# FULLY PRACTICAL, CONSTRAINT PRESERVING, IMPLICIT APPROXIMATION OF HARMONIC MAP HEAT FLOW INTO SPHERES

SÖREN BARTELS* AND ANDREAS PROHL

ABSTRACT. This article discusses an implicit approximation scheme for the harmonic map heat flow into the unit sphere $S^2$. The proposed scheme preserves a unit length constraint at the nodes of a lowest order finite element discretization and is unconditionally stable and convergent. The influence of the error introduced by an approximation of the nonlinear system of equations in each time step on the global convergence behaviour to a weak solution of the harmonic map heat flow problem is investigated.

## 1. INTRODUCTION

Critical points of the Dirichlet energy

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \tag{1.1}$$

among maps $u : \Omega \to S^2$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded Lipschitz domain and $S^2 \subset \mathbb{R}^3$ denotes the unit sphere, are known as harmonic maps into spheres. This energy minimization serves as a model problem for continuum models in ferromagnetics [12] and liquid crystal theory [16, 13]. At present, only few algorithms are available for its reliable numerical approximation [1, 3]. The main difficulties in the design of convergent numerical methods are the nonconvexity of the constraint, $|u| = 1$ almost everywhere (a.e.) in $\Omega$, limited regularity and nonuniqueness of minimizers, as well as confined flexibility of typical finite element methods.

An alternative strategy to study critical points of (1.1) is to consider the long-time behavior of the harmonic map heat flow into spheres,

$$u_t - \Delta u = |\nabla u|^2 u \text{ in } \Omega_T, \quad \frac{\partial u}{\partial n} = 0 \text{ on } (0,T)\times\partial\Omega, \quad |u| = 1 \text{ a.e. in } \Omega_T, \quad u(0,\cdot) = u_0 \tag{1.2}$$

for $T > 0$ and with $\Omega_T := (0,T)\times\Omega$. The Cauchy problem (1.2) characterizes the $L^2$-gradient flow of (1.1), and solutions to this problem have been studied intensively over the last fifteen years [14, 7, 8, 10]. We refer the reader to [15] for an extensive survey and only mention here that weak solutions exist but that they cannot be expected to be unique or regular.

The class of weak solutions to (1.2) that we consider satisfy the partial differential equation in (1.2) in distributional sense, the initial condition in (1.2) in the sense of traces for given $u_0 \in H^1(\Omega, \mathbb{R}^3)$ with $|u_0| = 1$ almost everywhere in $\Omega$, and the energy inequality

$$E\big(u(T')\big) + \int_0^{T'} ||u_t||^2_{L^2(\Omega)} \,\mathrm{d}t \le E(u_0) \tag{1.3}$$

for almost all $T' \in (0,T)$. It is well-known that there exist (sub-) sequences $\big(t_k\big)_{k\ge 0} \subset \mathbb{R}$ with $t_k \to \infty$ such that $u^* = \lim_{k\to\infty} u(t_k,\cdot)$ exists in an appropriate sense and is a harmonic map, i.e., is a stationary point of (1.1). In order to establish existence of a weak solution to (1.2), the

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problem is often modified to first finding a solution $u^\varepsilon : \Omega_T \to \mathbb{R}^3$ of the (unconstrained) penalized formulation, defined through a small parameter $\varepsilon > 0$,

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{2\varepsilon}\big(|u^\varepsilon|^2 - 1\big)u^\varepsilon = 0 \text{ in } \Omega_T, \quad \frac{\partial u^\varepsilon}{\partial n} = 0 \text{ on } (0,T)\times\partial\Omega, \quad u^\varepsilon(0,\cdot) = u_0, \tag{1.4}$$

cf. [7, 8]. Considering appropriate limits as $\varepsilon \to 0$ for solutions to (1.4) leads to weak solutions of (1.2) which satisfy (1.3). Apart from its use as an analytical tool, (1.4) is often the starting point for the construction of convergent numerical schemes for (1.2); however, the penalization parameter requires a careful balancing with numerical parameters and often leads to highly diffused structures in practice.

From this background, we aim at designing convergent, implicit discretizations of (1.2) that employ lowest order conforming finite elements and satisfy the constraint in an appropriate discrete sense. Classical solutions of the partial differential equation in (1.2) satisfy $|u| = 1$ almost everywhere in $\Omega_T$ provided that $|u_0| = 1$ almost everywhere in $\Omega$. This property is not valid for straightforward discretizations, due to a damping property of most implicit temporal discretization schemes, and restricted flexibility of employed finite element functions. Motivated by the scheme proposed in [2], an explicit, fully discrete method for the approximation of the more general $p$-harmonic map heat flow into spheres (resulting from an exponent $p > 1$ instead of 2 in (1.1)) is designed in [5]. The resulting approximate solutions satisfy the constraint at every vertex of the underlying triangulation. Convergence towards a weak solution is established under certain conditions on the discretization parameters, which for $p = 2$ reduce to $k = o\big(h^{1+d/2}\big)$, where $k$ and $h$ are the time-step size and the mesh size, respectively.

The goal of this paper is to construct unconditionally convergent schemes for the approximation of (1.2) that preserve the unit length constraint at every vertex of a triangulation. A similar goal was achieved in [4] for the closely related Landau-Lifshitz-Gilbert equation. Unfortunately, the arguments used there do not carry over to the present case, since there, the top order nonlinearity $-u \times (u \times \Delta u)$ in the Landau-Lifshitz formulation was deliberately replaced by the damping term $u \times u_t$ in the equivalent Gilbert formulation of the ferromagnetic problem.

In order to motivate our approximation scheme, we recast the differential equation in (1.2), by using the vector identity $a_1 \times (a_2 \times a_3) = (a_1 \cdot a_3)a_2 - (a_1 \cdot a_2)a_3$ for $a_1, a_2, a_3 \in \mathbb{R}^3$ and $|u|^2 = 1$, in the formally equivalent form

$$u_t + u \times (u \times \Delta u) = 0 \quad \text{in } \Omega_T. \tag{1.5}$$

This identity is the starting point for our discretization. With the lowest order finite element space $\mathcal{S}^1(\mathcal{T}_h) \subset H^1(\Omega)$ subordinate to a regular triangulation $\mathcal{T}_h$ of $\Omega$ into triangles or tetrahedra and a time-step size $k > 0$ our approximation scheme and analysis are motivated by the following (nonlinear) time-stepping scheme:

**Algorithm (A).** *Given $u^{(j)} \in \mathcal{S}^1(\mathcal{T}_h)^3$ find $u^{(j+1)} \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ there holds*

$$\big(d_t u^{(j+1)}, v_h\big)_h + \big(u^{(j+1/2)} \times (u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}), v_h\big)_h = 0. \tag{1.6}$$

Here, $(\cdot,\cdot)_h$ denotes a discrete version ("reduced integration") of the inner product in $L^2(\Omega,\mathbb{R}^3)$, $\tilde{\Delta}_h : H^1(\Omega) \to \mathcal{S}^1(\mathcal{T}_h)$ is a discrete version of the Laplacian, and we write $d_t\varphi^{(j+1)} = k^{-1}\big(\varphi^{(j+1)} - \varphi^{(j)}\big)$ and $\varphi^{(j+1/2)} = \big(\varphi^{(j)} + \varphi^{(j+1)}\big)/2$, $j \geq 1$, for a sequence $\big(\varphi^{(j)}\big)_{j\geq 0}$; we refer the reader to Section 2 for details.

The key observation in (1.6) is that the choice $v_h = u^{(j+1/2)}$ makes the second term on the left-hand side disappear; a binomial structure together with properties of the discrete inner product imply that $|u^{(j+1)}(z)|^2 = |u^{(j)}(z)|^2$ for all nodes $z \in \mathcal{N}_h$ of the triangulation.

Existence of a solution in each step of Algorithm $(A)$ can be established with Brouwer's fixed point theorem independently of the discretization parameters. Since in practice the discrete scheme requires the solution or approximation of a nonlinear system of equations in each time step we will analyze a larger class of schemes by allowing a (small) right-hand side with an appropriate vector product structure. We propose and analyze an iterative method for the approximation of the system of equations in (1.6) that converges provided that $k = O(h^2)$ and which introduces a residual that does not significantly influence the properties of discrete solutions. We remark that the approximate solution of the nonlinear system in (1.6) with a Newton iteration does not fit into our framework.

As will be shown in detail in Lemma 3.1, (approximate) solutions to (1.6) satisfy the sphere constraint exactly at the nodes of the triangulation and an approximate discrete energy law. It is well-known, that strong solutions to (1.2) solve (1.5) in distributional sense. In contrast, these relations need not hold for corresponding discretizations, due to competition of local and nonlocal aspects inherent to fully discrete finite-element based methods, cf. [4] for details for a related problem. The main contributions of this article are summarized in the following theorem.

**Theorem A.** *Let $k > 0$, $\varepsilon > 0$, and $\big(\mathcal{T}_h\big)_{h>0}$ be a family of regular triangulations of $\Omega$ with maximal mesh-size $h > 0$. Suppose that for $0 \le j \le J$ we are given $u^{(j)}, r^{(j)} \in \mathcal{S}^1(\mathcal{T}_h)^3$ satisfying $\|r^{(j)}\|_h \le \varepsilon$ and such that $|u^{(0)}(z)| = 1$ for all nodes $z \in \mathcal{N}_h$. Assume that for $0 \le j \le J-1$ and all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ there holds*

$$(1.7)\qquad \big(d_t u^{(j+1)}, v_h\big)_h + \big(u^{(j+1/2)} \times (u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}), v_h\big)_h = \big(u^{(j+1/2)} \times r^{(j+1)}, v_h\big)_h,$$

*Then $|u^{(j)}(z)| = 1$ for all $z \in \mathcal{N}_h$ and $0 \le j \le J$. Given $0 \le t \le Jk$ such that $t \in [jk, (j+1)k)$ for some $0 \le j \le J-1$ and $x \in \Omega$ let*

$$\hat{u}_{h,k,\varepsilon}(t,x) := \frac{t-jk}{k}\, u^{(j+1)}(x) + \frac{(j+1)k-t}{k}\, u^{(j)}(x)$$

*If $Jk \ge T$ and $u^{(0)} \to u_0$ in $H^1(\Omega)$ for $h \to 0$ then there exists a subsequence of $\big(\hat{u}_{h,k,\varepsilon}\big)$ as $(h,k,\varepsilon) \to 0$ which converges weakly in $H^1(\Omega_T)$ to a weak solution of the harmonic map heat flow problem.*

The remainder of this paper is organized as follows: Preliminaries and notation are stated and introduced in Section 2. Theorem A follows from three Lemmas proved in Section 3. A simple fixed-point iteration, that constructs $u^{(j)}$ and $r^{(j)}$ in Theorem A, is discussed in Section 4.

## 2. Preliminaries

2.1. **Weak solutions of** (1.2)**.** Our approximation scheme approximates weak solutions of (1.2) in the sense of [15].

**Definition 2.1.** *Given $u_0 \in H^1(\Omega, \mathbb{R}^3)$ such that $|u_0| = 1$ almost everywhere in $\Omega$, a function $u$ is called a* weak solution *of (1.2) if for all $T > 0$ there holds (i) $u \in H^1(\Omega_T; \mathbb{R}^3)$ with $u(0,\cdot) = u_0$ in the sense of traces, (ii) $|u| = 1$ almost everywhere in $\Omega_T$, (iii) for almost all $T' \in (0,T)$ there holds*

$$(2.1)\qquad \frac{1}{2}\int_\Omega |\nabla u(T',x)|^2 \,\mathrm{d}x + \int_0^{T'} \|\partial_t u\|^2 \,\mathrm{d}t \le \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 \,\mathrm{d}x,$$

*and, (iv) for all $\phi \in C^\infty(\overline{\Omega}_T, \mathbb{R}^3)$ there holds*

$$(2.2)\qquad \int_{\Omega_T} \partial_t u \cdot (u \times \phi) \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \nabla u \cdot \nabla(u \times \phi)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Existence of weak solutions has been established in [7, 8]. By choosing $\phi = u \times \psi$ in (2.2) one verifies with the properties of the vector product and with $|u|^2 = 1$ almost everywhere in $\Omega_T$, that a weak solution of (1.2) satisfies

$$\int_0^T \big(u_t, \psi\big) + \big(\nabla u, \nabla\psi\big)\,\mathrm{d}t = \int_0^T \big(|\nabla u|^2 u, \psi\big)\,\mathrm{d}t$$

for all $\psi \in C^\infty(\overline{\Omega}_T, \mathbb{R}^3)$.

2.2. **Finite element spaces.** Throughout this paper we assume that $\mathcal{T}_h$ is a regular triangulation of the polygonal or polyhedral bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ into triangles or tetrahedra of maximal diameter $h$ for $d = 2$ or $d = 3$, respectively. We let $\mathcal{S}^1(\mathcal{T}_h) \subseteq H^1(\Omega)$ denote the lowest order finite element space on $\mathcal{T}_h$, i.e., $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)$ if and only if $\phi_h \in C(\overline{\Omega})$ and $\phi_h|_K$ is affine for each $K \in \mathcal{T}_h$. Given the set of all nodes (or vertices) $\mathcal{N}_h$ in $\mathcal{T}_h$ and letting $\big(\varphi_z : z \in \mathcal{N}_h\big)$ denote the nodal basis in $\mathcal{S}^1(\mathcal{T}_h)$ we define the nodal interpolation operator $\mathcal{I}_h : C(\overline{\Omega}) \to \mathcal{S}^1(\mathcal{T}_h)$ by

$$\mathcal{I}_h\psi := \sum_{z\in\mathcal{N}_h} \psi(z)\varphi_z$$

for $\psi \in C(\overline{\Omega})$. We write $(f, g) = \int_\Omega f \cdot g \,\mathrm{d}x$ for $f, g \in L^2(\Omega; \mathbb{R}^\ell)$ and abbreviate $||f|| = ||f||_{L^2(\Omega)}$. Similarly, we use the notation $||v||_{W^{k,p}} = ||v||_{W^{k,p}(\Omega)}$ for $v \in W^{k,p}(\Omega)$. For functions $\phi, \psi \in C(\overline{\Omega})$ a discrete inner product is defined by

$$\big(\phi, \psi\big)_h := \int_\Omega \mathcal{I}_h[\phi\,\psi] \,\mathrm{d}x = \sum_{z\in\mathcal{N}_h} \beta_z \phi(z) \cdot \psi(z),$$

where $\beta_z = \int_\Omega \varphi_z \,\mathrm{d}x$ for all $z \in \mathcal{N}_h$; we define $||\psi||_h^2 := (\psi, \psi)_h$. We remark that there holds

$$||\psi_h|| \le ||\psi_h||_h \le (d + 2)^{1/2}||\psi_h||$$

for all $\psi_h \in \mathcal{S}^1(\mathcal{T}_h)$. Basic interpolation estimates yield that

$$\big|(\phi_h, \psi_h)_h - (\phi_h, \psi_h)\big| \le Ch||\phi_h||\,||\nabla\psi_h||$$

for all $\phi_h, \psi_h \in \mathcal{S}^1(\mathcal{T}_h)$, where here and throughout this paper $C > 0$ denotes an $(h, k, \varepsilon)$-independent constant. We define a discrete Laplace operator $\tilde{\Delta}_h : H^1(\Omega) \to \mathcal{S}^1(\mathcal{T}_h)$ by requiring

$$-(\tilde{\Delta}_h\phi, \chi_h)_h = (\nabla\phi, \nabla\chi_h)$$

for all $\chi_h \in \mathcal{S}^1(\mathcal{T}_h)$. We note that there exists a constant $c_1 > 0$ such that for all $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)$ there holds, with $h_{min} = \min\{\mathrm{diam}(K) : K \in \mathcal{T}_h\}$,

$$||\tilde{\Delta}_h\phi_h||_h \le c_1 h_{min}^{-2}||\phi_h||_h \quad\text{and}\quad ||\tilde{\Delta}_h\phi_h||_{L^\infty} \le c_1 h_{min}^{-2}||\phi_h||_{L^\infty}. \tag{2.3}$$

The proof of the first estimate in (2.3) follows directly from the inverse estimate $||\nabla\phi_h|| \le Ch_{min}^{-1}||\phi_h||$. In order to verify the second estimate, let $z \in \mathcal{N}_h$ be such that $||\tilde{\Delta}_h\phi_h||_{L^\infty} = |\tilde{\Delta}_h\phi_h(z)|$. Choosing $\chi_h = \tilde{\Delta}_h\phi_h(z)\varphi_z$ in the definition of $\tilde{\Delta}_h\phi_h$ yields that

$$\begin{aligned}
|\tilde{\Delta}_h\phi_h(z)|^2 &= \beta_z^{-1}\big(\tilde{\Delta}_h\phi_h, \chi_h\big)_h = -|\tilde{\Delta}_h\phi_h(z)|\beta_z^{-1}\big(\nabla\phi_h, \nabla\varphi_z\big) \\
&= |\tilde{\Delta}_h\phi_h(z)|\beta_z^{-1} \sum_{y\in\mathcal{N}_h} \phi_h(y)\big(\nabla\varphi_y, \nabla\varphi_z\big) \le C|\tilde{\Delta}_h\phi_h(z)|\beta_z^{-1}||\phi_h||_{L^\infty}||\nabla\varphi_z||^2 \\
&\le C|\tilde{\Delta}_h\phi_h(z)|\beta_z^{-1}||\phi_h||_{L^\infty} h_{min}^{-2}||\varphi_z||^2 \le C|\tilde{\Delta}_h\phi_h(z)|h_{min}^{-2}||\phi_h||_{L^\infty},
\end{aligned}$$

where we used that the number of nodes $y \in \mathcal{N}_h$ such that $\big(\nabla\varphi_y, \nabla\varphi_z\big) \neq 0$ is bounded $h$-independently, that $||\nabla\varphi_y|| \le C||\nabla\varphi_z||$ for such $y$, and that $\beta_z^{-1}||\varphi_z||^2 \le C$.

2.3. **Discrete time-derivatives and interpolation.** Given a time-step size $k > 0$ and a sequence $(\varphi_j)_{j\geq 0}$ in some vector space $X$ we set

$$d_t\varphi^{(j+1)} := k^{-1}\big(\varphi^{(j+1)} - \varphi^{(j)}\big) \qquad \text{and} \qquad \varphi^{(j+1/2)} := \big(\varphi^{(j)} + \varphi^{(j+1)}\big)/2,$$

for $j \geq 0$. We note that there holds $\langle d_t\varphi^{(j+1)}, \varphi^{(j+1/2)}\rangle_X = \frac{1}{2}d_t||\varphi^{(j+1)}||_X^2$ if $X$ is a Hilbert space. Piecewise constant interpolations of $\varphi^{(j)}$ are defined for $0 \leq t \leq Jk$ such that $t \in [jk, (j+1)k)$ for some $j \in \{0, 1, ..., J-1\}$ by

$$\overline{\varphi}(t) := \varphi^{(j+1/2)} \qquad \text{and} \qquad \varphi^{+}(t) := \varphi^{(j+1)},$$

and a piecewise linear interpolation is defined through

$$\hat{\varphi}(t) := \frac{t - jk}{k}\varphi^{(j+1)} + \frac{(j+1)k - t}{k}\varphi^{(j)}.$$

Note that there holds $||\varphi^{+} - \hat{\varphi}||_X + ||\overline{\varphi} - \hat{\varphi}||_X \leq 2k||d_t\hat{\varphi}||_X$.

2.4. **Properties of the vector product.** We will frequently make use of the following properties of the vector product: For $a_1, a_2, a_3, a_4 \in \mathbb{R}^3$ there holds

$$(a_1 \times a_2)\cdot a_3 = -a_2 \cdot (a_1 \times a_3), \tag{2.4}$$

$$a_1 \times (a_2 \times a_3) = (a_1 \cdot a_3)a_2 - (a_1 \cdot a_2)a_3, \tag{2.5}$$

$$(a_1 \times a_2)\cdot(a_3 \times a_4) = (a_1 \cdot a_3)(a_2 \cdot a_4) - (a_2 \cdot a_3)(a_1 \cdot a_4). \tag{2.6}$$

## 3. Proof of Theorem A

**Lemma 3.1.** *Under the assumptions of Theorem A there holds* $|u^{(j)}(z)| = 1$ *for* $0 \leq j \leq J$ *and all* $z \in \mathcal{N}_h$. *If* $0 < \varepsilon \leq 1$ *then for all* $0 \leq J' \leq J$ *there holds*

$$\frac{1}{2}||\nabla u^{(J')}||^2 + (1-\varepsilon)\,k\sum_{j=0}^{J'-1}||u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}||_h^2 \leq \frac{1}{2}||\nabla u^{(0)}||^2 + (J'k)\varepsilon$$

*and*

$$\frac{1}{2}||\nabla u^{(J')}||^2 + (1-\varepsilon)^2 k\sum_{j=0}^{J'-1}||d_t u^{(j+1)}||_h^2 \leq \frac{1}{2}||\nabla u^{(0)}||^2 + (5/4)(J'k)\varepsilon.$$

*Proof.* Choosing $v_h = u^{(j+1/2)}(z)\varphi_z$ in (1.7) yields that

$$\frac{1}{2}d_t|u^{(j+1)}(z)|^2 = d_t u^{(j+1)}(z)\cdot u^{(j+1/2)}(z) = \beta_z^{-1}\big(d_t u^{(j+1)}, u^{(j+1/2)}(z)\varphi_z\big)_h = 0$$

and implies $|u^{(j+1)}(z)| = 1$ provided that $|u^{(j)}(z)| = 1$. With $v_h = \tilde{\Delta}_h u^{(j+1/2)}$ in (1.7) we deduce

$$\begin{aligned} &\frac{1}{2}d_t||\nabla u^{(j+1)}||^2 + ||u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}||_h^2 \\ &\quad = -\big(d_t u^{(j+1)}, \tilde{\Delta}_h u^{(j+1/2)}\big)_h - \big(u^{(j+1/2)} \times (u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}), \tilde{\Delta}_h u^{(j+1/2)}\big)_h \\ &\quad = -\big(u^{(j+1/2)} \times r^{(j+1)}, \tilde{\Delta}_h u^{(j+1/2)}\big)_h \\ &\quad \leq \frac{1}{4\varepsilon}||r^{(j+1)}||_h^2 + \varepsilon||u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}||_h^2. \end{aligned} \tag{3.1}$$

A summation over $j = 0, 1, ..., J' - 1$ and the bounds for $||r^{(j+1)}||_h$ imply the first estimate. We choose $v_h = d_t u^{(j+1)}$ in (1.7) to verify with $||u^{(j+1/2)}||_{L^\infty} \le 1$ and Young's inequality, that

$$\begin{aligned} ||d_t u^{(j+1)}||_h^2 &= \big(u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}, u^{(j+1/2)} \times d_t u^{(j+1)}\big)_h + \big(u^{(j+1/2)} \times r^{(j+1)}, d_t u^{(j+1)}\big)_h \\ &\le ||u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}||_h ||u^{(j+1/2)}||_{L^\infty} ||d_t u^{(j+1)}||_h + ||u^{(j+1/2)}||_{L^\infty} ||d_t u^{(j+1)}||_h ||r^{(j+1)}||_h \\ &\le \frac{1}{2} ||u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}||_h^2 + \frac{1}{2\varepsilon} ||r^{(j+1)}||_h^2 + \frac{1}{2}(1+\varepsilon) ||d_t u^{(j+1)}||_h^2. \end{aligned}$$

An application of (3.1) and a summation over $j = 0, 1, ..., J' - 1$ finish the proof of the lemma. □

**Lemma 3.2.** *Under the assumptions of Theorem A there exists a subsequence (not relabeled) of* $(\hat{u})_{(h,k,\varepsilon)}$ *as* $(h, k, \varepsilon) \to 0$ *and* $u \in H^1(\Omega_T, \mathbb{R}^3)$ *such that* $\partial_t \hat{u} \rightharpoonup \partial_t u$ *in* $L^2(\Omega_T)$, $\overline{u} \to u$ *in* $L^2(\Omega_T)$, $\nabla \overline{u} \rightharpoonup \nabla u$ *in* $L^2(\Omega_T)$, *and* $u^+ \rightharpoonup^* u$ *in* $L^\infty(0, T; H^1(\Omega))$. *In particular, there holds* $|u| = 1$ *almost everywhere in* $\Omega_T$, $u$ *satisfies* (2.1), *and there holds* $u(0, \cdot) = u_0$ *in the sense of traces.*

*Proof.* The first assertions follow from the bounds of Lemma 3.1 and the relations between $\overline{u}, \hat{u}, u^+$. Since $|u^+| = 1$ for all $z \in \mathcal{N}_h$ and almost all $t \in (0, T)$ we deduce with

$$\big\| |u^+|^2 - 1 \big\|_{L^2(K)} \le Ch \big\| \nabla \big[ |u^+|^2 - 1 \big] \big\|_{L^2(K)} \le Ch \| (u^+)^T \nabla u^+ \|_{L^2(K)}^2 \le Ch \| \nabla u^+ \|_{L^2(K)}^2$$

for all $K \in \mathcal{T}_h$ that $|u^+| \to 1$ almost everywhere in $\Omega_T$ and hence $|u| = 1$ almost everywhere. Weak lower semicontinuity of norms and a passage $\varepsilon \to 0$ in the second estimate of Lemma 3.1 imply that $u$ satisfies (2.1). Since the trace operator is bounded and linear, it is weakly continuous as an operator from $H^1(\Omega_T)$ into $L^2(\Omega)$ and we deduce that there holds $u(0, \cdot) = u_0$ in the sense of traces. □

**Lemma 3.3.** *For* $u$ *as in Lemma 3.2 and all* $\psi \in C^\infty(\overline{\Omega}_T, \mathbb{R}^3)$ *there holds*

$$\int_0^T \big(u_t, u \times \psi\big) + \big(\nabla u, \nabla[u \times \psi]\big) \, \mathrm{d}t = 0.$$

*Proof.* For $t \in (0, T)$ let $\psi_h(t, \cdot) = \mathcal{I}_h \psi(t, \cdot)$. There holds

$$\begin{aligned} \big(\hat{u}_t, \overline{u} \times \psi_h\big)_h - \big(u_t, u \times \psi\big) &= \Big[ \big(\hat{u}_t, \mathcal{I}_h[\overline{u} \times \psi_h]\big)_h - \big(\hat{u}_t, \mathcal{I}_h[\overline{u} \times \psi_h]\big) \Big] \\ &\quad + \big(\hat{u}_t, \mathcal{I}_h[\overline{u} \times \psi_h] - \overline{u} \times \psi_h\big) + \big(\hat{u}_t, \overline{u} \times [\psi_h - \psi]\big) + \big(\hat{u}_t, [\overline{u} - u] \times \psi\big) + \big(\hat{u}_t - u_t, u \times \psi\big). \end{aligned}$$

The properties of $(\cdot, \cdot)_h$, $H^1$-stability of $\mathcal{I}_h$, and $||\overline{u}||_{L^\infty} \le 1$ yield

$$\Big| \big(\hat{u}_t, \mathcal{I}_h[\overline{u} \times \psi_h]\big)_h - \big(\hat{u}_t, \mathcal{I}_h[\overline{u} \times \psi_h]\big) \Big| \le Ch ||\hat{u}_t|| \, ||\nabla \mathcal{I}_h[\overline{u} \times \psi_h]|| \le Ch ||\hat{u}_t|| \big(||\nabla \overline{u}|| + 1\big) ||\psi||_{W^{1,\infty}}.$$

A similar argumentation shows

$$\Big| \big(\hat{u}_t, \mathcal{I}_h[\overline{u} \times \psi_h] - \overline{u} \times \psi_h\big) \Big| + \Big| \big(\hat{u}_t, \overline{u} \times [\psi_h - \psi]\big) \Big| \le Ch ||\hat{u}_t|| \big(||\nabla \overline{u}|| + 1\big) ||\psi||_{W^{1,\infty}}.$$

A combination of the last three equations yields

$$\begin{aligned} I &:= \Big| \int_0^T \big(\hat{u}_t, \overline{u} \times \psi_h\big)_h - \big(u_t, u \times \psi\big) \, \mathrm{d}t \Big| \\ &\le Ch ||\hat{u}_t||_{L^2(\Omega_T)} \big(||\nabla \overline{u}||_{L^2(\Omega_T)} + 1 + ||\overline{u} - u||_{L^2(\Omega_T)}\big) ||\psi||_{L^\infty(0,T;W^{1,\infty})} + \Big| \int_0^T \big(\hat{u}_t - u_t, u \times \psi\big) \, \mathrm{d}t \Big|. \end{aligned}$$

Since $\overline{u} \to u$ in $L^2(\Omega_T)$ and $\hat{u}_t \rightharpoonup u_t$ in $L^2(\Omega_T)$ we infer that $I \to 0$ for $(h, k, \varepsilon) \to 0$. We have, using that $\nabla u \cdot \nabla[u \times \psi] = \nabla u \cdot [u \times \nabla \psi]$ and $\nabla \overline{u} \cdot \nabla[\overline{u} \times \psi_h] = \nabla \overline{u} \cdot [\overline{u} \times \nabla \psi_h]$,

$$\begin{aligned} \big(\nabla \overline{u}, \nabla \mathcal{I}_h[\overline{u} \times \psi_h]\big) - \big(\nabla u, \nabla[u \times \psi]\big) &= \big(\nabla \overline{u}, \nabla \{\mathcal{I}_h[\overline{u} \times \psi_h] - \overline{u} \times \psi_h\}\big) \\ &\quad + \big(\nabla \overline{u}, \overline{u} \times \nabla[\psi_h - \psi]\big) + \big(\nabla \overline{u}, [\overline{u} - u] \times \nabla \psi\big) + \big(\nabla[\overline{u} - u], u \times \nabla \psi\big). \end{aligned}$$

Interpolation estimates and $D^2\overline{u}|_K = 0$ for all $K \in \mathcal{T}_h$ imply that

$$\Big|\big(\nabla\overline{u}, \nabla\{\mathcal{I}_h[\overline{u}\times\psi_h] - \overline{u}\times\psi_h\}\big)\Big| + \Big|\big(\nabla\overline{u}, \overline{u}\times\nabla[\psi_h - \psi]\big)\Big| \le Ch||\nabla\overline{u}||\big(||\nabla\overline{u}|| + 1\big)||\psi||_{W^{2,\infty}}.$$

We combine the previous two equations to verify that

$$\begin{aligned} II :=& \Big|\int_0^T \big(\nabla\overline{u}, \nabla\mathcal{I}_h[\overline{u}\times\psi_h]\big) - \big(\nabla u, \nabla[u\times\psi]\big)\,\mathrm{d}t\Big| \\ \le& Ch||\nabla\overline{u}||_{L^2(\Omega_T)}\big(||\nabla\overline{u}||_{L^2(\Omega_T)} + 1\big)||\psi||_{L^\infty(0,T;W^{2,\infty})} \\ &+ ||\nabla\overline{u}||_{L^2(\Omega_T)}||\overline{u} - u||_{L^2(\Omega_T)}||\psi||_{L^\infty(0,T;W^{1,\infty})} + \Big|\int_0^T \big(\nabla[\overline{u} - u], u\times\nabla\psi\big)\,\mathrm{d}t\Big|. \end{aligned}$$

Using that $\overline{u} \to u$ in $L^2(\Omega_T)$ and $\nabla\overline{u} \rightharpoonup \nabla u$ in $L^2(\Omega_T)$ we deduce that $II \to 0$ for $(h,k,\varepsilon) \to 0$. Using $||r^+||_h \le \varepsilon$ for almost all $t \in (0,T)$ we verify that

$$III := \Big|\int_0^T \big(\overline{u}\times r^+, \overline{u}\times\psi_h\big)_h\,\mathrm{d}t\Big| \le ||r^+||_{L^2(\Omega_T)}||\psi||_{L^2(\Omega_T)} \le T\varepsilon||\psi||_{L^2(\Omega_T)}.$$

There holds

$$\big(\big[1 - |\overline{u}|^2\big]\tilde{\Delta}_h\overline{u}, \overline{u}\times\psi_h\big)_h \le ||\overline{u}\times\tilde{\Delta}_h\overline{u}||_h\big\|1 - |\overline{u}|^2\big\|_h||\psi_h||_{L^\infty}$$

and using that $\big|1 - |\overline{u}|^2\big| = \big|(u - \overline{u})\cdot(u + \overline{u})\big| \le 2|u - \overline{u}|$ we deduce

$$IV := \Big|\int_0^T \big(\big[1 - |\overline{u}|^2\big]\tilde{\Delta}_h\overline{u}, \overline{u}\times\psi_h\big)_h\,\mathrm{d}t\Big| \le C||\overline{u}\times\tilde{\Delta}_h\overline{u}||_{L^2(\Omega_T)}||\psi||_{L^\infty}\|u - \overline{u}\|_{L^2(\Omega_T)}.$$

With the bounds of Lemma 3.1 and since $\overline{u} \to u$ in $L^2(\Omega_T)$ we verify that $IV \to 0$ as $(h,k,\varepsilon) \to 0$. In order to verify the assertion of the lemma we rewrite (1.7) as

$$\big(\hat{u}_t, \phi_h\big)_h + \big(\overline{u}\times(\overline{u}\times\tilde{\Delta}_h\overline{u}), \phi_h\big)_h = \big(\overline{u}\times r^+, \phi_h\big)_h$$

for $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ and almost all $t \in (0,T)$. The choice $\phi_h(t,\cdot) = \mathcal{I}_h\big[(\overline{u}\times\psi_h)(t,\cdot)\big]$ leads to

$$\big(\hat{u}_t, \overline{u}\times\psi_h\big)_h + \big(\overline{u}\times(\overline{u}\times\tilde{\Delta}_h\overline{u}), \overline{u}\times\psi_h\big)_h = \big(\overline{u}\times r^+, \overline{u}\times\psi_h\big)_h.$$

The vector product identity (2.6) and the definition of $\tilde{\Delta}_h$ imply that

$$\big(\hat{u}_t, \overline{u}\times\psi_h\big)_h + \big(\nabla\overline{u}, \nabla\mathcal{I}_h[\overline{u}\times\psi_h]\big) = \big(\overline{u}\times r^+, \overline{u}\times\psi_h\big)_h + \big(\big[1 - |\overline{u}|^2\big]\tilde{\Delta}_h\overline{u}, \overline{u}\times\psi_h\big)_h.$$

Thereby we verify that

$$\Big|\int_0^T \big(u_t, u\times\psi\big) + \big(\nabla u, \nabla[u\times\psi]\big)\,\mathrm{d}t\Big| \le I + II + III + IV \to 0$$

as$(h,k,\varepsilon) \to 0$. □

*Proof of Theorem A.* Theorem A follows from a combination of Lemmas 3.1, 3.2, and 3.3. □

## 4. Construction of $u^{(j)}$ and $r^{(j)}$

The following algorithm approximates equation (1.6) and provides the sequences $(u^{(j)})_{j=0,\dots,J}$ and $(r^{(j)})_{j=0,\dots,J}$ in Theorem A.

**Algorithm** $(B)$**.** *Input:* a time-step size $k > 0$, a positive integer $J$, a regular triangulation $\mathcal{T}_h$ of $\Omega$, a parameter $\varepsilon > 0$, and $u^{(0)} \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that $|u^{(0)}(z)| = 1$ for all $z \in \mathcal{N}_h$.

(a) Set $j = 0$, $r^{(0)} = 0$.
(b) Set $w^{(j+1,0)} = u^{(j)}$.
  (b1) Set $\ell = 0$.
  (b2) Compute $w^{(j+1,\ell+1)} \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that

$$\frac{2}{k}\big(w^{(j+1,\ell+1)}, v_h\big)_h + \big(w^{(j+1,\ell+1)} \times (w^{(j+1,\ell)} \times \tilde{\Delta}_h w^{(j+1,\ell)}), v_h\big)_h = \frac{2}{k}\big(u^{(j)}, v_h\big)_h$$

  for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. Set $e^{(j+1,\ell+1)} = w^{(j+1,\ell+1)} - w^{(j+1,\ell)}$ and

$$r^{(j+1)} = w^{(j+1,\ell+1)} \times \tilde{\Delta}_h e^{(j+1,\ell+1)} + e^{(j+1,\ell+1)} \times \tilde{\Delta}_h w^{(j+1,\ell)}.$$

  (b3) Go to (c) if $||r^{(j+1)}||_h \le \varepsilon$; set $\ell = \ell + 1$ and continue with (b2) otherwise.
(c) Set $u^{(j+1)} = 2w^{(j+1,\ell+1)} - u^{(j)}$.
(d) Stop if $j + 1 = J$; set $j = j + 1$ and go to (b) otherwise.

*Output:* Sequences $(u^{(j)})_{j=0,1,..,J}$ and $(r^{(j)})_{j=0,1,...,J}$.

The following theorem shows that all steps in Algorithm $(B)$ are well-defined and that the algorithm terminates if $k = O(h^2)$.

**Theorem 4.1.** *(i) Let* $0 \le j \le J-1$ *and* $u^{(j)} \in \mathcal{S}^1(\mathcal{T}_h)^3$ *such that* $|u^{(j)}(z)| = 1$ *for all* $z \in \mathcal{N}_h$*. Then, for all* $\ell \ge 0$ *the system in (b2) admits a unique solution* $w^{(j+1,\ell+1)} \in \mathcal{S}^1(\mathcal{T}_h)^3$ *such that* $\big|w^{(j+1,\ell+1)}(z)\big| \le 1$ *and* $\big|\big(2w^{(j+1,\ell+1)} - u^{(j)}\big)(z)\big| = 1$ *for all* $z \in \mathcal{N}_h$*. Moreover, there holds*

$$||e^{(j+1,\ell+1)}||_h \le c_1 k h_{min}^{-2} ||e^{(j+1,\ell)}||_h. \tag{4.1}$$

*(ii) If for all* $0 \le j \le J-1$ *the iteration (b1)-(b3) converges then there holds* (1.7).

*Proof.* The left-hand side in (b2) defines a continuous bilinear form $a\big(w^{(j+1,\ell+1)}, v_h\big)$ on $[\mathcal{S}^1(\mathcal{T}_h)^3]^2$. The choice $v_h = w^{(j+1,\ell+1)}$ shows that $a$ is elliptic. Hence, there exists a unique solution $w^{(j+1,\ell+1)}$ in (b2). On choosing $v_h = w^{(j+1,\ell+1)}(z)\varphi_z$ for $z \in \mathcal{N}_h$ we verify that $|w^{(j+1,\ell+1)}(z)| \le |u^{(j)}(z)| = 1$. Defining $\tilde{u}^{(j+1)} = 2w^{(j+1,\ell+1)} - u^{(j)}$, (b2) implies for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ that

$$\frac{1}{k}\big(\tilde{u}^{(j+1)} - u^{(j)}, v_h\big)_h + \big(w^{(j+1,\ell+1)} \times (w^{(j+1,\ell)} \times \tilde{\Delta}_h w^{(j+1,\ell)}), v_h\big)_h = 0.$$

Choosing $v_h = w^{(j+1,\ell+1)}(z)\varphi_z$ for $z \in \mathcal{N}_h$ in (b2) and noting that $w^{(j+1,\ell+1)} = \big(\tilde{u}^{(j+1)} + u^{(j)}\big)/2$ yields that $|\tilde{u}^{(j+1)}(z)|^2 = |u^{(j)}(z)|^2 = 1$. We subtract two subsequent equations in (b2) and choose $v_h = e^{(j+1,\ell+1)}$ to verify that for $\ell \ge 1$ there holds

$$\begin{aligned}
\frac{2}{k}||e^{(j+1,\ell+1)}||_h^2 &= -\big(w^{(j+1,\ell)} \times (e^{(j+1,\ell)} \times \tilde{\Delta}_h w^{(j+1,\ell)}), e^{(j+1,\ell+1)}\big)_h \\
&\quad - \big(w^{(j+1,\ell)} \times (w^{(j+1,\ell-1)} \times \tilde{\Delta}_h e^{(j+1,\ell)}), e^{(j+1,\ell+1)}\big)_h \\
&\le ||w^{(j+1,\ell)}||_{L^\infty} ||e^{(j+1,\ell)}||_h ||\tilde{\Delta}_h w^{(j+1,\ell)}||_{L^\infty} ||e^{(j+1,\ell+1)}||_h \\
&\quad + ||w^{(j+1,\ell)}||_{L^\infty} ||w^{(j+1,\ell-1)}||_{L^\infty} ||\tilde{\Delta}_h e^{(j+1,\ell)}||_h ||e^{(j+1,\ell+1)}||_h.
\end{aligned}$$

Employing the estimates in (2.3) and using $||w^{(j+1,\ell)}||_{L^\infty} \le 1$ we deduce (4.1). Suppose that for some $\ell \ge 0$ we have $u^{(j+1)} = 2w^{(j+1,\ell+1)} - u^{(j)}$, in particular $u^{(j+1/2)} = w^{(j+1,\ell+1)}$. Then, the system in (b2) implies that for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ there holds

$$\begin{aligned}
&\big(d_t u^{(j+1)}, v_h\big)_h + \big(u^{(j+1/2)} \times (u^{(j+1/2)} \times \tilde{\Delta}_h u^{(j+1/2)}), v_h\big)_h \\
&\quad = \big(u^{(j+1/2)} \times (w^{(j+1,\ell+1)} \times \tilde{\Delta}_h w^{(j+1,\ell+1)}), v_h\big)_h - \big(u^{(j+1/2)} \times (w^{(j+1,\ell)} \times \tilde{\Delta}_h w^{(j+1,\ell)}), v_h\big)_h \\
&\quad = \big(u^{(j+1/2)} \times (e^{(j+1,\ell+1)} \times \tilde{\Delta}_h w^{(j+1,\ell+1)}), v_h\big)_h + \big(u^{(j+1/2)} \times (w^{(j+1,\ell)} \times \tilde{\Delta}_h e^{(j+1,\ell+1)}), v_h\big)_h \\
&\quad = \big(u^{(j+1/2)} \times r^{(j+1)}, v_h\big)_h,
\end{aligned}$$

which proves (ii) and finishes the proof of the Theorem. □

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Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany
*E-mail address*: `sba@math.hu-berlin.de`

Department of Mathematics, ETH Zurich, CH-8092 Zurich, Switzerland
*E-mail address*: `apr@math.ethz.ch`