CONVERGENCE OF AN IMPLICIT FINITE ELEMENT METHOD FOR THE LANDAU-LIFSHITZ-GILBERT EQUATION

SÖREN BARTELS* AND ANDREAS PROHL

ABSTRACT. The Landau-Lifshitz-Gilbert equation describes dynamics of ferromagnetism, where strong nonlinearity, nonconvexity are hard to tackle: so far, existing schemes to approximate weak solutions suffer from severe time-step restrictions. In this paper, we propose an implicit fully discrete scheme and verify unconditional convergence.

1. INTRODUCTION

The phenomenological Landau-Lifschitz-Gilbert equation (LLG) describes dynamics of ferromagnetism; let $\alpha \geq 0$ denote the damping parameter, then the magnetization $\mathbf{m} : (0,T) \times \Omega \to S^2$, for $S^2 = \{\mathbf{x} \in \mathbb{R}^3 | \, |\mathbf{x}| = 1\}$, solves

$$\mathbf{m}_t = -\alpha\, \mathbf{m} \times (\mathbf{m} \times \Delta \mathbf{m}) + \mathbf{m} \times \Delta \mathbf{m}\,, \tag{1.1}$$

supplemented by initial and boundary conditions, $\mathbf{m}(0) = \mathbf{m}_0 \in W^{1,2}(\Omega; S^2)$, and $\partial_{\mathbf{n}} \mathbf{m} = 0$ on $(0,T) \times \partial\Omega$. A proper definition of weak solutions is given below. Limiting equations are the Heisenberg equation ($\alpha \to 0$), and heat flow for harmonic maps ($\alpha \to \infty$), see [1, Propositions 5.1, 5.2],

$$\mathbf{m}_t = \mathbf{m} \times \Delta \mathbf{m} \quad (\alpha \to 0)\,, \qquad \mathbf{m}_t = \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m} \quad (\alpha \to \infty)\,. \tag{1.2}$$

The construction of convergent schemes for (1.1) is a nontrivial task, due to the nonconvex side-constraint $|\mathbf{m}| = 1$ a.e. in $(0,T) \times \Omega$, which is difficult to realize in a numerical approximation scheme. A first explicit scheme is proposed in [2], where also (weak sub-) convergence towards weak solutions is verified; this program is continued in [3], where $k = o\big(\alpha^2 h^{1+\frac{n}{2}}\big)$ is identified to be sufficient for stability and convergence; sharpness of these restrictions is evidenced by computational studies in [3]. From this background, we look for an implicit scheme exempted from restricting requirements for numerical parameters, and higher flexibility with respect to (small) choices of $\alpha > 0$. The construction of our discretization is based on a reformation of (1.1) by Gilbert (see, e.g. [5]),

$$\mathbf{m}_t - \alpha\, \mathbf{m} \times \mathbf{m}_t = (1 + \alpha^2)\, \mathbf{m} \times \Delta \mathbf{m}\,.$$

Given the lowest order finite element space $\mathbf{V}_h \subset W^{1,2}(\Omega; \mathbb{R}^3)$ subordinate to a triangulation $\mathcal{T}_h$ of Ω and a time-step size $k > 0$, our approximation scheme reads as follows:

Algorithm 1.1. *Let* $\mathbf{m}_h^0 \in \mathbf{V}_h$. *Given* $j \geq 0$ *and* $\mathbf{m}_h^j \in \mathbf{V}_h$ *determine* $\mathbf{m}_h^{j+1} \in \mathbf{V}_h$ *from*

$$(d_t \mathbf{m}_h^{j+1}, \boldsymbol{\phi}_h)_h + \alpha\, (\mathbf{m}_h^j \times d_t \mathbf{m}_h^{j+1}, \boldsymbol{\phi}_h)_h = (1 + \alpha^2)(\overline{\mathbf{m}}_h^{j+1/2} \times \tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\phi}_h)_h \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h\,.$$

1991 *Mathematics Subject Classification.* 35K55, 65M12, 65M15, 68U10, 94A08.

Key words and phrases. ferromagnetism, Landau-Lifshitz-Gilbert equation, nonconvexity, finite elements, convergence.

*Supported by Deutsche Forschungsgemeinschaft through the DFG Research Center MATHEON 'Mathematics for key technologies' in Berlin

Here, $(\cdot,\cdot)_h$ denotes a discrete version (reduced integration) of the inner product in $L^2(\Omega;\mathbb{R}^3)$, $\tilde{\Delta}_h : W^{1,2}(\Omega;\mathbb{R}^3) \to \mathbf{V}_h$ is a discrete version of the Laplace operator, and we use $d_t\varphi^j := k^{-1}\big(\varphi^j - \varphi^{j-1}\big)$ for $j \geq 1$ and $\overline{\varphi}^{j+1/2} := \frac{1}{2}\big(\varphi^{j+1} + \varphi^j\big)$ for $j \geq 0$ and a sequence $\{\varphi^j\}_{j\geq 0}$; we refer the reader to Section 2 for details.

Remark 1.1. *The (linear) second term in Algorithm 1.1 is motivated by the identity*

$$\mathbf{m}_h^j \times d_t\mathbf{m}_h^{j+1} = \big(\overline{\mathbf{m}}_h^{j+1/2} - \frac{k}{2} d_t\mathbf{m}_h^{j+1}\big) \times d_t\mathbf{m}_h^{j+1} = \overline{\mathbf{m}}_h^{j+1/2} \times d_t\mathbf{m}_h^{j+1} .$$

It is well-known, that weak solutions to (1.1) solve

$$\mathbf{m}_t = \operatorname{div}\big(\mathbf{m} \times \nabla\mathbf{m}\big) + \alpha\big(\Delta\mathbf{m} + |\nabla\mathbf{m}|^2\mathbf{m}\big)$$

in distributional sense; cf. [1, 4]. Corresponding relations need not hold for discretizations, due to competition of local and nonlocal aspects inherent to fully discrete finite-element based methods. Lemma 6.1 below shows that solutions of Algorithm 1.1 satisfy

$$\begin{aligned}(d_t\mathbf{m}_h^{j+1}, \boldsymbol{\phi}_h)_h + \alpha\big(\nabla\overline{\mathbf{m}}_h^{j+1/2}, \nabla\boldsymbol{\phi}_h\big) &= \alpha\big(|\nabla\overline{\mathbf{m}}_h^{j+1/2}|^2\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\phi}_h\big) \\ &\quad - \big(\overline{\mathbf{m}}_h^{j+1/2} \times \nabla\overline{\mathbf{m}}_h^{j+1/2}, \nabla\boldsymbol{\phi}_h\big) + \mathrm{Corr}\,,\end{aligned} \tag{1.3}$$

for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and a correcting term 'Corr'.

Lemma 3.1 below states conservation of $|\mathbf{m}_h^j| = 1$ at the nodes of the triangulation $\mathcal{T}_h$ and verifies a discrete energy law for solutions to Algorithm 1.1. This indicates that the forcing correction term 'Corr' serves to balance the damping effect of the implicit Euler method with employed reduced integration and local averaging tools in Algorithm 1.1. The unconditional stability of Algorithm 1.1 allows to prove subconvergence to a weak solution of (1.1).

The remainder of this paper is organized as follows: Preliminaries are stated in Section 2. Our main result is Theorem 3.1 which verifies unconditional convergence for Algorithm 1.1; a simple fixed-point iteration is proposed in Algorithm 4.1, whose convergence is established for $k = \mathcal{O}(h^2)$, uniformly for values $\alpha \leq C$, in Section 4. We discuss numerical experiments with finite-time blow-up in Section 5, allowing for direct comparison with results for values $\alpha = \mathcal{O}(1)$ in [3], and study the limiting case $\alpha \to 0$. Section 6 proves (1.3) and illustrates difficulties in the construction of convergent implicit finite element schemes.

2. Preliminaries

Throughout this paper we assume that $\mathcal{T}_h$ is a quasiuniform regular triangulation of the polygonal or polyhedral bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ into triangles or tetrahedra for $n = 2$ or $n = 3$, respectively. We define the lowest order finite element space $\mathbf{V}_h \subset W^{1,2}(\Omega;\mathbb{R}^3)$ by

$$\mathbf{V}_h = \big\{\boldsymbol{\phi}_h \in C(\overline{\Omega};\mathbb{R}^3) : \boldsymbol{\phi}_h|_K \in \mathcal{P}_1(K;\mathbb{R}^3) \quad \forall\, K \in \mathcal{T}_h\big\}\,,$$

where $\mathcal{P}_1(K;\mathbb{R}^3)$ denotes the set of polynomials of total degree less or equal than one restricted to the element $K \in \mathcal{T}_h$. Given the set of nodes $\big\{\mathbf{x}_\ell : \ell \in L\big\}$ of the triangulation $\mathcal{T}_h$, the nodal interpolation operator $\mathcal{I}_h : C(\overline{\Omega};\mathbb{R}^3) \to \mathbf{V}_h$ satisfies $\mathcal{I}_h\boldsymbol{\phi}(\mathbf{x}_\ell) = \boldsymbol{\phi}(\mathbf{x}_\ell)$ for all $\ell \in L$. Given functions $\mathbf{f}, \mathbf{g} \in L^2(\Omega;\mathbb{R}^m)$ and letting $\langle\cdot,\cdot\rangle$ denote the inner product in $\mathbb{R}^m$ we set

$$\big(\mathbf{f}, \mathbf{g}\big) = \int_\Omega \langle\mathbf{f}, \mathbf{g}\rangle \,\mathrm{d}\mathbf{x}\,.$$

For continuous functions $\boldsymbol{\phi}$, $\mathfrak{z} \in C(\overline{\Omega};\mathbb{R}^3)$ we define

$$\big(\boldsymbol{\phi}, \mathfrak{z}\big)_h = \int_\Omega \mathcal{I}_h\big(\langle\boldsymbol{\phi}, \mathfrak{z}\rangle\big)\,\mathrm{d}\mathbf{x} = \sum_{\ell\in L} \beta_\ell\langle\boldsymbol{\phi}(\mathbf{x}_\ell), \mathfrak{z}(\mathbf{x}_\ell)\rangle\,,$$

for certain weights $\beta_\ell > 0$, $\ell \in L$. If for each $\ell \in L$ we denote by $\varphi_\ell \in C(\overline{\Omega})$ the nodal basis function which is $\mathcal{T}_h$-elementwise affine and satisfies $\varphi_\ell(\mathbf{x}_\ell) = 1$ and $\varphi_\ell(\mathbf{x}_m) = 0$ for all $m \in L \setminus \{\ell\}$ then we have $\beta_\ell = \int_\Omega \varphi_\ell \,\mathrm{d}\mathbf{x}$. We define $||\boldsymbol{\phi}||_h^2 = \big(\boldsymbol{\phi}, \boldsymbol{\phi}\big)_h$ and notice that

$$\| \boldsymbol{\phi}_h \|_{L^2}^2 \leq \| \boldsymbol{\phi}_h \|_h^2 \leq (n+2) \| \boldsymbol{\phi}_h \|_{L^2}^2 ,$$

for all $\boldsymbol{\phi}_h \in V_h$. We define a discrete Laplace operator $\tilde{\Delta}_h : W^{1,2}(\Omega;\mathbb{R}^3) \to \mathbf{V}_h$ by

$$-(\tilde{\Delta}_h \boldsymbol{\phi}, \boldsymbol{\chi}_h)_h = (\nabla \boldsymbol{\phi}, \nabla \boldsymbol{\chi}_h) \quad \forall \boldsymbol{\chi}_h \in \mathbf{V}_h \,. \tag{2.1}$$

It is well known that there exists a constant $c_1 > 0$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ there holds

$$||\nabla \boldsymbol{\phi}_h||_{L^2} \leq c_1 h^{-1} ||\boldsymbol{\phi}_h||_{L^2}, \tag{2.2}$$

where h is the maximal mesh-size in $\mathcal{T}_h$, i.e., $h = \max\{\mathrm{diam}(K) : K \in \mathcal{T}_h\}$. Choosing $\boldsymbol{\chi}_h = \tilde{\Delta}_h \boldsymbol{\phi}_h$ in (2.1) and using (2.2) we observe that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ there holds

$$||\tilde{\Delta}_h \boldsymbol{\phi}_h||_h^2 = -\big(\nabla \boldsymbol{\phi}_h, \nabla \tilde{\Delta}_h \boldsymbol{\phi}_h\big) \leq ||\nabla \boldsymbol{\phi}_h||_{L^2} ||\nabla \tilde{\Delta}_h \boldsymbol{\phi}_h||_{L^2} \leq c_1 h^{-1} ||\nabla \boldsymbol{\phi}_h||_{L^2} ||\tilde{\Delta}_h \boldsymbol{\phi}_h||_h. \tag{2.3}$$

Given $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and a node $\mathbf{x}_\ell$ for some $\ell \in L$ we obtain from using $\boldsymbol{\chi}_h = \varphi_\ell \tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_\ell)$ in (2.1) that

$$\begin{aligned} |\tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_\ell)|^2 &= \beta_\ell^{-1} \big(\tilde{\Delta}_h \boldsymbol{\phi}_h, \boldsymbol{\chi}_h\big)_h = -\beta_\ell^{-1} \big(\nabla \boldsymbol{\phi}_h, \nabla \boldsymbol{\chi}_h\big) \\ &= -\beta_\ell^{-1} \sum_{\substack{m \in L : \exists K \in \mathcal{T}, \\ \mathbf{x}_m, \mathbf{x}_\ell \in K}} \langle \boldsymbol{\phi}_h(\mathbf{x}_m), \tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_\ell) \rangle (\nabla \varphi_m, \nabla \varphi_\ell) \leq c_2 h^{-2} ||\boldsymbol{\phi}_h||_{L^\infty} |\tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_\ell)|, \end{aligned} \tag{2.4}$$

where we used (2.2), that given a node $\mathbf{x}_\ell$ the cardinality of the set $\{m \in L : \exists K,\, \mathbf{x}_m, \mathbf{x}_\ell \in K\}$ is bounded h-independently, and that $||\varphi_m||_{L^2} \leq c \beta_\ell^{1/2}$ for all $m \in L$.

3. Unconditional Convergence

We first recall the definition of a weak solution to (LLG). Throughout this section we abbreviate $\Omega_T = (0,T) \times \Omega$.

Definition 3.1. *Let $\mathbf{m}_0 \in W^{1,2}(\Omega; S^2)$, then $\mathbf{m}$ is called weak solution to (LLG), if for all $T > 0$*

1. $\mathbf{m} \in W^{1,2}(\Omega_T;\mathbb{R}^3)$ *such that* $|\,\mathbf{m}\,| = 1$ *almost everywhere in* Ω_T*;*
2. *for all* $\boldsymbol{\phi} \in C^\infty\big(\Omega_T;\mathbb{R}^3\big)$ *holds*
$$\int_{\Omega_T} \langle \mathbf{m}_t, \boldsymbol{\phi} \rangle \,\mathrm{d}\mathbf{x}\mathrm{d}t + \alpha \int_\Omega \langle \mathbf{m} \times \mathbf{m}_t, \boldsymbol{\phi} \rangle \,\mathrm{d}\mathbf{x}\mathrm{d}t = -(1+\alpha^2) \int_{\Omega_T} \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla \boldsymbol{\phi} \rangle \,\mathrm{d}\mathbf{x}\mathrm{d}t\,;$$
3. $\mathbf{m}(0) = \mathbf{m}_0$ *in the sense of traces;*
4. *for almost all* $T' \in (0,T)$ *there holds*
$$\frac{1}{2} \int_\Omega |\,\nabla \mathbf{m}(T')\,|^2 \,\mathrm{d}\mathbf{x} + \frac{\alpha}{1+\alpha^2} \int_{(0,T')\times\Omega} |\,\mathbf{m}_t\,|^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \leq \frac{1}{2} \int_\Omega |\,\nabla \mathbf{m}_0\,|^2 \,\mathrm{d}\mathbf{x}\,.$$

The following lemma provides discrete counterparts of (1) and (4). We remark that well-posedness of Algorithm 1.1, i.e., the existence of a unique sequence $\{\mathbf{m}_h^j\}_{j\geq 0}$ that solves Algorithm 1.1, can be deduced from a classical argument, see e.g. [1, Section 3].

Lemma 3.1. *Suppose that $|\mathbf{m}_h^0(\mathbf{x}_\ell)| = 1$ for all $\ell \in L$. Then the sequence $\{\mathbf{m}_h^j\}_{j\geq 0}$ produced by Algorithm 1.1 satisfies for all $j \geq 0$*

$$\begin{aligned} &\text{(i)} \qquad |\,\mathbf{m}_h^{j+1}(\mathbf{x}_\ell)\,| = 1 \quad \forall \ell \in L\,, \\ &\text{(ii)} \qquad \frac{1}{2} d_t \|\,\nabla \mathbf{m}_h^{j+1}\,\|_{L^2}^2 + \frac{\alpha}{1+\alpha^2} \|\,d_t \mathbf{m}_h^{j+1}\,\|_h^2 = 0\,. \end{aligned}$$

Proof. Verification of (i) follows from choosing $\boldsymbol{\phi}_h = \varphi_\ell \overline{\mathbf{m}}_h^{j+1/2}(\mathbf{x}_\ell) \in \mathbf{V}_h$ for $\ell \in L$ in Algorithm 1.1. In order to verify (ii), we first choose $\boldsymbol{\phi}_h = -\tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2}$ and find

$$\frac{1}{2} d_t \| \nabla \mathbf{m}_h^{j+1} \|_{L^2}^2 + \alpha \big(\overline{\mathbf{m}}_h^{j+1/2} \times d_t \mathbf{m}_h^{j+1}, \tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2} \big)_h = 0 \, .$$

Choosing $\boldsymbol{\phi}_h = d_t \mathbf{m}_h^{j+1}$ yields

$$\frac{\alpha}{1+\alpha^2} \| d_t \mathbf{m}_h^{j+1} \|_h^2 = \alpha \big(\overline{\mathbf{m}}_h^{j+1/2} \times \tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2}, d_t \mathbf{m}_h^{j+1} \big)_h .$$

A combination of the two identities proves (ii) and finishes the proof of the lemma. □

Definition 3.2. *For* $\mathbf{x} \in \Omega$ *and* $t \in [t_j, t_{j+1})$ *define*

$$\mathbf{M}(t,\mathbf{x}) := \frac{t-t_j}{k} \mathbf{m}_h^{j+1}(\mathbf{x}) + \frac{t_{j+1}-t}{k} \mathbf{m}_h^j(\mathbf{x}) \, ,$$
$$\mathbf{M}^-(t,\mathbf{x}) := \mathbf{m}_h^j(\mathbf{x}) \, , \quad \mathbf{M}^+(t,\mathbf{x}) := \mathbf{m}_h^{j+1}(\mathbf{x}) \, , \quad \overline{\mathbf{M}}(t,\mathbf{x}) := \overline{\mathbf{m}}_h^{j+1/2} \, .$$

Given any $T' > 0$ equation (ii) in Lemma 3.1 may be rewritten as

$$\frac{1}{2} ||\nabla \mathbf{M}^+(T')||_{L^2}^2 + \frac{\alpha}{1+\alpha^2} \int_0^{T'} ||\mathbf{M}_t||_h^2 \, \mathrm{d}t \le \frac{1}{2} ||\nabla \mathbf{M}(0)||_{L^2}^2 .$$

This bound yields the existence of some $\mathbf{m} \in W^{1,2}(\Omega;\mathbb{R}^3)$ which is the weak limit (as $k, h \to 0$) of a subsequence such that

$$\mathbf{M} \rightharpoonup \mathbf{m} \;\text{ in } W^{1,2}(\Omega_T) \, , \quad \nabla \mathbf{M}^-, \nabla \mathbf{M}^+, \nabla \overline{\mathbf{M}} \rightharpoonup \nabla \mathbf{m} \;\text{ in } L^2(\Omega) \, , \quad \mathbf{M}^-, \mathbf{M}^+, \overline{\mathbf{M}} \to \mathbf{m} \;\text{ in } L^2(\Omega_T) \, .$$

Since $|\mathbf{M}^-(t,\mathbf{x}_\ell)| = 1$ for every $\ell \in L$ and almost all $t \in (0,T)$, there holds for every $K \in \mathcal{T}_h$,

$$\| \, |\mathbf{M}^-|^2 - 1 \, \|_{L^2(K)} \le Ch \, \| \nabla \big(|\mathbf{M}^-|^2 - 1 \big) \|_{L^2(K)} = Ch \, \| 2 (\nabla \mathbf{M}^-) \mathbf{M}^- \|_{L^2(K)} \le 2Ch \, \| \nabla \mathbf{M}^- \|_{L^2(K)} \, ,$$

which implies $|\mathbf{M}^-| \to 1$ in $L^2(\Omega_T;\mathbb{R}^3)$, and hence $|\mathbf{m}| = 1$ almost everywhere in Ω_T. Algorithm 1.1 may be written as follows: taking $\boldsymbol{\phi}_h(t) := \mathcal{I}_h \boldsymbol{\phi}(t,\cdot)$, for $\boldsymbol{\phi} \in C^\infty(\Omega_T;\mathbb{R}^3)$, there holds

$$\int_0^T (\mathbf{M}_t, \boldsymbol{\phi}_h)_h \, \mathrm{d}t + \alpha \int_0^T (\mathbf{M}^- \times \mathbf{M}_t, \boldsymbol{\phi}_h)_h \, \mathrm{d}t = (1+\alpha^2) \int_0^T \big(\overline{\mathbf{M}} \times \tilde{\Delta}_h \overline{\mathbf{M}}, \boldsymbol{\phi}_h \big)_h \, \mathrm{d}t \, . \tag{3.1}$$

Effects of reduced integration are controlled using the fact that for all $\boldsymbol{\chi}_h, \boldsymbol{\eta}_h \in \mathbf{V}_h$ there holds

$$| (\boldsymbol{\chi}_h, \boldsymbol{\eta}_h)_h - (\boldsymbol{\chi}_h, \boldsymbol{\eta}_h) | \le Ch \, \| \boldsymbol{\chi}_h \|_{L^2} \| \nabla \boldsymbol{\eta}_h \|_{L^2} .$$

This implies that for almost all $t \in (0,T)$ we have

$$\big| (\mathbf{M}_t, \boldsymbol{\phi}_h)_h - (\mathbf{M}_t, \boldsymbol{\phi}_h) \big| \le Ch \, \| \mathbf{M}_t \|_{L^2} \| \nabla \boldsymbol{\phi}_h \|_{L^2}$$

and allows to prove that

$$\int_0^T (\mathbf{M}_t, \boldsymbol{\phi}_h)_h \, \mathrm{d}t \to \int_0^T (\mathbf{m}_t, \boldsymbol{\phi}) \, \mathrm{d}t.$$

Using that for $\boldsymbol{\chi}_h \in \mathbf{V}_h$ and $\boldsymbol{\eta} \in C(\overline{\Omega};\mathbb{R}^3)$ there holds $(\boldsymbol{\chi}_h, \boldsymbol{\eta})_h = (\boldsymbol{\chi}_h, \mathcal{I}_h \boldsymbol{\eta})_h$ and employing a triangle inequality and standard estimates for nodal interpolation results in

$$\big| (\mathbf{M}_t, \mathbf{M}^- \times \boldsymbol{\phi}_h)_h - \big(\mathbf{M}_t, (\mathrm{Id} \pm \mathcal{I}_h)(\mathbf{M}^- \times \boldsymbol{\phi}_h) \big) \big| \le Ch \, \| \mathbf{M}_t \|_{L^2} \| \nabla (\mathbf{M}^- \times \boldsymbol{\phi}_h) \|_{L^2} \, .$$

This yields that

$$\int_0^T (\overline{\mathbf{M}} \times \mathbf{M}_t, \boldsymbol{\phi}_h)_h \mathrm{d}t \to \int_0^T (\mathbf{m} \times \mathbf{m}_t, \boldsymbol{\phi}) \mathrm{d}t \, .$$

The only troublesome limit is for the last term in (3.1). We write

$$\big(\overline{\mathbf{M}}\times\tilde\Delta_h\overline{\mathbf{M}},\boldsymbol{\phi}_h\big)_h = (\overline{\mathbf{M}}\times\boldsymbol{\phi}_h,\tilde\Delta_h\overline{\mathbf{M}})_h = \big((\mathrm{Id}-\mathcal{I}_h)(\overline{\mathbf{M}}\times\boldsymbol{\phi}_h),\tilde\Delta_h\overline{\mathbf{M}}\big)_h$$
$$+\big(\nabla(\mathcal{I}_h-\mathrm{Id})(\overline{\mathbf{M}}\times\boldsymbol{\phi}_h),\nabla\overline{\mathbf{M}}\big)+\big(\nabla(\overline{\mathbf{M}}\times\boldsymbol{\phi}_h),\nabla\overline{\mathbf{M}}\big) =: I+II+III\,.$$

Control of I uses the bound $\|\,\tilde\Delta_h\boldsymbol{\chi}\,\|_{L^2}\le c_1h^{-1}\,\|\,\nabla\boldsymbol{\chi}\,\|_{L^2}$ and estimates for nodal intepolation,

$$I\le Ch^2h^{-1}\sum_{K\in\mathcal{T}_h}\|\,D^2(\overline{\mathbf{M}}\times\boldsymbol{\phi}_h)\,\|_{L^2(K)}\,\|\,\nabla\overline{\mathbf{M}}\,\|_{L^2(K)}\le Ch\|\,\nabla\overline{\mathbf{M}}\,\|_{L^2}\|\,\nabla\boldsymbol{\phi}_h\,\|_{L^\infty}\|\,\nabla\overline{\mathbf{M}}\,\|_{L^2}\,.$$

A similar argumentation proves

$$II\le Ch\sum_{K\in\mathcal{T}_h}\|\,D^2\big(\overline{\mathbf{M}}\times\boldsymbol{\phi}_h\big)\,\|_{L^2(K)}\|\,\nabla\overline{\mathbf{M}}\,\|_{L^2(K)}\le Ch\|\,\nabla\overline{\mathbf{M}}\,\|_{L^2}\|\,\nabla\boldsymbol{\phi}_h\,\|_{L^\infty}\|\,\nabla\overline{\mathbf{M}}\,\|_{L^2}.$$

We use that given any $\boldsymbol{\mathfrak{z}},\boldsymbol{\chi}\in W^{1,2}(\Omega;\mathbb{R}^3)$ there holds $\langle\nabla\boldsymbol{\mathfrak{z}},\nabla(\boldsymbol{\mathfrak{z}}\times\boldsymbol{\chi})\rangle=\langle\nabla\boldsymbol{\mathfrak{z}},\boldsymbol{\mathfrak{z}}\times\nabla\boldsymbol{\chi}\rangle$ to verify

$$III=\big(\nabla(\overline{\mathbf{M}}\times\boldsymbol{\phi}_h),\nabla\overline{\mathbf{M}}\big)=\big(\overline{\mathbf{M}}\times\nabla\boldsymbol{\phi}_h,\nabla\overline{\mathbf{M}}\big).$$

A combination of the last four assertions shows

$$\int_0^T\big(\overline{\mathbf{M}}\times\tilde\Delta_h\overline{\mathbf{M}},\boldsymbol{\phi}_h\big)_h\,\mathrm{d}t\to\int_0^T(\mathbf{m}\times\nabla\boldsymbol{\phi},\nabla\mathbf{m})\,\mathrm{d}t=\int_0^T\big(\nabla(\mathbf{m}\times\boldsymbol{\phi}),\nabla\mathbf{m}\big)\,\mathrm{d}t\,.$$

This proves our main theorem.

Theorem 3.1. *Suppose $|\mathbf{m}_h^0(\mathbf{x}_\ell)|=1$ for all $\ell\in L$ and let $\{\mathbf{m}_h^j\}_{j\ge0}$ solve Algorithm 1.1. Assume that $\mathbf{m}_h^0\to\mathbf{m}_0$ in $W^{1,2}(\Omega)$ for $h\to0$. For $k,h\to0$ there exists $\mathbf{m}\in W^{1,2}(\Omega_T;\mathbb{R}^3)$ such that $\mathbf{M}$ subconverges to $\mathbf{m}$ in $W^{1,2}(\Omega_T)$ and $\mathbf{m}$ is a weak solution of (LLG).*

4. Solving the nonlinear system

In the numerical experiments reported below we employ the following fixed-point iteration to solve the nonlinear system in Algorithm 1.1:

Algorithm 4.1. *Set $\mathbf{m}_h^{j+1,0}:=\mathbf{m}_h^j$ and $\ell:=0$.*
(i) Compute $\mathbf{m}_h^{j+1,\ell+1}\in\mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h\in\mathbf{V}_h$ there holds

$$\begin{aligned}
&\frac{1}{k}(\mathbf{m}_h^{j+1,\ell+1},\boldsymbol{\phi}_h)_h+\frac{\alpha}{k}(\mathbf{m}_h^j\times\mathbf{m}_h^{j+1,\ell+1},\boldsymbol{\phi}_h)_h-\frac{1+\alpha^2}{4}(\mathbf{m}_h^{j+1,\ell+1}\times\tilde\Delta_h\mathbf{m}_h^{j+1,\ell},\boldsymbol{\phi}_h)_h\\
&\quad-\frac{1+\alpha^2}{4}(\mathbf{m}_h^{j+1,\ell+1}\times\tilde\Delta_h\mathbf{m}_h^j,\boldsymbol{\phi}_h)_h-\frac{1+\alpha^2}{4}(\mathbf{m}_h^j\times\tilde\Delta_h\mathbf{m}_h^{j+1,\ell+1},\boldsymbol{\phi}_h)_h\\
&\quad=\frac{1}{k}(\mathbf{m}_h^j,\boldsymbol{\phi}_h)_h+\frac{1+\alpha^2}{4}(\mathbf{m}_h^j\times\tilde\Delta_h\mathbf{m}_h^j,\boldsymbol{\phi}_h)_h
\end{aligned}\tag{4.1}$$

(ii) If $||\mathbf{m}_h^{j+1,\ell+1}-\mathbf{m}_h^{j+1,\ell}||_h\le\varepsilon$ then stop and set $\mathbf{m}_h^{j+1}:=\mathbf{m}_h^{j+1,\ell+1}$.
(iii) Set $\ell:=\ell+1$ and go to (i).

The following lemma shows that the iteration converges provided that $k\le ch^2/(1+\alpha^2)$, for an (h,k,α)-independent constant factor $c>0$ that only depends on the geometry of $\mathcal{T}_h$.

Lemma 4.1. *Suppose that $\gamma:=\sqrt{5}(1+\alpha^2)c_1^2h^{-2}k/4<1$ and $|\mathbf{m}_h^j(\mathbf{x}_m)|=1$ for all $m\in L$. Then, for all $\ell\ge0$ there exists a unique solution $\mathbf{m}_h^{j+1,\ell+1}$ to (4.1). For all $\ell\ge1$ there holds*

$$||\mathbf{m}_h^{j+1,\ell+1}-\mathbf{m}_h^{j+1,\ell}||_h\le\Theta\frac{\gamma}{1-\gamma}||\mathbf{m}_h^{j+1,\ell}-\mathbf{m}_h^{j+1,\ell-1}||_h,$$

provided that $\Theta := \frac{1+\rho}{1-\rho} > 0$ *for* $\rho := (1+\alpha^2)c_2 k h^{-2}/4$. *Moreover, for all* $\ell \geq 0$ *and all* $\boldsymbol{\phi}_h \in \mathbf{V}_h$ *there holds*

$$\begin{aligned}\big|(d_t\mathbf{m}_h^{j+1,\ell+1},\boldsymbol{\phi}_h)_h + \alpha(\mathbf{m}_h^j \times d_t\mathbf{m}_h^{j+1,\ell+1},\boldsymbol{\phi}_h)_h - (1+\alpha^2)(\overline{\mathbf{m}}_h^{j+1/2,\ell+1} \times \tilde{\Delta}_h\overline{\mathbf{m}}_h^{j+1/2,\ell+1},\boldsymbol{\phi}_h)_h\big| \\ \leq \Theta\sqrt{5}\frac{1+\alpha^2}{4}c_1^2 h^{-2}||\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell}||_h||\boldsymbol{\phi}_h||_h,\end{aligned}$$

where $d_t\mathbf{m}_h^{j+1,\ell+1} = k^{-1}(\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^j)$ *and* $\overline{\mathbf{m}}_h^{j+1,\ell+1/2} = \frac{1}{2}(\mathbf{m}_h^{j+1,\ell+1} + \mathbf{m}_h^j)$.

Proof. We abbreviate $\mu = (1+\alpha^2)/4$. For $\boldsymbol{\phi}_h = \mathbf{m}_h^{j+1,\ell+1}$ the left-hand side of (4.1) is bounded from below by

$$\begin{aligned}&\frac{1}{k}||\mathbf{m}_h^{j+1,\ell+1}||_h^2 - \mu(\mathbf{m}_h^j \times \tilde{\Delta}_h\mathbf{m}_h^{j+1,\ell+1},\mathbf{m}_h^{j+1,\ell+1})_h \\ &\quad\geq \frac{1}{k}||\mathbf{m}_h^{j+1,\ell+1}||_h^2 - \mu||\mathbf{m}_h^j||_{L^\infty}||\tilde{\Delta}_h\mathbf{m}_h^{j+1,\ell+1}||_h||\mathbf{m}_h^{j+1,\ell+1}||_h \geq \big(\frac{1}{k} - \mu\sqrt{5}c_1^2h^{-2}\big)||\mathbf{m}_h^{j+1,\ell+1}||_h^2,\end{aligned}$$

where we used $||\mathbf{m}_h^j||_{L^\infty} = 1$ and $||\tilde{\Delta}_h\mathbf{m}_h^{j+1,\ell+1}||_h \leq c_1^2\sqrt{5}h^{-2}||\mathbf{m}_h^{j+1,\ell+1}||_h$. Therefore, the bilinear form defined by the left-hand side of (4.1) is positive definite on $\mathbf{V}_h \times \mathbf{V}_h$ if $\gamma < 1$ and then (4.1) admits a unique solution. Let $m \in L$ be such that $||\mathbf{m}_h^{j+1,\ell}||_{L^\infty} = |\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)|$. Choosing $\boldsymbol{\phi}_h = \phi_m\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)$ in the equation for $\mathbf{m}_h^{j+1,\ell}$ proves

$$\begin{aligned}\frac{1}{k}|\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)|^2 &\leq \mu|\mathbf{m}_h^j(\mathbf{x}_m)|\,|\tilde{\Delta}_h\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)|\,|\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)| \\ &\quad + \frac{1}{k}|\mathbf{m}_h^j(\mathbf{x}_m)|\,|\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)| + \mu|\mathbf{m}_h^j(\mathbf{x}_m)|\,|\tilde{\Delta}_h\mathbf{m}_h^j(\mathbf{x}_m)|\,|\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)| \\ &= \mu|\tilde{\Delta}_h\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)|\,|\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)| + \frac{1}{k}|\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)| + \mu|\tilde{\Delta}_h\mathbf{m}_h^j(\mathbf{x}_m)|\,|\mathbf{m}_h^{j+1,\ell}(\mathbf{x}_m)|\end{aligned}$$

There holds $|\tilde{\Delta}_h\boldsymbol{\phi}_h(\mathbf{x}_m)| \leq c_2h^{-2}||\boldsymbol{\phi}_h||_{L^\infty}$ for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and hence

$$||\mathbf{m}_h^{j+1,\ell}||_{L^\infty} \leq \frac{1+k\mu c_2h^{-2}}{1-k\mu c_2h^{-2}} = \Theta.$$

Subtraction of two subsequent equations in the fixed-point iteration yields

$$\begin{aligned}&\frac{1}{k}(\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell},\boldsymbol{\phi}_h)_h + \frac{\alpha}{k}(\mathbf{m}_h^j \times [\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell}],\boldsymbol{\phi}_h)_h \\ &\quad - \mu([\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell}] \times \tilde{\Delta}_h\mathbf{m}_h^{j+1,\ell},\boldsymbol{\phi}_h)_h - \mu(\mathbf{m}_h^{j+1,\ell} \times \tilde{\Delta}_h[\mathbf{m}_h^{j+1,\ell} - \mathbf{m}_h^{j+1,\ell-1}],\boldsymbol{\phi}_h)_h \\ &\qquad - \mu([\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell}] \times \tilde{\Delta}_h\mathbf{m}_h^j,\boldsymbol{\phi}_h)_h - \mu(\mathbf{m}_h^j \times \tilde{\Delta}_h[\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell}],\boldsymbol{\phi}_h)_h = 0\end{aligned}$$

for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$. Choosing $\boldsymbol{\phi}_h := \mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell}$ shows

$$||\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell}||_h \leq k\mu\Theta||\tilde{\Delta}_h[\mathbf{m}_h^{j+1,\ell} - \mathbf{m}_h^{j+1,\ell-1}]||_h + k\mu||\tilde{\Delta}_h[\mathbf{m}_h^{j+1,\ell+1} - \mathbf{m}_h^{j+1,\ell}]||_h.$$

Using $||\tilde{\Delta}_h\boldsymbol{\phi}_h||_h \leq c_1^2\sqrt{5}h^{-2}||\boldsymbol{\phi}_h||_h$ for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ we deduce the first estimate of the lemma. In order to verify the second estimate we notice that owing to (4.1), $\mathbf{m}_h^j \times \mathbf{m}_h^j = 0$, and the above estimate $||\mathbf{m}_h^{j+1,\ell+1}||_{L^\infty} \leq \Theta$ there holds for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$

$$
\begin{aligned}
(d_t\mathbf{m}_h^{j+1,\ell+1}, \boldsymbol{\phi}_h)_h &+ \alpha(\mathbf{m}_h^j \times d_t\mathbf{m}_h^{j+1,\ell+1}, \boldsymbol{\phi}_h)_h - \mu(\overline{\mathbf{m}}_h^{j+1/2,\ell+1} \times \tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2,\ell+1}, \boldsymbol{\phi}_h)_h \\
=&\frac{1}{k}(\mathbf{m}_h^{j+1,\ell+1}, \boldsymbol{\phi}_h)_h - \frac{1}{k}(\mathbf{m}_h^j, \boldsymbol{\phi}_h)_h + \frac{\alpha}{k}(\mathbf{m}_h^j \times \mathbf{m}_h^{j+1,\ell+1}, \boldsymbol{\phi}_h)_h - \frac{\alpha}{k}(\mathbf{m}_h^j \times \mathbf{m}_h^j, \boldsymbol{\phi}_h)_h \\
&- \mu(\mathbf{m}_h^{j+1,\ell+1} \times \tilde{\Delta}_h \mathbf{m}_h^{j+1,\ell+1}, \boldsymbol{\phi}_h)_h - \mu(\mathbf{m}_h^{j+1,\ell+1} \times \tilde{\Delta}_h \mathbf{m}_h^j, \boldsymbol{\phi}_h)_h \\
&- \mu(\mathbf{m}_h^j \times \tilde{\Delta}_h \mathbf{m}_h^{j+1,\ell+1}, \boldsymbol{\phi}_h)_h - \mu(\mathbf{m}_h^j \times \tilde{\Delta}_h \mathbf{m}_h^j, \boldsymbol{\phi}_h)_h \\
=&\mu(\mathbf{m}_h^{j+1,\ell+1} \times \tilde{\Delta}_h \mathbf{m}_h^{j+1,\ell}, \boldsymbol{\phi}_h)_h - \mu(\mathbf{m}_h^{j+1,\ell+1} \times \tilde{\Delta}_h \mathbf{m}_h^{j+1,\ell+1}, \boldsymbol{\phi}_h)_h \\
\leq&\mu||\mathbf{m}_h^{j+1,\ell+1}||_{L^\infty}||\tilde{\Delta}_h(\mathbf{m}_h^{j+1,\ell} - \mathbf{m}_h^{j+1,\ell+1})||_h||\boldsymbol{\phi}_h||_h \\
\leq&\Theta\mu c_1^2\sqrt{5}h^{-2}||\mathbf{m}_h^{j+1,\ell} - \mathbf{m}_h^{j+1,\ell+1}||_h||\boldsymbol{\phi}_h||_h
\end{aligned}
$$

which finishes the proof of the lemma. □

5. Numerical Experiments

The implementation of Algorithms 1.1 and 4.1 was performed in MATLAB with an assemblation of the stiffness matrices in C. We set $\varepsilon = h^4$ for the termination criterion in Algorithm 4.1, and it terminated after at most 5 iterations in all of our experiments. The experiments are defined through the following example which is taken from [3].

Example 5.1. *Let $\Omega = (-1/2, 1/2)^2$ and let $\mathbf{m}_0 : \Omega \to S^2$ be defined by*

$$
\mathbf{m}_0(\mathbf{x}) = \begin{cases} (0, 0, -1) & \text{for } |\mathbf{x}| \geq 1/2, \\ (2\mathbf{x}A, A^2 - |\mathbf{x}|^2)/(A^2 + |\mathbf{x}|^2) & \text{for } |\mathbf{x}| \leq 1/2, \end{cases}
$$

where $A := (1 - 2|\mathbf{x}|)^4/s$ for some $s > 0$. The triangulations $\mathcal{T}_\ell$ used in the numerical simulations are defined through a positive integer ℓ and consist of $2^{2\ell+1}$ halved squares with edge length $h = 2^{-\ell}$. Motivated by Lemma 4.1 we use $k = h^2/(10(1+\alpha^2))$ As discrete initial data we employ the nodal interpolant of $\mathbf{m}_0$, i.e., we set $\mathbf{m}^0 = \mathcal{I}_{\mathcal{T}_\ell}\mathbf{m}_0$ in all experiments.

Figures 1 and 2 display snapshots of the numerical approximation provided by Algorithm 1.1 with $\alpha = 1$, $s = 1$, and $\ell = 4$. The plots in Figure 1 display the first two components of the vector field $\mathbf{M}$ at the nodes of the triangulation (after an appropriate rescaling) and at various times. Figure 2 shows a zoom towards the origin and reveals that in this experiment regularity of the exact solution cannot be expected. At time $t \approx 0.0529$ the vector at the origin points in another direction than all surrounding vectors resulting in a large (maximal) $W^{1,\infty}$ norm.

Figures 3 and 4 show similar snapshots for $\alpha = 1/64$, $s = 1$, and $\ell = 4$. Owing to the significantly smaller stabilization corresponding to the small value of α, the numerical solution is even less regular than in the previous experiment and fails to become stationary for times $t \leq 1/2$.

For fixed $\alpha = 1$ and $s = 4$ we used $\ell = 4, 5, 6$ in Example 5.1. In Figure 5 we displayed the energy

$$
E(\mathbf{M}(t)) = \frac{1}{2}\int_\Omega |\nabla\mathbf{M}(t, \cdot)|^2 \, \mathrm{d}\mathbf{x}
$$

and the $W^{1,\infty}$ semi-norm $|\mathbf{M}(t)|_{1,\infty} = ||\nabla\mathbf{M}(t)||_{L^\infty}$ as functions of t for $t \in (0, 6/100)$ for $\ell = 4, 5, 6$. For each $\ell = 4, 5, 6$ the function $||\nabla\mathbf{M}(t)||_{L^\infty}$ assumes the maximum value $2\sqrt{2}h^{-1}$ (among functions $\boldsymbol{\phi}_h \in \mathbf{V}_h$ with $|\boldsymbol{\phi}_h(\mathbf{x}_m)| = 1$ for all nodes $\mathbf{x}_m$). We observe that for decreasing mesh-size h the blow-up time (the time at which $||\nabla\mathbf{M}(t)||_{L^\infty}$ assumes its maximum) approaches $t \approx 0.03$.

In order to study the dependence of blow-up behaviour on the parameter α we ran Algorithm 1.1 in Example 5.1 for fixed $\ell = 5$, $s = 1$, and for $\alpha = 1$, $1/4$, $1/16$, $1/64$, $1/256$. The plot in Figure 6

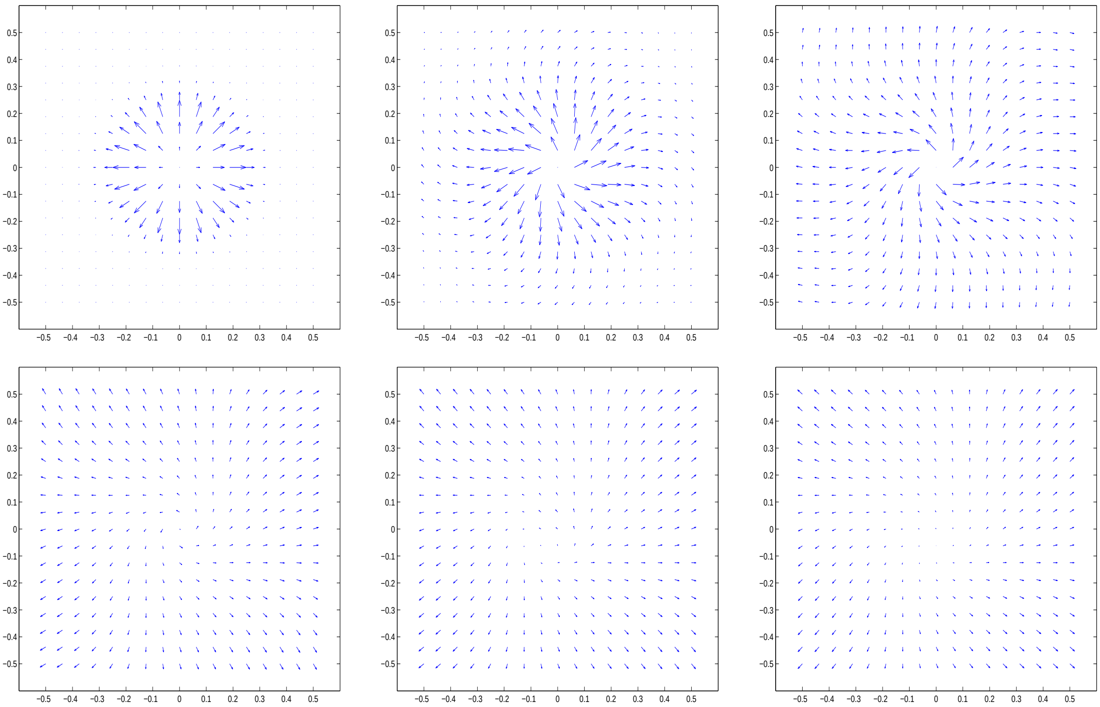

FIGURE 1. Numerical approximation $\mathbf{M}(t,\cdot)$ in Example 5.1 with $s=1$, $\ell=4$, and $\alpha=1$ for $t=0$, 0.0119, 0.0295, 0.0529, 0.0588, 0.0646.

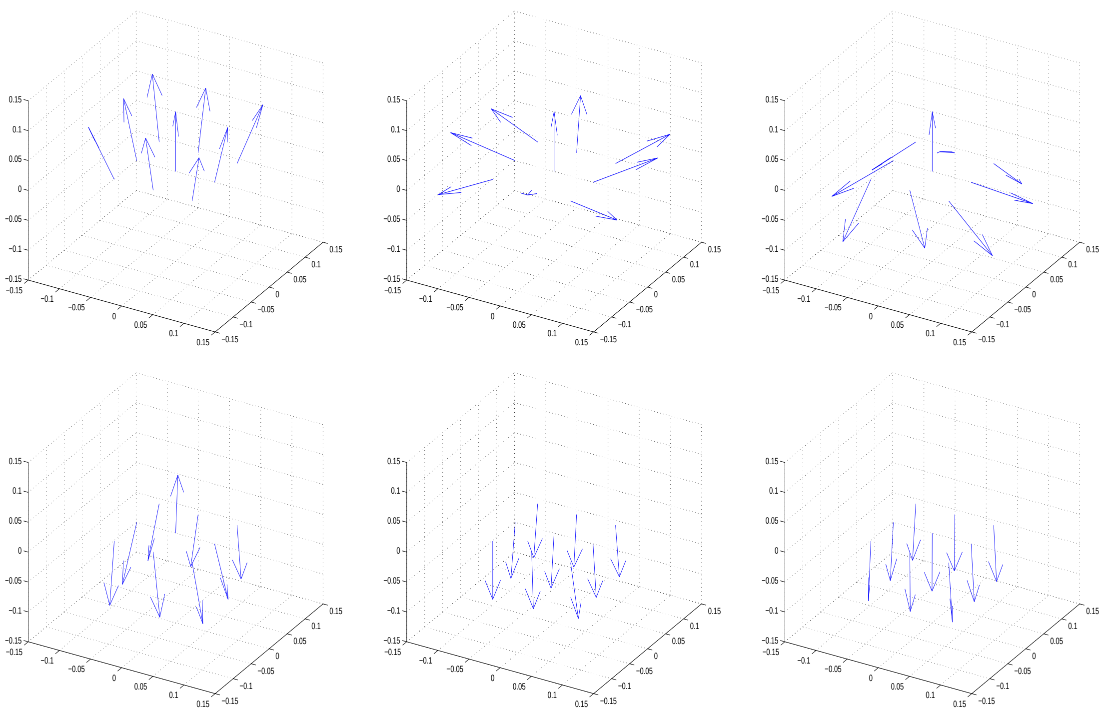

FIGURE 2. Nodal values $\mathbf{M}(t,\mathbf{x}_m)$ for nodes $\mathbf{x}_m$ close to the origin in Example 5.1 with $s=1$, $\ell=4$, and $\alpha=1$ for $t=0$, 0.0119, 0.0295, 0.0529, 0.0588, 0.0646.

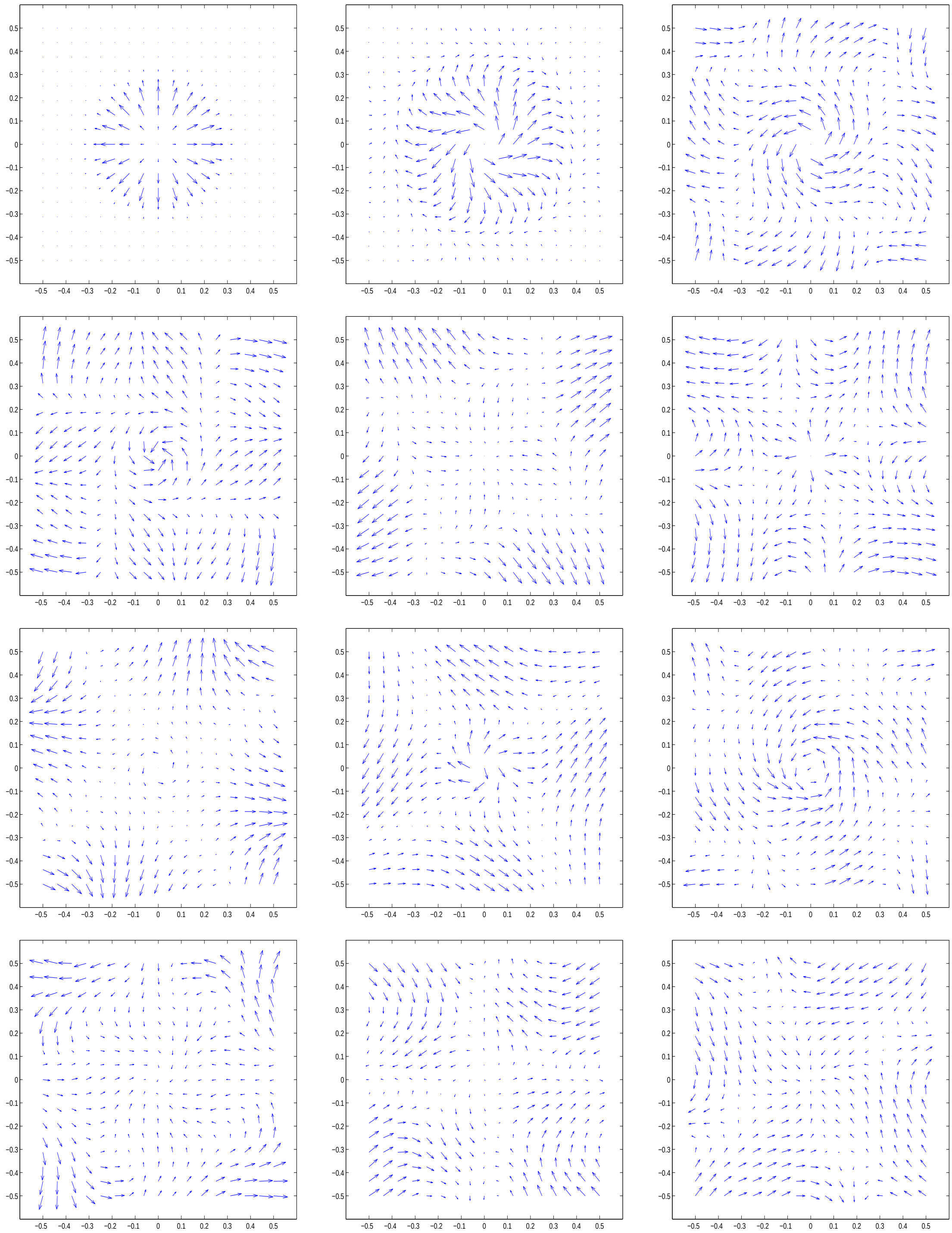

FIGURE 3. Numerical approximation $\mathbf{M}(t,\cdot)$ in Example 5.1 with $s=1$, $\ell=4$, and $\alpha=1/64$ for $t=0$, 0.0102, 0.0297, 0.0492, 0.0687, 0.1078, 0.1371, 0.1664, 0.2054, 0.2347, 0.2738, 0.3031.

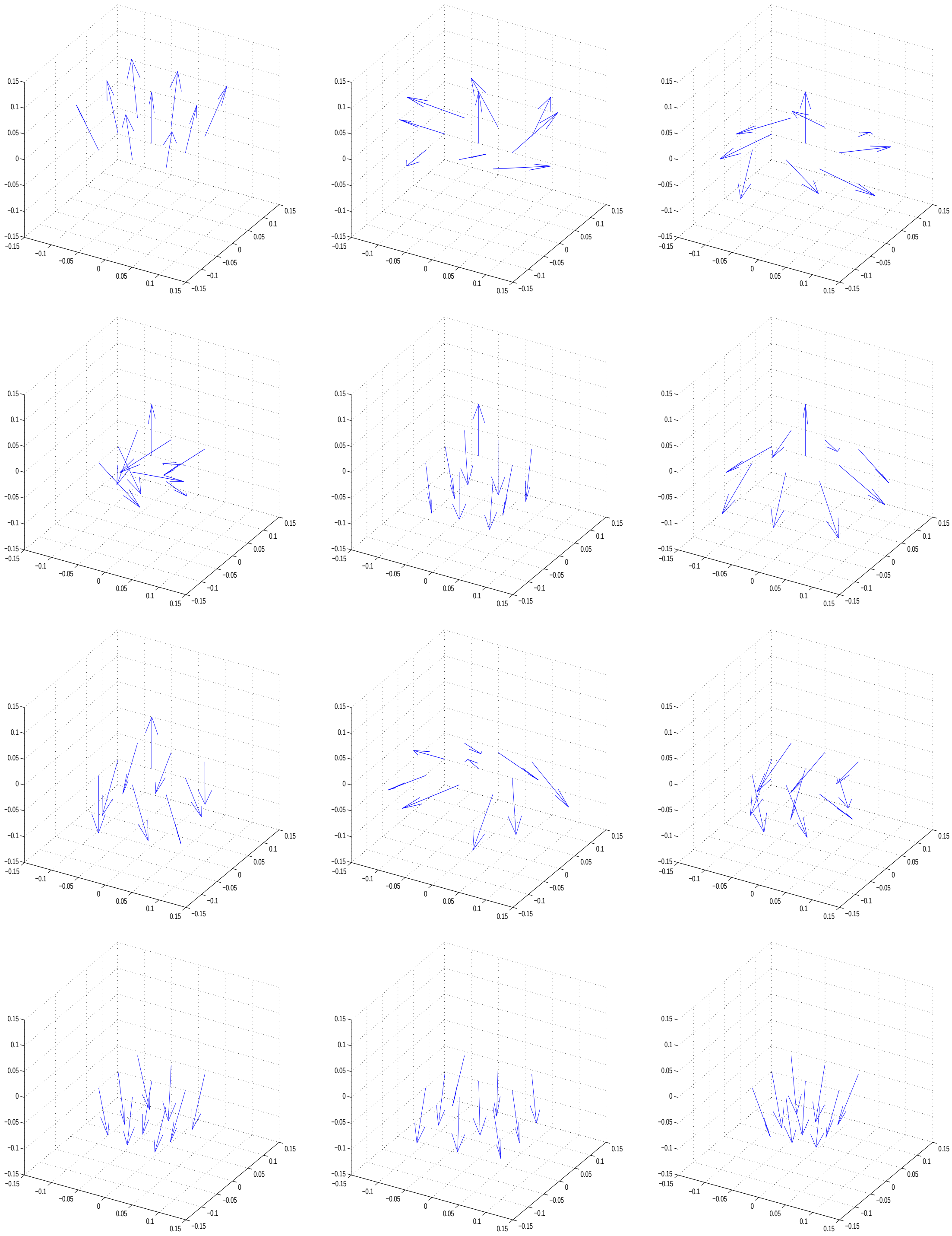

FIGURE 4. Nodal values $\mathbf{M}(t, \mathbf{x}_m)$ for nodes $\mathbf{x}_m$ close to the origin in Example 5.1 with $s = 1$, $\ell = 4$, and $\alpha = 1/64$ for $t = 0$, 0.0102, 0.0297, 0.0492, 0.0687, 0.1078, 0.1371, 0.1664, 0.2054, 0.2347, 0.2738, 0.3031.

indicates that the blow-up time approaches the time $t \approx 0.06$ for decreasing α. The experimental values for $\alpha = 1/64$ and $\alpha = 1/256$ almost coincide.

We remark that the results of our experiments for $\alpha = 1,\ 1/4,\ 1/16$ are similar to the results obtained in [3] with an explicit scheme. The implicit scheme of this article allows to use smaller values for α which lead to too restrictive conditions on the time step size for the explicit scheme of [3]. For the triangulations employed here and for $\alpha = 1$ the total runtimes of the explicit scheme (using reduced integration) and the implicit scheme are comparable.

6. Proof of (1.3)

Lemma 6.1. *Assume that* $|\mathbf{m}_h^0(\mathbf{x}_\ell)| = 1$ *for all* $\ell \in L$ *and let* $\{\mathbf{m}_h^j\}_{j\geq 0}$ *solve Algorithm 1.1. There holds for all* $\boldsymbol{\phi}_h \in \mathbf{V}_h$,

$$\begin{aligned}(d_t\mathbf{m}_h^{j+1}, \boldsymbol{\phi}_h)_h + \alpha\big(\nabla\overline{\mathbf{m}}_h^{j+1/2}, \nabla\boldsymbol{\phi}_h\big) &= \alpha\big(|\,\nabla\overline{\mathbf{m}}_h^{j+1/2}\,|^2\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\phi}_h\big)\\ &\quad -\big(\overline{\mathbf{m}}_h^{j+1/2} \times \nabla\overline{\mathbf{m}}_h^{j+1/2}, \nabla\boldsymbol{\phi}_h\big) + \mathrm{Corr}\,,\end{aligned}$$

for a correcting term $\mathrm{Corr} = \mathrm{Corr}_{\mathrm{A}} + \mathrm{Corr}_{\mathrm{B}}$*, with*

$$\begin{aligned}\mathrm{Corr}_{\mathrm{A}} \;:=\;& \frac{\alpha}{2}\big(\nabla|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2, \nabla\langle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\phi}_h\rangle\big) + \frac{\alpha^2}{1+\alpha^2}\Big(d_t\mathbf{m}_h^{j+1}, [1-|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2]\boldsymbol{\phi}_h\Big)_h\\ &+\alpha\Big(\nabla\overline{\mathbf{m}}_h^{j+1/2}, \nabla[(1-|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2)\boldsymbol{\phi}_h]\Big)\,,\end{aligned}$$

and $\mathrm{Corr}_{\mathrm{B}} = \sum_{i=1}^3 \mathrm{Corr}_{\mathrm{B}_i}$ *given in the proof below.*

Proof. Given any $\boldsymbol{\mathfrak{z}}_h \in \mathbf{V}_h$ choose $\boldsymbol{\phi}_h = \mathcal{I}_h(\overline{\mathbf{m}}_h^{j+1/2} \times \boldsymbol{\mathfrak{z}}_h)$ in Algorithm 1.1, then the properties of $(\cdot,\cdot)_h$ imply

$$\begin{aligned}&\big(\overline{\mathbf{m}}_h^{j+1/2} \times d_t\mathbf{m}_h^{j+1}, \boldsymbol{\mathfrak{z}}_h\big)_h + \alpha\big(\overline{\mathbf{m}}_h^{j+1/2} \times (\overline{\mathbf{m}}_h^{j+1/2} \times d_t\mathbf{m}_h^{j+1}), \boldsymbol{\mathfrak{z}}_h\big)_h\\ &\qquad= (1+\alpha^2)\big(\overline{\mathbf{m}}_h^{j+1/2} \times (\overline{\mathbf{m}}_h^{j+1/2} \times \tilde{\Delta}_h\overline{\mathbf{m}}_h^{j+1/2}), \boldsymbol{\mathfrak{z}}_h\big)_h\,.\end{aligned} \tag{6.1}$$

Owing to $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \langle\mathbf{a}, \mathbf{c}\rangle\mathbf{b} - \langle\mathbf{a}, \mathbf{b}\rangle\mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, the second term on the left-hand side in (6.1) may be rewritten as

$$\begin{aligned}&\alpha\big(\langle\overline{\mathbf{m}}_h^{j+1/2}, d_t\mathbf{m}_h^{j+1}\rangle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\big)_h - \alpha\big(|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2 d_t\mathbf{m}_h^{j+1}, \boldsymbol{\mathfrak{z}}_h\big)_h\\ &\quad= \frac{\alpha}{2}\big((d_t|\,\mathbf{m}_h^{j+1}\,|^2)\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\big)_h - \alpha\big(|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2 d_t\mathbf{m}_h^{j+1}, \boldsymbol{\mathfrak{z}}_h\big)_h\,,\end{aligned}$$

and the first term on the left-hand side vanishes owing to Lemma 3.1 below. We again use the above vector identity to recast the right-hand side of (6.1) as

$$(1+\alpha^2)\big(\langle\overline{\mathbf{m}}_h^{j+1/2}, \tilde{\Delta}_h\overline{\mathbf{m}}_h^{j+1/2}\rangle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\big)_h - (1+\alpha^2)\big(|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2\tilde{\Delta}_h\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\big)_h\,. \tag{6.2}$$

We proceed independently with arising two terms: intermitting the Lagrange interpolant for the nonlinear term in the first case to benefit from (2.1) yields

$$\begin{aligned}&\Big((\mathrm{Id} \pm \mathcal{I}_h)\big(\langle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\rangle\overline{\mathbf{m}}_h^{j+1/2}\big), \tilde{\Delta}_h\overline{\mathbf{m}}_h^{j+1/2}\big)\Big)_h = \Big((\mathrm{Id} - \mathcal{I}_h)\big(\langle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\rangle\overline{\mathbf{m}}_h^{j+1/2}\big), \tilde{\Delta}_h\overline{\mathbf{m}}_h^{j+1/2}\big)\Big)_h\\ &\quad + \Big(\nabla\big((\mathrm{Id} - \mathcal{I}_h)(\langle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\rangle\overline{\mathbf{m}}_h^{j+1/2})\big), \nabla\overline{\mathbf{m}}_h^{j+1/2}\Big) - \Big(\nabla\big(\langle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\rangle\overline{\mathbf{m}}_h^{j+1/2}\big), \nabla\overline{\mathbf{m}}_h^{j+1/2}\Big)\,,\end{aligned}$$

where the first two terms on the right-hand side are referred to as Corr_{B_1}. For the last term, we resume

$$\big(\nabla\overline{\mathbf{m}}_h^{j+1/2}, \nabla(\langle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\rangle\overline{\mathbf{m}}_h^{j+1/2})\big) = \big(|\,\nabla\overline{\mathbf{m}}_h^{j+1/2}\,|^2\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\big) + \frac{1}{2}\big(\nabla|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2, \nabla\langle\overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\rangle\big)\,.$$

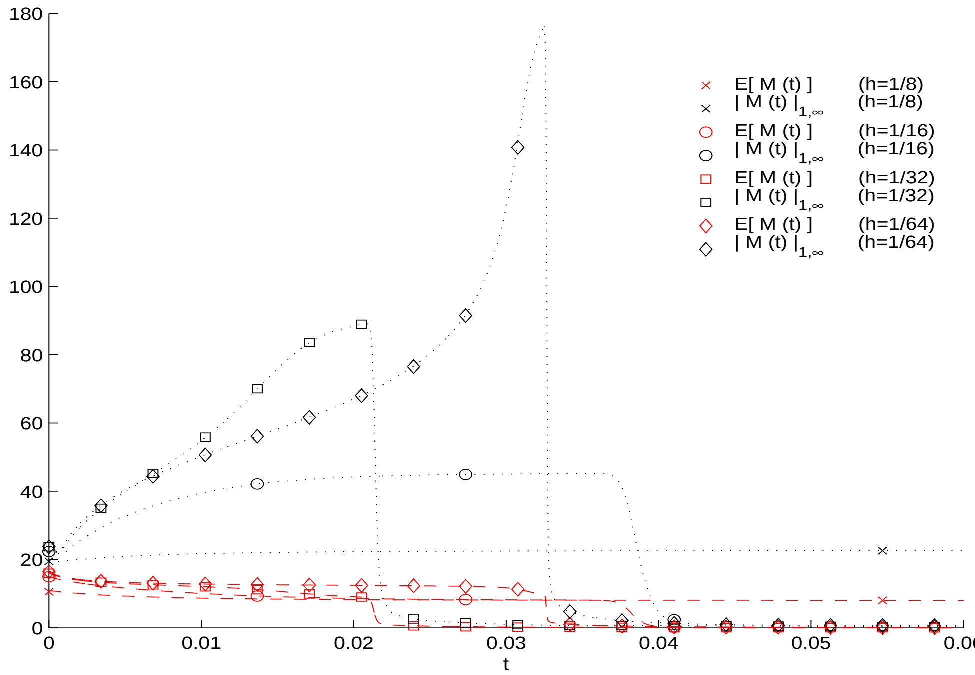

FIGURE 5. Energy and $W^{1,\infty}$ semi-norm for decreasing mesh-sizes in Example 5.1 with $\alpha = 1$ and $s = 4$.

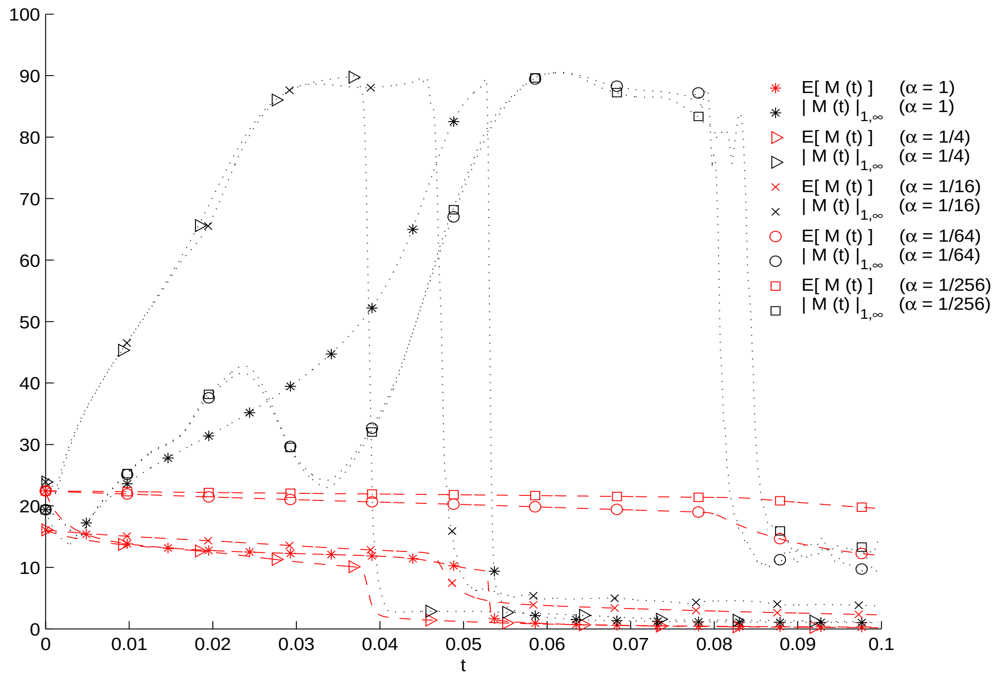

FIGURE 6. Energy and $W^{1,\infty}$ semi-norm in Example 5.1 for $\ell = 5$, $s = 1$, and $\alpha = 1,\ 1/4,\ 1/16,\ 1/64,\ 1/256$.

Similarly, we account for effects of reduced integration and local averaging inherent to the scheme for the second term in (6.2),

$$\Big((\mathrm{Id} \pm \mathcal{I}_h)\big(|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2 \boldsymbol{\mathfrak{z}}_h\big), \tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2}\Big)_h = \Big((\mathrm{Id} - \mathcal{I}_h)\big(|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2 \boldsymbol{\mathfrak{z}}_h\big), \tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2}\Big)_h$$
$$+\Big(\nabla\big((\mathrm{Id} - \mathcal{I}_h)(|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2 \boldsymbol{\mathfrak{z}}_h)\big), \nabla \overline{\mathbf{m}}_h^{j+1/2}\Big) - \big(\nabla(|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2 \boldsymbol{\mathfrak{z}}_h), \nabla \overline{\mathbf{m}}_h^{j+1/2}\big)\,,$$

where the first two terms on the right-hand side are gathered in Corr_{B_2}. Finally, by Algorithm 1.1, and $\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \rangle = -\langle \mathbf{a} \times \mathbf{c}, \mathbf{b} \rangle$, the first term in (6.1) is identical to

$$-\frac{1}{\alpha}\big(d_t \mathbf{m}_h^{j+1}, \boldsymbol{\mathfrak{z}}_h\big)_h + \frac{1+\alpha^2}{\alpha}\big(\overline{\mathbf{m}}_h^{j+1/2} \times \tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\big)_h$$
$$= -\frac{1}{\alpha}\big(d_t \mathbf{m}_h^{j+1}, \boldsymbol{\mathfrak{z}}_h\big)_h + \frac{1+\alpha^2}{\alpha}\big(\nabla(\overline{\mathbf{m}}_h^{j+1/2} \times \boldsymbol{\mathfrak{z}}_h), \nabla \overline{\mathbf{m}}_h^{j+1/2}\big) + \mathrm{Corr}_{B_3}\,,$$

for $\mathrm{Corr}_{B_3} = -\Big((\mathrm{Id} - \mathcal{I}_h)(\overline{\mathbf{m}}_h^{j+1/2} \times \boldsymbol{\mathfrak{z}}_h), \tilde{\Delta}_h \overline{\mathbf{m}}_h^{j+1/2}\Big)_h - \Big(\nabla\big((\mathrm{Id} - \mathcal{I}_h)(\overline{\mathbf{m}}_h^{j+1/2} \times \boldsymbol{\mathfrak{z}}_h)\big), \nabla \overline{\mathbf{m}}_h^{j+1/2}\Big)$. Reassembling (6.1) then yields to

$$\big((1+\alpha^2|\,\mathbf{m}_h^{j+1/2}\,|^2) d_t \mathbf{m}_h^{j+1}, \boldsymbol{\mathfrak{z}}_h\big)_h = (1+\alpha^2)\big(\nabla(\overline{\mathbf{m}}_h^{j+1/2} \times \boldsymbol{\mathfrak{z}}_h), \nabla \overline{\mathbf{m}}_h^{j+1/2}\big)$$
$$+\alpha(1+\alpha^2)\Big[\big(|\,\nabla \overline{\mathbf{m}}_h^{j+1/2}\,|^2 \overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\big) + \frac{1}{2}\big(\nabla|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2, \nabla\langle \overline{\mathbf{m}}_h^{j+1/2}, \boldsymbol{\mathfrak{z}}_h\rangle\big)$$
$$-\big(\nabla \overline{\mathbf{m}}_h^{j+1/2}, \nabla(|\,\overline{\mathbf{m}}_h^{j+1/2}\,|^2 \boldsymbol{\mathfrak{z}}_h)\big)\Big] + \alpha\Big(\mathrm{Corr}_{\mathrm{A}} + \big(1+\alpha^2\big)\,\mathrm{Corr}_{\mathrm{B}}\Big)\,.$$

Rearranging terms then yields to the assertion. □

Acknowledgment: Part of the work was written when S. B. visited 'Forschungsinstitut für Mathematik' (ETH Zürich) in January 2005. S. B. gratefully acknowledges hospitality of the Department of Mathematics at the University of Maryland at College Park.

References

[1] F. Alouges, A. Soyeur, *On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness*, Nonl. Analysis, Theory, Meth. & Appl. **18**, pp. 1071–1084 (1992).

[2] F. Alouges, P. Jaisson *Convergence of a finite elements discretization for the Landau Lifshitz equations*, downloadable at `mahery.math.u-psud.fr/alouges/` (2004).

[3] S. Bartels, J. Ko, A. Prohl, *Numerical approximation of Landau-Lifshitz equations and finite time blow up of weak solutions*, manuscript (2005).

[4] B. Guo, M.-C. Hong, *The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps*, Calc. Var. **1**, pp. 311–334 (1993).

[5] M. Kružík, A. Prohl, *Recent Developments in Modeling, Analysis and Numerics of Ferromagnetism*, SIAM Review (accepted), also downloadable at: `www.fim.math.ethz.ch/preprints/` (2005).

Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany.

Department of Mathematics, ETHZ, CH-8092 Zürich, Switzerland