, *Math. Program.*, **92** (2002), 103–118.
- [25] B. He, L.-Z. Liao and S. Wang, Self-adaptive operator splitting methods for monotone variational inequalities, *Numer. Math.*, **94** (2003), 715–737.
- [26] B. S. He, H. Yang and S. L. Wang, Alternating direction method with self-adaptive penalty parameters for monotone variational inequalities, *Journal of Optimization and Applications*, **106** (2000), 337–356.
- [27] B. He and H. Yang, Some convergence properties of a method of multipliers for linearly constrained monotone variational inequalities, *Operations Research Letters*, **23** (1998), 151–161.
- [28] B. He and X. Yuan, On the $\mathcal{O}(1/n)$ convergence rate of the Douglas-Rachford alternating direction method, *SIAM J. Numer. Anal.*, **50** (2012), 700–709.
- [29] B. He and X. Yuan, On non-ergodic convergence rate of Douglas-Rachford alternating direction method of multipliers, *Numerische Mathematik*, **130** (2015), 567–577.
- [30] M. R. Hestenes, Multiplier and gradient methods, *Journal of Optimization Theory and Applications*, **4** (1969), 303–320.
- [31] M. Hintermüller, K. Ito and K. Kunisch, The primal-dual active set strategy as a semi-smooth Newton method, *SIAM J. Optim.*, **13** (2003), 865–888.
- [32] B. S. Kontogiorgis and R. R. Meyer, A variable-penalty alternating directions method for convex optimization, *Math. Program.*, **83** (1998), 29–53.
- [33] P. L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, **16** (1979), 964–979.
- [34] Y. Nesterov, A method for solving a convex programming problem with convergence rate $o(1/k^2)$, *Soviet Math. Dokl.*, **27** (1983), 372–376.
- [35] Y. Nesterov, Smooth minimization of non-smooth functions, *Math. Program.*, **103** (2005), 127–152.
- [36] M. J. D. Powell, A method for nonlinear constraints in minimization problems, in *Optimization*, Academic Press (ed. R. Fletcher), (1969), 283–298.
- [37] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, New Jersey, 1970.
- [38] R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control. Optim.*, **14** (1976), 877–898.
- [39] L. I. Rudin, S. Osher and E. Fatemi, Nonlinear total variation based noise removal algorithms, *Physica D*, **60** (1992), 259–268.
- [40] Y. Shen and M. Xu, On the $\mathcal{O}(1/t)$ convergence rate of Ye-Yuan’s modified alternating direction method of multipliers, *Applied Mathematics and Computation*, **226** (2014), 367–373.
- [41] C. Ye and X. Yuan, A descent method for structured monotone variational inequalities, *Optim. Methods Softw.*, **22** (2007), 329–338.

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