CONVERGENCE OF AN IMPLICIT, CONSTRAINT PRESERVING FINITE ELEMENT DISCRETIZATION OF p-HARMONIC HEAT FLOW INTO SPHERES

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ABSTRACT. We propose an implicit discretization of p-harmonic map heat flow into the sphere $\mathbf{S}^2$ that enjoys a discrete energy inequality and converges under a mild mesh constraint to a weak solution. A fully practical iterative scheme that approximates the solution of the nonlinear system of equations in each time step is proposed and analyzed.

1. INTRODUCTION

Given $1 < p < \infty$, a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, and $\mathbf{u}_0 : \Omega \to \mathbf{S}^2$, the p-harmonic heat flow into the sphere $\mathbf{u} : \Omega_T \to \mathbf{S}^2$ solves

$$\mathbf{u}_t - \Delta_p \mathbf{u} = |\nabla \mathbf{u}|^p \mathbf{u} \quad \text{on } \Omega_T\,, \qquad \partial_{\mathbf{n}} \mathbf{u} = 0 \quad \text{on } \partial\Omega_T\,, \tag{1.1}$$

$$|\mathbf{u}| = 1 \quad \text{in } \Omega_T\,, \qquad \mathbf{u}(0,\cdot) = \mathbf{u}_0 \quad \text{on } \Omega\,, \tag{1.2}$$

for $T > 0$. Here, $\Omega_T := (0,T) \times \Omega$ and $\partial\Omega_T := (0,T) \times \partial\Omega$, with $\partial\Omega$ being the boundary of Ω and $\mathbf{n} \in \mathbf{S}^{N-1}$ denoting the outer unit normal to $\partial\Omega$. Solutions to this problem have been studied intensively over the last fifteen years, starting with the case $p = 2$ [11], followed by $p > 2$ (existence [12, 22], nonuniqueness [20]), and $1 < p < 2$ (existence and nonuniqueness [14, 24]). Weak solutions to (1.1)–(1.2) satisfy (1.1) in a distributional sense, the initial condition in (1.2) in the sense of traces for $\mathbf{u}_0 \in W^{1,p}(\Omega, \mathbf{S}^2)$, and the energy inequality (cf. [26])

$$\int_0^t \| \mathbf{u}_t(s,\cdot) \|_{L^2}^2 \, \mathrm{d}s + E_p\big(\mathbf{u}(t,\cdot)\big) \le E_p(\mathbf{u}_0) \quad \text{for a.e. } t \in (0,T)\,, \tag{1.3}$$

where $E_p(\boldsymbol{\varphi}) = \frac{1}{p} \int_\Omega |\nabla \boldsymbol{\varphi}|^p \, \mathrm{d}\mathbf{x}$. For the limit $t \to \infty$, (1.3) motivates the conjecture that there exists a subsequence $\{t_{k'}\} \subset \{t_k\}$, for $t_k \to \infty$, such that $\mathbf{u}^* = \lim_{k' \to \infty} \mathbf{u}(t_{k'}, \cdot)$ is a (weakly) p-harmonic map (not necessarily energy minimizing). This is known for the case $p = 2$ and for $1 < p < \infty$ in the case of small initial data [15].

In this paper, we propose a direct implicit discretization of (1.1)–(1.2) that converges under at most a mild mesh constraint and uses lowest order conforming finite elements. It is known that classical solutions to (1.1)–(1.2) conserve $|\mathbf{u}| = 1$ in Ω_T if $|\mathbf{u}_0| = 1$ in Ω. Unfortunately, this property is not valid any more for its discretization, due to damping character of most implicit temporal discretization schemes and restricted flexibility of used finite element functions.

As is shown in [10], existing weak solutions may show finite-time blow-up behavior, even for $p = N = 2$. In [7], and motivated by [2], an explicit fully discrete method which satisfies the side constraint at every spatial mesh node is developed, but its convergence towards weak solutions of

1991 *Mathematics Subject Classification.* 65M12, 65M60, 35K55, 35Q35.

Key words and phrases. heat flow p-harmonic map, finite element method, fully discrete scheme, convergence.

*Supported by Deutsche Forschungsgemeinschaft through the DFG Research Center MATHEON 'Mathematics for key technologies' in Berlin

(1.1)–(1.2) is established for (i) $p \in (1,\infty)$ in case $N=2$ and (ii) $p \in (1,2] \cup [3,\infty)$ in case $N=3$, provided that

$$(1.4)\quad k \leq o\Big(\min\big\{h^{\frac{p}{p-1}}, h^{p+\frac{N}{2}}\big\}\Big) \quad \text{for } 1<p<2\,, \quad k \leq o\Big(\min\big\{h^{p}, h^{1+N(1-\frac{1}{p})}\big\}\Big) \quad \text{for } 2 \leq p < \infty\,,$$

where k and h denote the time-step size and the maximal mesh-size of the triangulation $\mathcal{T}_h$ of Ω, respectively. Finite-time blow-up behavior is computationally evidenced for the analytically open case $p<2$.

We use the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \langle \mathbf{a},\mathbf{c}\rangle\mathbf{b} - \langle \mathbf{a},\mathbf{b}\rangle\mathbf{c}$, for all $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbf{R}^3$, together with $(1.2)_1$, to restate problem (1.1)–(1.2) as

$$(1.5)\qquad \mathbf{u}_t - \mathbf{u}\times(\mathbf{u}\times\Delta_p\mathbf{u}) = 0 \qquad \text{in } \Omega_T.$$

Given a time-step size $k>0$ and the lowest order finite element space $\mathbf{V}_h \subset W^{1,p}(\Omega;\mathbb{R}^3)$ subordinate to a quasiuniform regular triangulation $\mathcal{T}_h$ of Ω into triangles or tetrahedra of maximal mesh-size $h>0$, our approximation scheme reads as follows:

Algorithm 1.1. *Let $\mathbf{U}^0 \in \mathbf{V}_h$ be given. For every $j \geq 0$ let $\mathbf{U}^{j+1} \in \mathbf{V}_h$ solve*

$$(1.6)\qquad (d_t\mathbf{U}^{j+1},\mathbf{\Phi})_h + \Big(\overline{\mathbf{U}}^{j+1/2}\times(\overline{\mathbf{U}}^{j+1/2}\times\tilde{\Delta}_{p,h}\mathbf{U}^{j+1}),\mathbf{\Phi}\Big)_h = 0 \quad \forall \mathbf{\Phi}\in\mathbf{V}_h\,.$$

Here, $(\cdot,\cdot)_h$ denotes a discrete version (reduced integration) of the inner product in $L^2(\Omega,\mathbb{R}^3)$, $\tilde{\Delta}_{p,h} : W^{1,p}(\Omega,\mathbb{R}^3) \to \mathbf{V}_h$ is a discrete version of the p-Laplacian; furthermore, we use $d_t\boldsymbol{\varphi}^j := k^{-1}\big(\boldsymbol{\varphi}^j - \boldsymbol{\varphi}^{j-1}\big)$ and $\overline{\boldsymbol{\varphi}}^{j-1/2} := \frac{1}{2}(\boldsymbol{\varphi}^j + \boldsymbol{\varphi}^{j-1})$ for $j \geq 1$ and a sequence $\{\boldsymbol{\varphi}^j\}_{j\geq 0} \subset \mathbf{V}_h$; we refer the reader to Section 2 for details.

Strong solutions to (1.1)-(1.2) solve (1.5). In contrast, this property need not hold for corresponding discretizations, due to competition of local and nonlocal aspects inherent to fully discrete finite-element based methods, cf. [6]. In this paper, we construct discretizations, where iterates converge under a mild mesh constraint to a weak solution of (1.1)–(1.2). In [5], unconditional convergence of a fully implicit discretization of (1.1)–(1.2) is verified for $p=2$. The case $1<p<\infty$ is more involved: Lemma 3.1 below states conservation of the sphere constraint at nodes of the spatial mesh $\mathcal{T}_h$ and verifies a discrete energy inequality for solutions to Algorithm 1.1 (opposed to an equality for $p=2$). It turns out that these properties, together with a compactness argument (Lemma 2.3) are sufficient to achieve convergence of solutions of Algorithm 1.1 on right-angled meshes $\mathcal{T}_h$, and in case of a mild mesh constraint. We note that the result can be significantly improved if $p=2$, cf. [5].

Assumption 1.1. *Suppose that $p \in \big(\frac{2N}{N+2},2\big] \cup [N,\infty)$ and that $\mathcal{T}_h$ is a quasiuniform triangulation of Ω into triangles or tetrahedra. If $p=N=3$ or $\frac{2N}{N+2} < p < 2$ assume that each $K \in \mathcal{T}_h$ is right-angled.*

Theorem 1.1. *Suppose that Assumption 1.1 holds, let $T>0$, and assume $\mathbf{U}^0 \to \mathbf{u}^0$ strongly in $W^{1,p}(\Omega,\mathbb{R}^3)$ as $h \to 0$. For a pair (h,k) let $\boldsymbol{\mathcal{U}} \equiv \boldsymbol{\mathcal{U}}_{h,k}$ be obtained by linear interpolation of the existing iterates $\mathbf{U}^j$ of Algorithm 1.1. Assume $k = o\big(h^{1+N\min\{0,\frac{2-p}{2p}\}}\big)$. Then there exists a subsequence (not relabeled) of $\big(\boldsymbol{\mathcal{U}}\big)$, such that for $h,k \to 0$,*

$$\boldsymbol{\mathcal{U}} \rightharpoonup \mathbf{u} \quad \textit{weakly* in } L^\infty\big(0,T;W^{1,p}(\Omega,\mathbb{R}^3)\big)\,, \qquad \boldsymbol{\mathcal{U}}_t \rightharpoonup \mathbf{u}_t \quad \textit{weakly in } L^2\big(\Omega_T,\mathbb{R}^3\big)\,,$$

where $\mathbf{u} : \Omega_T \to \mathbb{R}^3$ is a weak solution to (1.1)–(1.2).

The remainder of this paper is organized as follows: Preliminaries are stated in Section 2. Theorem 1.1 is verified in Section 3; a linear, iterative scheme with stopping criterion is proposed in Section 4 and convergence is established. Computational experiments are reported in Section 5, and conclusions are drawn in Section 6.

2. Preliminaries

We define the nonlinear Sobolev space

$$W^{1,p}(\Omega,\mathbf{S}^2)=\Big\{\mathbf{v}\in W^{1,p}(\Omega,\mathbb{R}^3)\big|\ \mathbf{v}(\mathbf{x})\in\mathbf{S}^2\quad\text{for almost all }\mathbf{x}\in\Omega\Big\}\,,$$

which is equipped with the topology inherited from the one of $W^{1,p}(\Omega,\mathbb{R}^3)$. Critical points $\mathbf{u}\in W^{1,p}(\Omega,\mathbf{S}^2)$ of E_p for $p\in(1,\infty)$ can be characterized as solutions to the Euler-Lagrange equation

$$-\Delta_p\mathbf{u}=|\nabla\mathbf{u}|^p\mathbf{u}\quad\text{on }\Omega\,,\qquad\partial_{\mathbf{n}}\mathbf{u}=0\quad\text{on }\partial\Omega\,. \tag{2.1}$$

If a map $\mathbf{u}\in W^{1,p}(\Omega,\mathbf{S}^2)$ satisfies (2.1) in the sense of distributions, $\mathbf{u}$ is called a weakly p-harmonic map; regularity properties of energy minimizing p-harmonic maps are reviewed in [16]. The p-harmonic flow (1.1)–(1.2) was first studied in [12, 21]. We now make precise what we mean by a weak solution to (1.1)–(1.2).

Definition 2.1. *Let $\mathbf{u}_0\in W^{1,p}(\Omega,\mathbf{S}^2)$ and $1<p<\infty$, then $\mathbf{u}$ is a weak solution to (1.1)–(1.2) if $\mathbf{u}:\mathbb{R}^+\times\Omega\to\mathbb{R}^3$ satisfies*

1. $\mathbf{u}\in L^\infty\big(0,T;W^{1,p}(\Omega,\mathbb{R}^3)\big)\cap H^1\big(0,T;L^2(\Omega,\mathbb{R}^3)\big)$ *for all $T>0$;*
2. (1.1) *holds in the sense of distributions;*
3. $|\mathbf{u}|=1$ *a.e. on $\mathbb{R}^+\times\Omega$;*
4. $\mathbf{u}(0,\cdot)=\mathbf{u}_0(\cdot)$ *in the sense of traces.*

Verification of the existence of a weak solution to (1.1)–(1.2) uses monotonicity arguments for a penalization approach to approximate the p-harmonic flow on the space $W^{1,p}(\Omega,\mathbb{R}^3)$. A compactness argument then identifies limits of terms of a wedged version of the penalized problem with corresponding terms appearing in a wedged version of (1.1). The (possibly non-unique [24]) weak solution is known to satisfy the energy inequality (1.3); see [26, 19] for further details. Our proof of Theorem 1.1 can be considered a constructive proof of existence of weak solutions to (1.1)–(1.2) in a non-penalized setting. The following technical results (i), (ii) may be found in [9], and (iii) is proved in [25, Lemma 2.6, (iii)] (for $\delta=1$).

Lemma 2.1. *For $\gamma\geq 0$ there exist positive constants $C_i=C_i(p,N)$, $i=1,2,3$, such that for all $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{3N}$ there holds*

$$\text{(i)}\qquad \big||\mathbf{P}|^{p-2}\mathbf{P}-|\mathbf{Q}|^{p-2}\mathbf{Q}\big|\leq C_1\big(|\mathbf{P}|+|\mathbf{Q}|\big)^{p-2+\gamma}\big|\mathbf{P}-\mathbf{Q}\big|^{1-\gamma}\,,$$

$$\text{(ii)}\qquad \big\langle|\mathbf{P}|^{p-2}\mathbf{P}-|\mathbf{Q}|^{p-2}\mathbf{Q},\mathbf{P}-\mathbf{Q}\big\rangle\geq C_2\Big(|\mathbf{P}|+|\mathbf{Q}|\Big)^{p-2-\gamma}\big|\mathbf{P}-\mathbf{Q}\big|^{2+\gamma}\,,$$

and if $1<p<2$ and $\delta\in(0,1)$,

$$\text{(iii)}\quad \Big|\big(|\mathbf{P}|^2+\delta\big)^{\frac{p-2}{2}}\mathbf{P}-\big(|\mathbf{Q}|^2+\delta\big)^{\frac{p-2}{2}}\mathbf{Q}\Big|\leq C_3\left(\delta^{1/2}+|\mathbf{P}|+|\mathbf{Q}-\mathbf{P}|\right)^{p-2}|\mathbf{P}-\mathbf{Q}|\,.$$

The effect of regularization in the singular regime $1<p<2$ is considered next.

Lemma 2.2. *Suppose $\mathbf{w}\in L^p(\Omega,\mathbb{R}^{3N})$, $1<p<2$. Then*

$$\Big\|\Big(|\mathbf{w}|^{p-2}-\big(|\mathbf{w}|^2+\delta\big)^{\frac{p-2}{2}}\Big)\mathbf{w}\Big\|_{L^\infty}\leq C\delta^{(p-1)/2}\,.$$

Proof. Let $\mathcal{B}_\sigma:=\big\{\mathbf{x}\in\Omega:\ |\mathbf{w}(\mathbf{x})|\geq\sigma\big\}\subset\Omega$, with $0<\sigma=\delta^\gamma$ and $\gamma>0$. On $\Omega\setminus\mathcal{B}_\sigma$ the asserted estimate follows from

$$|\mathbf{w}|^{p-1}+\frac{|\mathbf{w}|}{\big(|\mathbf{w}|^2+\delta\big)^{\frac{2-p}{2}}}\leq 2|\mathbf{w}|^{p-1}\leq C\delta^{\gamma(p-1)}\,.$$

On $\mathcal{B}_\sigma$, we may use the mean value theorem to get an upper bound,

$$\frac{(|\,\mathbf{w}\,|^2+\delta)^{\frac{2-p}{2}} - |\,\mathbf{w}\,|^{2-p}}{(|\,\mathbf{w}\,|^2+\delta)^{\frac{2-p}{2}}}|\,\mathbf{w}\,|^{p-1} \leq |\,\mathbf{w}\,|^{p-2}\frac{2-p}{2}\,|\,\mathbf{w}\,|^{-p}\delta\,|\,\mathbf{w}\,|^{p-1} \leq \frac{2-p}{2}\,\delta^{1+\gamma(p-3)}\,,$$

and hence $0 < \gamma < \frac{1}{3-p}$ is sufficient for convergence of this term; optimal balancing leads to $\gamma = 1/2$. $\square$

Given a quasiuniform, regular triangulation $\mathcal{T}_h$ of Ω we define the lowest order finite element space $\mathbf{V}_h \subset W^{1,\infty}(\Omega,\mathbb{R}^3)$ by

$$\mathbf{V}_h = \Big\{\boldsymbol{\phi}_h \in C(\overline{\Omega},\mathbb{R}^3) : \boldsymbol{\phi}_h|_K \in \mathcal{P}_1(K,\mathbb{R}^3) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,$$

where $\mathcal{P}_1(K,\mathbb{R}^3)$ denotes the set of polynomials of total degree less or equal to one restricted to the element $K \in \mathcal{T}_h$. Given the set of nodes $\big\{\mathbf{x}_\ell : \ell \in L\big\}$ of the triangulation $\mathcal{T}_h$, the nodal interpolation operator $\boldsymbol{\mathcal{I}}_h : C(\overline{\Omega},\mathbb{R}^3) \to \mathbf{V}_h$ satisfies $\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}(\mathbf{x}_\ell) = \boldsymbol{\phi}(\mathbf{x}_\ell)$ for all $\ell \in L$. Given functions $\mathbf{f},\mathbf{g} \in L^2(\Omega,\mathbb{R}^3)$ and letting $\langle\cdot,\cdot\rangle$ denote the inner product in $\mathbb{R}^M$, $M>0$, we set

$$(\mathbf{f},\mathbf{g}) = \int_\Omega \langle \mathbf{f},\mathbf{g}\rangle\,\mathrm{d}\mathbf{x}\,.$$

For continuous functions $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in C(\overline{\Omega},\mathbb{R}^3)$ we define

$$\big(\boldsymbol{\phi},\,\boldsymbol{\psi}\big)_h = \int_\Omega \mathcal{I}_h\big(\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle\big)\,\mathrm{d}\mathbf{x} = \sum_{\ell\in L}\beta_\ell\big\langle\boldsymbol{\phi}(\mathbf{x}_\ell),\boldsymbol{\psi}(\mathbf{x}_\ell)\big\rangle\,,$$

for certain weights $\beta_\ell > 0$, $\ell \in L$. If for each $\ell \in L$ we denote by $\varphi_\ell \in C(\overline{\Omega})$ the nodal basis function which is $\mathcal{T}_h$ elementwise affine and satisfies $\varphi_\ell(\mathbf{x}_\ell) = 1$ and $\varphi_\ell(\mathbf{x}_m) = 0$ for all $m \in L\setminus\{\ell\}$, then we have $\beta_\ell = \int_\Omega \varphi_\ell\,\mathrm{d}\mathbf{x}$. We define $||\boldsymbol{\phi}||_h^2 = \big(\boldsymbol{\phi},\,\boldsymbol{\phi}\big)_h$ and note that

$$\|\,\boldsymbol{\phi}_h\,\|_{L^2}^2 \leq \|\,\boldsymbol{\phi}_h\,\|_h^2 \leq (N+2)\,\|\,\boldsymbol{\phi}_h\,\|_{L^2}^2, \tag{2.2}$$

$$\big|(\boldsymbol{\chi}_h,\boldsymbol{\eta}_h)_h - (\boldsymbol{\chi}_h,\boldsymbol{\eta}_h)\big| \leq Ch\,\|\,\boldsymbol{\chi}_h\,\|_{L^2}\|\,\nabla\boldsymbol{\eta}_h\,\|_{L^2}\,, \tag{2.3}$$

for all $\boldsymbol{\phi}_h,\boldsymbol{\eta}_h,\boldsymbol{\chi}_h \in \mathbf{V}_h$. The discrete p-Laplacian $\tilde{\Delta}^\delta_{p,h} : W^{1,p}(\Omega,\mathbb{R}^3) \to \mathbf{V}_h$, where $\delta = 0$ in case $p \geq 2$ and $\delta \geq 0$ else, is defined through the identity

$$(\tilde{\Delta}^\delta_{p,h}\boldsymbol{\phi},\boldsymbol{\chi}_h)_h = -\Big(\big[|\,\nabla\boldsymbol{\phi}\,|^2+\delta\big]^{\frac{p-2}{2}}\nabla\boldsymbol{\phi},\nabla\boldsymbol{\chi}_h\Big) \quad \forall\boldsymbol{\chi}_h \in \mathbf{V}_h\,; \tag{2.4}$$

for notational brevity, we let $|\cdot|_\delta^2 := \big(|\cdot|^2+\delta\big)$ and $\tilde{\Delta}_{p,h} := \tilde{\Delta}^0_{p,h}$. Choosing $\boldsymbol{\chi}_h = \tilde{\Delta}^\delta_{p,h}\boldsymbol{\phi}_h$ in (2.4) and using (2.2), an inverse estimate yields to

$$\begin{aligned} ||\tilde{\Delta}^\delta_{p,h}\boldsymbol{\phi}_h||_h^2 &= -\Big(|\,\nabla\boldsymbol{\phi}_h\,|_\delta^{p-2}\nabla\boldsymbol{\phi}_h,\nabla\tilde{\Delta}^\delta_{p,h}\boldsymbol{\phi}_h\Big) \\ &\leq C\,\big\||\,\nabla\boldsymbol{\phi}_h\,|^{p-1}\big\|_{L^{\frac{p}{p-1}}}\,||\nabla\tilde{\Delta}^\delta_{p,h}\boldsymbol{\phi}_h||_{L^p} \leq Ch^{-1}\,||\nabla\boldsymbol{\phi}_h||_{L^p}^{p-1}\,||\tilde{\Delta}^\delta_{p,h}\boldsymbol{\phi}_h||_{L^p}\,, \end{aligned}$$

and hence

$$\|\,\tilde{\Delta}^\delta_{p,h}\boldsymbol{\phi}_h\,\|_h \leq Ch^{-1+N\min\{0,\frac{2-p}{2p}\}}\,\|\,\nabla\boldsymbol{\phi}_h\,\|_{L^p}^{p-1}\,. \tag{2.5}$$

Using an inverse estimate in (2.5) implies

$$\|\,\tilde{\Delta}^\delta_{p,h}\boldsymbol{\phi}_h\,\|_{L^\infty} \leq Ch^{-(1+\frac{N}{2})+N\min\{0,\frac{2-p}{2p}\}}\,\|\,\nabla\boldsymbol{\phi}_h\,\|_{L^p}^{p-1}\,. \tag{2.6}$$

Here and below, $C = C(\Omega,\mathcal{T}_h,p) > 0$ denotes a generic constant, which depends on $\Omega \subset \mathbb{R}^N$, the geometry of $\mathcal{T}_h$, p, and $\mathbf{u}_0$.

The following lemma may be regarded as a discrete version of a compactness result in [12, 24].

Lemma 2.3. *Suppose that Assumption 1.1 holds. Let the sequences* $\big(\boldsymbol{\mathcal{U}}_t\big)_{h,k} \subset L^2\big(\Omega_T,\mathbb{R}^3\big)$ *and* $(\boldsymbol{\mathcal{U}}^+)_{h,k} \subset L^\infty\big(0,T;W^{1,p}(\Omega,\mathbb{R}^3)\big)$ *be such that for each* (h,k) *and almost all* $t \in (0,T)$ *there holds* $\boldsymbol{\mathcal{U}}^+(t,\cdot), \boldsymbol{\mathcal{U}}_t(t,\cdot) \in \mathbf{V}_h$ *and* $|\boldsymbol{\mathcal{U}}^+(t,\mathbf{x}_\ell)| = 1$ *for all* $\ell \in L$. *Suppose that for all* $\boldsymbol{\phi}_h \in L^\infty\big(0,T;\mathbf{V}_h\big)$,

$$\Big|\int_0^T \Big[(\boldsymbol{\mathcal{U}}_t, \boldsymbol{\mathcal{I}}_h[\boldsymbol{\mathcal{U}}^+ \times \boldsymbol{\phi}_h]) + \big(|\,\nabla \boldsymbol{\mathcal{U}}^+\,|^{p-2}\nabla\boldsymbol{\mathcal{U}}^+, \nabla \boldsymbol{\mathcal{I}}_h[\boldsymbol{\mathcal{U}}^+ \times \boldsymbol{\phi}_h]\big)\Big]\,\mathrm{d}t\Big| \leq \mathcal{H}(\|\,\boldsymbol{\phi}_h\,\|_{L^\infty(\Omega_T)})\,, \tag{2.7}$$

where $\mathcal{H} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is a homogeneous function. Assume that for* $k, h \to 0$,

(i) $\big(\boldsymbol{\mathcal{U}}\big)$ *converges weakly* $*$ *in* $L^\infty\big(0,T;W^{1,p}(\Omega,\mathbb{R}^3)\big)$,

(ii) $\big(\boldsymbol{\mathcal{U}}_t\big)$ *converges weakly in* $L^2\big(\Omega_T,\mathbb{R}^3\big)$.

Then, $\big(\boldsymbol{\mathcal{U}}^+\big)$ *is precompact in* $L^q\big(0,T;W^{1,q}(\Omega,\mathbb{R}^3)\big)$ *for all* $1 \leq q < p$.

A proof of Lemma 2.3 is technical and follows the ones of Lemmata 4.2, 4.3 in [7] on a line-by-line basis. Major part of it uses Assumption 1.1, properties (i)–(ii), and $|\boldsymbol{\mathcal{U}}^\pm(t,\mathbf{x}_\ell)\,| = 1$, for all $\ell \in L$ and $t \geq 0$, whereas (2.7) is used once to replace the weak form of the p-Laplacian of $(\boldsymbol{\mathcal{U}}^+)$.

3. Proof of Theorem 1.1

Theorem 1.1 is a consequence of the following assertion, putting $\varepsilon = \delta = 0$.

Theorem 3.1. *Suppose that Assumption 1.1 holds and let* $T > 0$. *Let* $k > 0$, $\varepsilon \geq 0$, *and* $\delta = 0$ *for* $p \geq 2$. *Suppose that we are given* $\{\mathbf{U}^j\}$, $\{\mathbf{R}^j\} \subset \mathbf{V}_h$ *satisfying for all* $0 \leq j \leq J$ *that* $\|\,\mathbf{R}^j\,\|_h \leq \varepsilon$, $|\,\mathbf{U}^0(\mathbf{x}_\ell)\,| = 1$ *for all* $\ell \in L$, $j = 0,1,...,J-1$, *and for all* $\boldsymbol{\Phi} \in \mathbf{V}_h$ *that*

$$(d_t\mathbf{U}^{j+1}, \boldsymbol{\Phi})_h + \Big(\overline{\mathbf{U}}^{j+1/2} \times (\overline{\mathbf{U}}^{j+1/2} \times \tilde{\Delta}^\delta_{p,h}\mathbf{U}^{j+1}), \boldsymbol{\Phi}\Big)_h = \big(\overline{\mathbf{U}}^{j+1/2} \times \mathbf{R}^{j+1}, \boldsymbol{\Phi}\big)_h\,. \tag{3.1}$$

Then, $|\,\mathbf{U}^j(\mathbf{x}_\ell)\,| = 1$ *for all* $\ell \in L$ *and* $j = 0,1,...,J$. *Given* $0 \leq t \leq T \leq Jk$ *such that* $t \in \big[jk,(j+1)k\big)$, *for* $0 \leq j \leq J-1$, *let*

$$\boldsymbol{\mathcal{U}}(t,\cdot) \equiv \boldsymbol{\mathcal{U}}_{h,k,\varepsilon,\delta}(t,\cdot) = \frac{t-jk}{k}\mathbf{U}^{j+1} + \frac{(j+1)k-t}{k}\mathbf{U}^j\,, \quad \boldsymbol{\mathcal{U}}^-(t,\cdot) := \mathbf{U}^j\,, \quad \boldsymbol{\mathcal{U}}^+(t,\cdot) := \mathbf{U}^{j+1}\,,$$

and $\overline{\boldsymbol{\mathcal{U}}} := \frac{1}{2}\big(\boldsymbol{\mathcal{U}}^- + \boldsymbol{\mathcal{U}}^+\big)$. *Let* $\mathbf{u}_0 \in \mathbf{W}^{1,p}(\Omega,\mathbb{R}^3)$, *and suppose* $\mathbf{U}^0 \to \mathbf{u}_0$ *in* $\mathbf{W}^{1,p}(\Omega,\mathbb{R}^3)$ *for* $h \to 0$. *If*

$$k = o\big(h^{1+N\min\{0,\frac{2-p}{2p}\}}\big) \qquad \text{and} \qquad \delta = o(h^{2/(p-1)})$$

then there exists a subsequence of $\big(\boldsymbol{\mathcal{U}}\big)$ *which converges weakly* in* $L^\infty\big(0,T;W^{1,p}(\Omega,\mathbb{R}^3)\big)$ *to a weak solution of* (1.1)–(1.2) *as* $(k,h,\varepsilon,\delta) \to 0$.

Existence of a sequence $\{\mathbf{U}^j\}_{j\geq 0}$ that solves (3.1), e.g., with $\mathbf{R}^j \equiv 0$ for $j = 0,1,...,J$, can be deduced with Brouwer's fixed point theorem; cf. e.g. [18, Corollary 1.1, p. 279]. The mesh constraint is necessary in (3.11) below to apply the compactness result of Lemma 2.3. The following lemma verifies discrete counterparts of an energy inequality valid for solutions to (3.1).

Lemma 3.1. *Suppose that* $|\mathbf{U}^0(\mathbf{x}_\ell)| = 1$ *for all* $\ell \in L$ *and* $\delta \geq 0$. *Then the sequences* $\{\mathbf{U}^j\}_{j\geq 0}$, $\{\mathbf{R}^j\}_{j\geq 0}$ *from* (3.1) *satisfy for all* $j \geq 0$

(i) $|\,\mathbf{U}^{j+1}(\mathbf{x}_\ell)\,| = 1 \quad \forall \ell \in L\,,$

(ii) $\displaystyle\frac{1}{p}d_t\|\,|\,\nabla\mathbf{U}^{j+1}\,|_\delta\,\|^p_{L^p} + (1-\varepsilon)\Big\|\overline{\mathbf{U}}^{j+1/2} \times \tilde{\Delta}^\delta_{p,h}\mathbf{U}^{j+1}\Big\|^2_h \leq \frac{1}{4\varepsilon}\,\|\,\mathbf{R}^{j+1}\,\|^2_h\,,$

(iii) $\displaystyle\frac{1}{p}\|\,|\,\nabla\mathbf{U}^{j+1}\,|_\delta\,\|^p_{L^p} + (1-\varepsilon)^2 k\sum_{n=0}^{j}\|\,d_t\mathbf{U}^{n+1}\,\|^2_h \leq \frac{1}{p}\|\,|\,\nabla\mathbf{U}^0\,|_\delta\,\|^p_{L^p} + \frac{5}{4\varepsilon}k\sum_{n=0}^{j}\|\,\mathbf{R}^{n+1}\,\|^2_h\,.$

Proof. Assertion (i) follows from choosing $\mathbf{\Phi}_h = \varphi_\ell \overline{\mathbf{U}}^{j+1/2}(\mathbf{x}_\ell) \in \mathbf{V}_h$, $\ell \in L$ in Algorithm 1.1, binomial formula, and the sphere condition for spatially discretized initial data at nodes. In order to verify (ii), we choose $\boldsymbol{\phi}_h = -\tilde{\Delta}^\delta_{p,h}\mathbf{U}^{j+1}$, and find by convexity of $|\nabla\cdot|^p_\delta$,

$$\frac{1}{p} d_t \| \, |\nabla \mathbf{U}^{j+1}|^p_\delta \|_{L^1} - \Big(\overline{\mathbf{U}}^{j+1/2} \times (\overline{\mathbf{U}}^{j+1/2} \times \tilde{\Delta}^\delta_{p,h}\mathbf{U}^{j+1}), \tilde{\Delta}^\delta_{p,h}\mathbf{U}^{j+1} \Big)_h \leq - \big(\overline{\mathbf{U}}^{j+1/2} \times \mathbf{R}^{j+1}, \tilde{\Delta}^\delta_{p,h}\mathbf{U}^{j+1} \big)_h .$$

Thanks to $(\mathbf{a}\times\mathbf{b},\mathbf{c}) = -(\mathbf{a}\times\mathbf{c},\mathbf{b})$, for all $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$, this verifies (ii). The estimate (iii) follows from (i)–(ii) by choosing $\boldsymbol{\phi}_h = d_t\mathbf{U}^{j+1}$ and using Young's inequality. ☐

In the sequel, let $\boldsymbol{\mathcal{R}}^+(t,\cdot) := \mathbf{R}^{j+1}$, for $jk \le t < (j+1)k$. Given any $0 < T' < T$, inequality (ii) in Lemma 3.1 may be rewritten as

$$\begin{aligned} &\frac{1}{p}\| \, |\nabla \boldsymbol{\mathcal{U}}^+(T',\cdot)|_\delta \|^p_{L^p} + (1-\varepsilon)\int_0^{T'} \Big\| \overline{\boldsymbol{\mathcal{U}}} \times \tilde{\Delta}^\delta_{p,h}\boldsymbol{\mathcal{U}}^+ \Big\|^2_h \,\mathrm{d}t \\ &\qquad\qquad \le \frac{1}{p}\| \, |\nabla \boldsymbol{\mathcal{U}}^+(0,\cdot)|_\delta \|^p_{L^p} + \frac{1}{4\varepsilon}\int_0^T \| \boldsymbol{\mathcal{R}}^+ \|^2_h \,\mathrm{d}t . \end{aligned} \tag{3.2}$$

This bound, together with Lemma 3.1 (iii), yields the existence of $\mathbf{u} \in L^\infty\big(0,T;W^{1,p}(\Omega;\mathbb{R}^3)\big)$ which is the weak limit (as $k,h,\varepsilon,\delta \to 0$) of a subsequence such that

$$\begin{aligned} &\boldsymbol{\mathcal{U}}_t \rightharpoonup \mathbf{u}_t \ \text{ in } L^2\big(\Omega_T,\mathbb{R}^3\big)\,, \quad \boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{U}}^\pm,\overline{\boldsymbol{\mathcal{U}}} \overset{*}{\rightharpoonup} \mathbf{u} \ \text{ in } L^\infty\big(0,T;W^{1,p}(\Omega,\mathbb{R}^3)\big)\,, \\ &\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{U}}^\pm,\overline{\boldsymbol{\mathcal{U}}} \to \mathbf{u} \ \text{ in } L^r(\Omega_T,\mathbb{R}^3), \quad r<\infty \text{ for } p \le N \text{ and } r=\infty \text{ for } p > N\,, \end{aligned} \tag{3.3}$$

by Kondrachov's compactness theorem [12], as well as

$$\nabla\boldsymbol{\mathcal{U}}^\pm, \nabla\boldsymbol{\mathcal{U}}, \nabla\overline{\boldsymbol{\mathcal{U}}} \rightharpoonup \nabla\mathbf{u} \ \text{ in } L^p(\Omega_T,\mathbb{R}^{3N})\,.$$

We use Lemma 3.1, (i), a Poincaré argument, and inverse estimates to prove that for all $K \in \mathcal{T}_h$ there holds

$$\begin{aligned} \big\| |\boldsymbol{\mathcal{U}}^\pm(t,\cdot)|^2 - 1 \big\|_{L^p(K)} &\le Ch \, \| \nabla\big(|\boldsymbol{\mathcal{U}}^\pm(t,\cdot)|^2\big) \|_{L^p(K)} \\ &\le Ch \, \| \nabla\boldsymbol{\mathcal{U}}^\pm(t,\cdot)) \|_{L^p(K)}\,. \end{aligned}$$

and hence $\| \, |\boldsymbol{\mathcal{U}}^\pm|^2 - 1 \|_{L^p(\Omega_T)} \le Ch$ which implies $|\mathbf{u}| = 1$ a.e. in Ω_T. Together with $|\overline{\boldsymbol{\mathcal{U}}}| \le 1$, and $|\,|\overline{\boldsymbol{\mathcal{U}}}|^2 - 1\,| = |\langle \mathbf{u} - \overline{\boldsymbol{\mathcal{U}}}, \mathbf{u} + \overline{\boldsymbol{\mathcal{U}}}\rangle| \le 2|\,\mathbf{u} - \overline{\boldsymbol{\mathcal{U}}}\,|$, we find

$$\| \, |\overline{\boldsymbol{\mathcal{U}}}|^2 - 1 \|_{L^2(\Omega_T)} \to 0 \qquad (k,h,\varepsilon,\delta \to 0)\,. \tag{3.4}$$

Equation (3.1) may be rewritten as follows: for all $\boldsymbol{\phi}_h(t) \in \mathbf{V}_h$ holds

$$\int_0^T (\boldsymbol{\mathcal{U}}_t, \boldsymbol{\phi}_h)_h \,\mathrm{d}t + \int_0^T \Big(\overline{\boldsymbol{\mathcal{U}}} \times (\overline{\boldsymbol{\mathcal{U}}} \times \tilde{\Delta}^\delta_{p,h}\boldsymbol{\mathcal{U}}^+), \boldsymbol{\phi}_h \Big)_h \,\mathrm{d}t = \int_0^T \big(\overline{\boldsymbol{\mathcal{U}}} \times \boldsymbol{\mathcal{R}}^+, \boldsymbol{\phi}_h \big)_h \,\mathrm{d}t\,. \tag{3.5}$$

Some manipulations show that we may apply Lemma 2.3 to the sequence $\big(\boldsymbol{\mathcal{U}}^+\big)$.

Lemma 3.2. *If $\delta = \mathcal{O}(h^{(2/(p-1)})$ and $k = \mathcal{O}(h^{1-N\min\{0,\frac{2-p}{2p}\}})$ then there holds for $\boldsymbol{\psi}_h(t) \in \mathbf{V}_h$*

$$\Big| \int_0^T \big(\boldsymbol{\mathcal{U}}_t, \boldsymbol{\mathcal{I}}_h[\boldsymbol{\mathcal{U}}^+ \times \boldsymbol{\psi}_h] \big) + \big(|\nabla\boldsymbol{\mathcal{U}}^+|^{p-2}\nabla\boldsymbol{\mathcal{U}}^+, \nabla\boldsymbol{\mathcal{I}}_h[\boldsymbol{\mathcal{U}}^+ \times \boldsymbol{\psi}_h] \big) \,\mathrm{d}t \Big| \le C \| \boldsymbol{\psi}_h \|_{L^\infty}\,. \tag{3.6}$$

Proof. The properties of the inner product $(\cdot,\cdot)_h$, $\|\boldsymbol{\mathcal{U}}^\pm\|_{L^\infty} = 1$ and (elementwise) H^1-stability of $\boldsymbol{\mathcal{I}}_h$ imply

$$\begin{aligned} \Big| \big(\boldsymbol{\mathcal{U}}_t, \boldsymbol{\mathcal{U}}^+ \times \boldsymbol{\psi}_h \big)_h - \big(\boldsymbol{\mathcal{U}}_t, \boldsymbol{\mathcal{I}}_h\big[\boldsymbol{\mathcal{U}}^+ \times \boldsymbol{\psi}_h\big] \big) \Big| &\le Ch \, \| \boldsymbol{\mathcal{U}}_t \|_{L^2} \, \| \nabla\boldsymbol{\mathcal{I}}_h\big[\boldsymbol{\mathcal{U}}^+ \times \boldsymbol{\psi}_h\big] \|_{L^2} \\ &\le C \| \boldsymbol{\mathcal{U}}_t \|_{L^2} \| \boldsymbol{\psi}_h \|_{L^\infty}\,. \end{aligned} \tag{3.7}$$

Lagrange's identity $\langle \mathbf{a}\times\mathbf{b},\mathbf{c}\times\mathbf{d}\rangle = \langle\mathbf{a},\mathbf{c}\rangle\langle\mathbf{b},\mathbf{d}\rangle - \langle\mathbf{b},\mathbf{c}\rangle\langle\mathbf{a},\mathbf{d}\rangle$ for $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{R}^3$ implies

$$\Big(\overline{\boldsymbol{\mathcal{U}}}\times(\overline{\boldsymbol{\mathcal{U}}}\times\tilde{\Delta}^{\delta}_{p,h}\boldsymbol{\mathcal{U}}^+),\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\Big)_h = \big(\overline{\boldsymbol{\mathcal{U}}}\times\tilde{\Delta}^{\delta}_{p,h}\boldsymbol{\mathcal{U}}^+,\boldsymbol{\psi}_h\big)_h + \Big([|\overline{\boldsymbol{\mathcal{U}}}|^2-1]\overline{\boldsymbol{\mathcal{U}}}\times\tilde{\Delta}^{\delta}_{p,h}\boldsymbol{\mathcal{U}}^+,\boldsymbol{\psi}_h\Big)_h. \tag{3.8}$$

The last term is estimated by terms that can be controlled with Lemma 3.1 and (3.4) by

$$\Big([|\overline{\boldsymbol{\mathcal{U}}}|^2-1]\overline{\boldsymbol{\mathcal{U}}}\times\tilde{\Delta}^{\delta}_{p,h}\boldsymbol{\mathcal{U}}^+,\boldsymbol{\psi}_h\Big)_h \le \big\||\overline{\boldsymbol{\mathcal{U}}}|^2-1\big\|_h\big\|\overline{\boldsymbol{\mathcal{U}}}\times\tilde{\Delta}^{\delta}_{p,h}\boldsymbol{\mathcal{U}}^+\big\|_h\|\boldsymbol{\psi}_h\|_{L^\infty}. \tag{3.9}$$

We rewrite the first term on the right-hand side of (3.8) as

$$\begin{aligned}\big(\overline{\boldsymbol{\mathcal{U}}}\times\tilde{\Delta}^{\delta}_{p,h}\boldsymbol{\mathcal{U}}^+,\boldsymbol{\psi}_h\big)_h &= \Big(|\nabla\boldsymbol{\mathcal{U}}^+|_\delta^{p-2}\nabla\boldsymbol{\mathcal{U}}^+,\nabla\boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]\Big)\\ &= \big(|\nabla\boldsymbol{\mathcal{U}}^+|^{p-2}\nabla\boldsymbol{\mathcal{U}}^+,\nabla\boldsymbol{\mathcal{I}}_h[\boldsymbol{\mathcal{U}}^+\times\boldsymbol{\psi}_h]\big) - I,\end{aligned} \tag{3.10}$$

where we use Lemma 2.2, $\overline{\boldsymbol{\mathcal{U}}} = \boldsymbol{\mathcal{U}}^+ - k\boldsymbol{\mathcal{U}}_t$, $|\overline{\boldsymbol{\mathcal{U}}}|\le 1$, and inverse estimates to bound

$$\begin{aligned}|I| \le&\Big|\Big(|\nabla\boldsymbol{\mathcal{U}}^+|^{p-2}\nabla\boldsymbol{\mathcal{U}}^+,\nabla\boldsymbol{\mathcal{I}}_h\big[(\overline{\boldsymbol{\mathcal{U}}}-\boldsymbol{\mathcal{U}}^+)\times\boldsymbol{\psi}_h\big]\Big)\Big|\\ &+\Big|\Big(|\nabla\boldsymbol{\mathcal{U}}^+|_\delta^{p-2}\nabla\boldsymbol{\mathcal{U}}^+ - |\nabla\boldsymbol{\mathcal{U}}^+|^{p-2}\nabla\boldsymbol{\mathcal{U}}^+,\nabla\boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]\Big)\Big|\\ \le& Ckh^{-1}\|\nabla\boldsymbol{\mathcal{U}}^+\|_{L^p}^{p-1}\|\boldsymbol{\mathcal{U}}_t\|_{L^p}\|\boldsymbol{\psi}_h\|_{L^\infty} + C\delta^{(p-1)/2}h^{-1}\|\boldsymbol{\psi}_h\|_{L^\infty}.\\ \le& Ckh^{-1}h^{\min\{0,N\frac{2-p}{2p}\}}\|\nabla\boldsymbol{\mathcal{U}}^+\|_{L^p}^{p-1}\|\boldsymbol{\mathcal{U}}_t\|_{L^2}\|\boldsymbol{\psi}_h\|_{L^\infty} + C\delta^{(p-1)/2}h^{-1}\|\boldsymbol{\psi}_h\|_{L^\infty}.\end{aligned}$$

A combination of (3.7)-(3.11) with (3.5) for $\boldsymbol{\phi}_h = \boldsymbol{\mathcal{I}}_h\big[\boldsymbol{\mathcal{U}}^+\times\boldsymbol{\psi}_h\big]$ implies the lemma. □

The choice $\boldsymbol{\phi}_h = \boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]$ where $\boldsymbol{\psi}_h = \boldsymbol{\mathcal{I}}_h\boldsymbol{\psi}$ for arbitrary $\boldsymbol{\psi}\in C^\infty(\overline{\Omega}_T,\mathbb{R}^3)$ in (3.5) leads to

$$\int_0^T(\boldsymbol{\mathcal{U}}_t,\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h)_h\,\mathrm{d}t + \int_0^T\Big(\overline{\boldsymbol{\mathcal{U}}}\times(\overline{\boldsymbol{\mathcal{U}}}\times\tilde{\Delta}^{\delta}_{p,h}\boldsymbol{\mathcal{U}}^+),\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\Big)_h\,\mathrm{d}t = \int_0^T\big(\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\mathcal{R}}^+,\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big)_h\,\mathrm{d}t. \tag{3.11}$$

We study limits of terms separately, starting with

$$\int_0^T(\boldsymbol{\mathcal{U}}_t,\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h)_h\,\mathrm{d}t \to \int_0^T(\mathbf{u}_t,\mathbf{u}\times\boldsymbol{\psi})\,\mathrm{d}t. \tag{3.12}$$

To verify this claim, consider

$$\begin{aligned}\big(\boldsymbol{\mathcal{U}}_t,\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big)_h - (\mathbf{u}_t,\mathbf{u}\times\boldsymbol{\psi}) =& \Big[\big(\boldsymbol{\mathcal{U}}_t,\boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]\big)_h - \big(\boldsymbol{\mathcal{U}}_t,\boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]\big)\Big]\\ &+\big(\boldsymbol{\mathcal{U}}_t,\boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]-\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big) + \big(\boldsymbol{\mathcal{U}}_t,\overline{\boldsymbol{\mathcal{U}}}\times\big[\boldsymbol{\psi}_h-\boldsymbol{\psi}\big]\big)\\ &+\big(\boldsymbol{\mathcal{U}}_t,\big[\overline{\boldsymbol{\mathcal{U}}}-\mathbf{u}\big]\times\boldsymbol{\psi}\big) + \big(\boldsymbol{\mathcal{U}}_t-\mathbf{u}_t,\mathbf{u}\times\boldsymbol{\psi}\big).\end{aligned} \tag{3.13}$$

Similar to the derivation of (3.7) we verify, using properties of nodal interpolation and $|\overline{\boldsymbol{\mathcal{U}}}|\le 1$,

$$\begin{aligned}&\Big|\big(\boldsymbol{\mathcal{U}}_t,\boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]\big)_h - \big(\boldsymbol{\mathcal{U}}_t,\boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]\big)\Big| + \Big|\Big(\boldsymbol{\mathcal{U}}_t,\boldsymbol{\mathcal{I}}_h\big[\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\big]-\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\Big)\Big| + \Big|\Big(\boldsymbol{\mathcal{U}}_t,\overline{\boldsymbol{\mathcal{U}}}\times\big[\boldsymbol{\psi}_h-\boldsymbol{\psi}\big]\Big)\Big|\\ &\quad\le Ch\|\boldsymbol{\mathcal{U}}_t\|_{L^2}\big(\|\nabla\overline{\boldsymbol{\mathcal{U}}}\|_{L^2}+1\big)\big(\|\boldsymbol{\psi}\|_{L^\infty}+\|\boldsymbol{\psi}\|_{W^{1,2}}\big)\\ &\quad\le Ch^{1+N\min\{0,\frac{p-2}{2p}\}}\|\boldsymbol{\mathcal{U}}_t\|_{L^2}\big(\|\nabla\overline{\boldsymbol{\mathcal{U}}}\|_{L^p}+1\big)\big(\|\boldsymbol{\psi}\|_{L^\infty}+\|\boldsymbol{\psi}\|_{W^{1,2}}\big).\end{aligned}$$

Since $\overline{\boldsymbol{\mathcal{U}}}\to\mathbf{u}$ in $L^2(\Omega_T)$ and $\boldsymbol{\mathcal{U}}_t\rightharpoonup\mathbf{u}_t$ in $L^2(\Omega_T,\mathbb{R}^3)$ the integral over $t\in(0,T)$ of the last two terms in (3.13) tends to zero for $(k,h,\varepsilon,\delta)\to 0$. This verifies (3.12), provided $p>\frac{2N}{2+N}$. We now verify

$$\int_0^T\Big(\overline{\boldsymbol{\mathcal{U}}}\times(\overline{\boldsymbol{\mathcal{U}}}\times\tilde{\Delta}^{\delta}_{p,h}\boldsymbol{\mathcal{U}}^+),\overline{\boldsymbol{\mathcal{U}}}\times\boldsymbol{\psi}_h\Big)_h\,\mathrm{d}t \to \int_0^T\Big(|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u},\nabla[\mathbf{u}\times\boldsymbol{\psi}]\Big)\,\mathrm{d}t. \tag{3.14}$$

As in the proof of Lemma 3.2 (equations (3.8) and (3.10)) we write

$$\Big(\overline{\mathcal{U}} \times (\overline{\mathcal{U}} \times \tilde{\Delta}^{\delta}_{p,h}\mathcal{U}^+), \overline{\mathcal{U}} \times \boldsymbol{\psi}_h\Big)_h = \Big([|\overline{\mathcal{U}}|^2 - 1]\overline{\mathcal{U}} \times \tilde{\Delta}^{\delta}_{p,h}\mathcal{U}^+, \boldsymbol{\psi}_h\Big)_h + \big(|\nabla\mathcal{U}^+|^{p-2}\nabla\mathcal{U}^+, \nabla\mathcal{I}_h[\mathcal{U}^+ \times \boldsymbol{\psi}_h]\big) - I. \tag{3.15}$$

By (3.2), (3.4), the integral over $t \in (0,T)$ of the first term on the right-hand side tends to zero for $(k,h,\varepsilon,\delta) \to 0$ while the term I can be bounded as in the proof of Lemma 3.2. The combination of Lemmata 2.3 and 3.2 yields

$$\mathcal{U}^{\pm}, \overline{\mathcal{U}} \to \mathbf{u} \quad \text{in } L^q\big(0,T; W^{1,q}(\Omega,\mathbb{R}^3)\big) \qquad (1 \le q < p) \tag{3.16}$$

and this together with Lemma 3.1 (ii) implies, cf. [24, Lemma 6],

$$|\nabla\mathcal{U}^+|^{p-2}\nabla\mathcal{U}^+ \rightharpoonup |\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u} \qquad \text{in } L^{\frac{p}{p-1}}(\Omega_T,\mathbb{R}^{3N}). \tag{3.17}$$

This additional property for $(\mathcal{U}^+)$, next to (3.3), is sufficient to verify (3.14). Thanks to $\langle\nabla\mathbf{u}, \nabla[\mathbf{u}\times\boldsymbol{\psi}]\rangle = \langle\nabla\mathbf{u}, \mathbf{u}\times\nabla\boldsymbol{\psi}\rangle$ and $\langle\nabla\mathcal{U}^+, \nabla[\mathcal{U}^+ \times \boldsymbol{\psi}_h]\rangle = \langle\nabla\mathcal{U}^+, \mathcal{U}^+ \times \nabla\boldsymbol{\psi}_h\rangle$, we conclude

$$\begin{aligned}
&\big(|\nabla\mathcal{U}^+|^{p-2}\nabla\mathcal{U}^+, \nabla\mathcal{I}_h[\mathcal{U}^+ \times \boldsymbol{\psi}_h]\big) - \big(|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}, \nabla[\mathbf{u}\times\boldsymbol{\psi}]\big)\\
&= \Big(|\nabla\mathcal{U}^+|^{p-2}\nabla\mathcal{U}^+, \nabla\big[\mathcal{I}_h[\mathcal{U}^+ \times \boldsymbol{\psi}_h] - \mathcal{U}^+ \times \boldsymbol{\psi}_h\big]\Big) + \Big(|\nabla\mathcal{U}^+|^{p-2}\nabla\mathcal{U}^+, \mathcal{U}^+ \times \nabla[\boldsymbol{\psi}_h - \boldsymbol{\psi}]\Big)\\
&\quad + \Big(|\nabla\mathcal{U}^+|^{p-2}_{\delta}\nabla\mathcal{U}^+, [\mathcal{U}^+ - \mathbf{u}] \times \nabla\boldsymbol{\psi}\Big) + \Big(|\nabla\mathcal{U}^+|^{p-2}\nabla\mathcal{U}^+ - |\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}, \mathbf{u}\times\nabla\boldsymbol{\psi}\Big).
\end{aligned} \tag{3.18}$$

The first two terms on the right-hand side may be controlled using properties of nodal interpolation. We use $\mathcal{U}^+ \to \mathbf{u}$ in $L^2(\Omega_T)$ and (3.17) to show that the integral over $t \in (0,T)$ of the third and fourth term tends to zero for $(k,h,\varepsilon,\delta) \to 0$. Since $|\mathbf{u}| = 1$, a.e. in Ω_T, the estimates imply that $\mathbf{u} : \Omega_T \to \mathbf{S}^2$ satisfies (1.1)–(1.2) in a weak sense, see Lemma 1.8 in [26], or Theorem 14 in [19], and hence Theorem 1.1 is proved.

4. Fixed point method for Algorithm 1.1

A realization of Algorithm 1.1 usually employs a fixed-point iteration, together with a stopping criterion to solve the nonlinear system in Algorithm 1.1. Here, we propose a strategy that preserves the unit-length constraint and provides sequences $\mathbf{U}^j$, $\mathbf{R}^j$, that satisfy the assumptions of Theorem 3.1. We assume that $\delta > 0$ if $p < 2$ and $\delta = 0$ otherwise.

Algorithm 4.1. *Set* $\tilde{\mathbf{U}}^0 := \mathbf{U}^0$ *and* $j := 0$.
1. Set $\mathbf{W}^{j+1,0} := \tilde{\mathbf{U}}^j$ *and* $\ell := 0$.
2. Compute $\mathbf{W}^{j+1,\ell+1} \in \mathbf{V}_h$ *such that for all* $\mathbf{\Phi} \in \mathbf{V}_h$

$$\frac{1}{k}\big(\mathbf{W}^{j+1,\ell+1}, \mathbf{\Phi}\big)_h + \frac{1}{4}\Big([\mathbf{W}^{j+1,\ell+1} + \tilde{\mathbf{U}}^j] \times ([\mathbf{W}^{j+1,\ell} + \tilde{\mathbf{U}}^j] \times \tilde{\Delta}^{\delta}_{p,h}\mathbf{W}^{j+1,\ell}), \mathbf{\Phi}\Big)_h = \frac{1}{k}\big(\tilde{\mathbf{U}}^j, \mathbf{\Phi}\big)_h. \tag{4.1}$$

3. If $\|\mathbf{R}^{j+1,\ell+1}\|_{L^2} \le \varepsilon$ *for*

$$\mathbf{R}^{j+1,\ell+1} := \frac{1}{2}[\mathbf{W}^{j+1,\ell+1} + \tilde{\mathbf{U}}^j] \times \Delta^{\delta}_{p,h}\mathbf{W}^{j+1,\ell+1} - \frac{1}{2}[\mathbf{W}^{j+1,\ell} + \tilde{\mathbf{U}}^j] \times \Delta^{\delta}_{p,h}\mathbf{W}^{j+1,\ell}, \tag{4.2}$$

set $\tilde{\mathbf{U}}^{j+1} := \mathbf{W}^{j+1,\ell+1} \in \mathbf{V}_h$, $j := j+1$, *and go to Step 1.*
4. Set $\ell := \ell + 1$ *and go to Step 2.*

Unconditional unique solvability of (4.1) for $\mathbf{W}^{j+1,\ell+1} \in \mathbf{L}^2$ follows from the Lax-Milgram theorem; moreover, $|\mathbf{W}^{j+1,\ell+1}(\mathbf{x}_\alpha)| = 1$ for $\ell \ge 0$ and $\alpha \in L$ follows from choosing $\mathbf{\Phi} =$

$\frac{1}{2}[\mathbf{W}^{j+1,\ell+1} + \tilde{\mathbf{U}}^j](\mathbf{x}_\alpha)\varphi_\alpha$, provided $|\,\tilde{\mathbf{U}}^j(\mathbf{x}_\alpha)\,| = 1$. Upon testing (4.1) with $\mathbf{\Phi} = -\tilde{\Delta}^\delta_{p,h}\mathbf{W}^{j+1,\ell+1}$, using the last result, and (2.5) lead to

$$\begin{aligned}
\|\,|\,\nabla\mathbf{W}^{j+1,\ell+1}\,|_\delta\,\|^p_{L^p} &= \big(|\,\nabla\mathbf{W}^{j+1,\ell+1}\,|^{p-2}_\delta\nabla\mathbf{W}^{j+1,\ell+1}, \nabla\mathbf{W}^{j+1,\ell+1}\big) + \int_\Omega |\,\nabla\mathbf{W}^{j+1,\ell+1}\,|^{p-2}_\delta \delta\,d\mathbf{x} \\
&\le k\,\|\,\tilde{\Delta}^\delta_{p,h}\mathbf{W}^{j+1,\ell}\,\|_{L^2}\|\,\tilde{\Delta}^\delta_{p,h}\mathbf{W}^{j+1,\ell+1}\,\|_{L^2} + \Big|\Big(\nabla\tilde{\mathbf{U}}^j, |\,\nabla\mathbf{W}^{j+1,\ell+1}\,|^{p-2}_\delta\nabla\mathbf{W}^{j+1,\ell+1}\Big)\Big| \\
&\qquad + \delta^{1/2}\|\,|\,\nabla\mathbf{W}^{j+1,\ell+1}\,|_\delta\,\|^{p-1}_{L^{p-1}} \\
&\le \tilde{C}\Big(kh^{2(-1+N\min\{0,\frac{2-p}{2p}\})}\|\,\nabla\mathbf{W}^{j+1,\ell}\,\|^{p-1}_{L^p} + \|\,\nabla\tilde{\mathbf{U}}^j\,\|_{L^p} + \delta^{1/2}\Big)\|\,|\,\nabla\mathbf{W}^{j+1,\ell+1}\,|_\delta\,\|^{p-1}_{L^p}\,.
\end{aligned}$$

An inductive argument proves the existence of $\hat{C} = \hat{C}(p; \|\,\nabla\tilde{\mathbf{U}}^j\,\|^p_{L^p}) > 1$ such that

$$\max_{\ell\ge 0}\|\,\nabla\mathbf{W}^{j+1,\ell}\,\|^p_{L^p} \le \hat{C} \qquad \text{if} \quad \tilde{C}\hat{C}kh^{2(-1+N\min\{0,\frac{2-p}{2p}\})} < 1\,.$$

The following theorem summarizes these results and discusses convergence of Algorithm 4.1 under certain mesh-size restrictions.

Theorem 4.1. *Let $j \ge 0$ and suppose $|\,\tilde{\mathbf{U}}^j(\mathbf{x}_\alpha)\,| = 1$ for all $\alpha \in L$. Assume that $\delta > 0$ if $p < 2$ and $\delta = 0$ otherwise. For all $\ell \ge 0$, there exists a unique solution $\mathbf{W}^{j+1,\ell+1} \in \mathbf{V}_h$ to* (4.1)*, such that $|\,\mathbf{W}^{j+1,\ell+1}(\mathbf{x}_\alpha)\,| = 1$ for every $\alpha \in L$. Moreover, there exists $\tilde{C} > 0$ such that if $\tilde{C}kh^{2(-1+N\min\{0,\frac{2-p}{2p}\})} < 1$ then*

$$\|\,\mathbf{W}^{j+1,\ell+1} - \mathbf{W}^{j+1,\ell}\,\|_{L^2} \le q\,\|\,\mathbf{W}^{j+1,\ell} - \mathbf{W}^{j+1,\ell-1}\,\|_{L^2},$$

where $q = Ck\big(h^{-1-N/2} + h^{-2}\delta^{(p-2)/2}\big)$ if $p < 2$ and $q = Ck\big(h^{-(2+N\frac{p-2}{p})} + h^{-(1+N\frac{p-1}{p})}\big)$ if $p \ge 2$. If for every $0 \le j \le J-1$ the iteration in Algorithm 4.1 converges, then there holds (3.1)*.*

Proof. (i) Contraction property. Let $\mathbf{E}^{\ell+1}_{j+1} := \mathbf{W}^{j+1,\ell+1} - \mathbf{W}^{j+1,\ell} \in \mathbf{V}_h$, $\ell \ge 0$. Subtraction of two subsequent equations (4.1) implies for every $\ell \ge 1$, and all $\mathbf{\Phi} \in \mathbf{V}_h$,

$$\begin{aligned}
\frac{1}{k}\Big(\mathbf{E}^{\ell+1}_{j+1}, \mathbf{\Phi}\Big)_h &+ \frac{1}{4}\Big(\mathbf{E}^{\ell+1}_{j+1} \times \big([\mathbf{W}^{j+1,\ell} + \tilde{\mathbf{U}}^j] \times \tilde{\Delta}^\delta_{p,h}\mathbf{W}^{j+1,\ell}\big), \mathbf{\Phi}\Big)_h \\
&+ \frac{1}{4}\Big([\mathbf{W}^{j+1,\ell} + \tilde{\mathbf{U}}^j] \times \big(\mathbf{E}^\ell_{j+1} \times \tilde{\Delta}^\delta_{p,h}\mathbf{W}^{j+1,\ell}\big), \mathbf{\Phi}\Big)_h \\
&+ \Big(\nabla\Big[(\mathbf{W}^{j+1,\ell-1} + \tilde{\mathbf{U}}^j) \times \big((\mathbf{W}^{j+1,\ell} + \tilde{\mathbf{U}}^j) \times \mathbf{\Phi}\big)\Big], \\
&\qquad |\,\nabla\mathbf{W}^{j+1,\ell}\,|^{p-2}_\delta\nabla\mathbf{W}^{j+1,\ell} - |\,\nabla\mathbf{W}^{j+1,\ell-1}\,|^{p-2}_\delta\nabla\mathbf{W}^{j+1,\ell-1}\Big) = 0\,.
\end{aligned} \tag{4.3}$$

We study the singular and degenerate cases independently.
Case 1: $1 < p < 2$. Choose $\mathbf{\Phi} = \mathbf{E}^{\ell+1}_{j+1}$ in (4.3). Using L^∞-, $W^{1,p}$-bounds for $\mathbf{W}^{j+1,\ell+r}$, $r \in \{-1,0,1\}$, Lemma 2.1 (iii), inverse estimates, and (2.6) we compute

$$\|\,\mathbf{E}^{\ell+1}_{j+1}\,\|_h \le Ck\,\|\,\tilde{\Delta}^\delta_{p,h}\mathbf{W}^{j+1,\ell}\,\|_{L^\infty}\|\,\mathbf{E}^\ell_{j+1}\,\|_h + Ckh^{-1}\big(\delta^{(p-2)/2} + 1\big)\,\|\,\nabla\mathbf{E}^\ell_{j+1}\,\|_h\,,$$

and hence

$$\|\,\mathbf{E}^{\ell+1}_{j+1}\,\|_h \le Ck\big[h^{-(1+\frac{N}{2})} + h^{-2}\delta^{(p-2)/2}\big]\|\,\mathbf{E}^\ell_{j+1}\,\|_h\,. \tag{4.4}$$

Case 2: $2 < p < \infty$. Note that in this case $\delta = 0$. We use Lemma 2.1 (i) (with $\gamma = 0$) to control the last term in (4.3), i.e., (for ease of reading we skip the index $j+1$)

$$
\begin{aligned}
&\Big(\nabla\Big[(\mathbf{W}^{\ell-1} + \tilde{\mathbf{U}}^j) \times \big((\mathbf{W}^{\ell} + \tilde{\mathbf{U}}^j) \times \boldsymbol{\Phi}\big)\Big], |\,\nabla \mathbf{W}^{\ell}\,|^{p-2}\nabla\mathbf{W}^{j+1,\ell} - |\,\nabla\mathbf{W}^{\ell-1}\,|^{p-2}\nabla\mathbf{W}^{\ell-1}\Big) \\
&\le C \max_{r,s\in\{-1,0\}} \int_\Omega \big(|\,\nabla\boldsymbol{\Phi}\,| + \big[|\,\nabla\mathbf{W}^{\ell+r}\,| + |\,\nabla\tilde{\mathbf{U}}^j\,|\big]|\,\boldsymbol{\Phi}\,|\big)|\,\nabla\mathbf{W}^{\ell+s}\,|^{p-2}|\,\nabla\mathbf{E}^{\ell}\,|\,\mathrm{d}\mathbf{x} \\
&\le C \max_{r,s\in\{-1,0\}} \big(\|\,\nabla\boldsymbol{\Phi}\,\|_{L^p} + \big[\|\,\nabla\mathbf{W}^{\ell+r}\,\|_{L^p} + \|\,\nabla\tilde{\mathbf{U}}^j\,\|_{L^p}\big]\|\,\boldsymbol{\Phi}\,\|_{L^\infty}\big) \\
&\qquad\qquad \times \|\,|\,\nabla\mathbf{W}^{\ell+s}\,|^{p-2}\,\|_{L^{\frac{p}{p-2}}}\|\,\nabla\mathbf{E}^{\ell}\,\|_{L^p} \qquad\qquad (4.5)\\
&\le C \max_{r,s\in\{-1,0\}} \Big(h^{-(1+N\frac{p-2}{2p})}\|\,\boldsymbol{\Phi}\,\|_{L^2} + \big[\|\,\nabla\mathbf{W}^{\ell+r}\,\|_{L^p} + \|\,\nabla\tilde{\mathbf{U}}^j\,\|_{L^p}\big]h^{-\frac{N}{2}}\|\,\boldsymbol{\Phi}\,\|_{L^2}\Big) \\
&\qquad\qquad \times \|\,\nabla\mathbf{W}^{\ell+s}\,\|_{L^p}^{p-2} h^{-(1+N\frac{p-2}{2p})}\|\,\mathbf{E}^{\ell}\,\|_{L^2} \\
&\le C\Big[h^{-(2+N\frac{p-2}{p})} + h^{-(1+N\frac{p-1}{p})}\Big]\|\,\boldsymbol{\Phi}\,\|_{L^2}\|\,\mathbf{E}^{\ell}\,\|_{L^2}\,.
\end{aligned}
$$

For $\boldsymbol{\Phi} = \mathbf{E}^{\ell+1}_{j+1}$ we verify the asserted contraction property.

(ii) Verification of (3.1). Suppose that for some $\ell \ge 0$ we have $\tilde{\mathbf{U}}^{j+1} = \mathbf{W}^{j+1,\ell+1}$. By (4.1), there holds for all $\boldsymbol{\Phi} \in \mathbf{V}_h$,

$$
\begin{aligned}
(d_t\tilde{\mathbf{U}}^{j+1}, \boldsymbol{\Phi})_h &+ \Big(\frac{1}{2}[\tilde{\mathbf{U}}^{j+1} + \tilde{\mathbf{U}}^j] \times \big(\frac{1}{2}[\tilde{\mathbf{U}}^{j+1} + \tilde{\mathbf{U}}^j] \times \tilde{\Delta}^{\delta}_{p,h}\tilde{\mathbf{U}}^{j+1}\big), \boldsymbol{\Phi}\Big)_h \\
&= -\Big([\frac{1}{2}[\tilde{\mathbf{U}}^{j+1} + \tilde{\mathbf{U}}^j] \times (\frac{1}{2}[\mathbf{W}^{j+1,\ell} + \tilde{\mathbf{U}}^j] \times \tilde{\Delta}^{\delta}_{p,h}\mathbf{W}^{j+1,\ell}), \boldsymbol{\Phi}\Big)_h \\
&\qquad + \Big(\frac{1}{2}[\tilde{\mathbf{U}}^{j+1} + \tilde{\mathbf{U}}^j] \times \big(\frac{1}{2}[\mathbf{W}^{j+1,\ell+1} + \tilde{\mathbf{U}}^j] \times \tilde{\Delta}^{\delta}_{p,h}\mathbf{W}^{j+1,\ell+1}\big), \boldsymbol{\Phi}\Big)_h \\
&= \Big(\frac{1}{2}[\tilde{\mathbf{U}}^{j+1} + \tilde{\mathbf{U}}^j] \times \mathbf{R}^{j+1}, \boldsymbol{\Phi}\Big)_h\,,
\end{aligned}
$$

and overall convergence follows from Theorem 3.1. □

5. Computational experiments

In this section we report on numerical experiments for (1.1),(1.2) with $p = 3/2$ and $p = 5/2$ and compare the results obtained with Algorithm 4.1 to those produced with an explicit projection scheme devised in [7]. The initial data is taken from [26, p. 270f.] and is known to lead to finite time blow up for $p = 2$ (with different boundary conditions and a pinned vector at the origin).

Example 5.1. *Let $\Omega := (-1,1)^2 \subseteq \mathbb{R}^2$, $T = 1$, and $\mathbf{u}_0 : \Omega \to \mathbf{S}^2$ for $\mathbf{x} \in \Omega$ be defined by*

$$
\mathbf{u}_0(\mathbf{x}) = \big(\frac{\mathbf{x}}{|\mathbf{x}|}\sin\phi(|\mathbf{x}|), \cos\phi(|\mathbf{x}|)\big)
$$

for $\phi(r) = 3\pi r^2/2$ if $r \le 1$ and $\phi(r) = 3\pi/2$ if $r \ge 1$. The employed triangulations $\mathcal{T}_h$ consist of 2048 and 8192 triangles which are halved squares (along the direction $(1,1)$) and have 1089 respectively 4225 nodes. Thus, $h = \sqrt{2}2^{-4}$ respectively $h = \sqrt{2}2^{-5}$ and we choose, with $s = 1/4$,

(a) $p = 3/2$, $k = h^{3+s}$, $\delta = h^{4+s}$, $\varepsilon = h^2$, $k_{expl} = k$;

(b) $p = 5/2$, $k = h^{16/5+s}$, $\delta = 0$, $\varepsilon = h^2$, $k_{expl} = h^{5/2+s}$.

We included the additional power h^s in order to guarantee that, e.g., for $p = 3/2$ there holds $k = o(h^3)$. The number k_{expl} is the time step size used for the explicit scheme in [7].

Our experience with Algorithm 4.1 shows that the sufficient conditions for convergence of Theorem 4.1 cannot be significantly improved: for $k = h^{12/5+s}$ in (a) of Example 5.1 we did not observe convergence.

Figures 2 and 4 display the energy $E_p\big(\boldsymbol{\mathcal{U}}(t,\cdot)\big)$ and the semi-norm in $W^{1,\infty}(\Omega)$ denoted by $\big|\boldsymbol{\mathcal{U}}(t,\cdot)\big|_{1,\infty}$ for $p = 3/2$ and $p = 5/2$, respectively, as functions of t for $t \in [0, 1]$ for $\boldsymbol{\mathcal{U}}$ obtained with Algorithm 4.1 (implicit scheme) for different mesh-sizes and with the approximation scheme of [7] (explicit scheme). We observe that the implicit and the explicit strategy lead to similar results. Snapshots of the numerical solution for $t = 0$, $t \approx 0.4$, and $t \approx 0.7$ for $p = 3/2$ are shown in Figure 1. For $p = 3/2$, the $W^{1,\infty}$ semi-norm assumes its maximum among functions in $\mathbf{V}_h$ that satisfy the unit length constraint at the nodes in $\mathcal{T}$. This indicates the finite time blow up of [26] for the slightly modified setting (Neumann type boundary conditions and a free vector at the origin) considered here. We refer the reader to [7] for a more detailed discussion of the behaviour of numerical approximations at the time of blow up. A zoom towards the origin is displayed in Figure 3, where snapshots of the numerical solution for (a) in a neighborhood of the origin are shown for various times before and after occurence of blow-up. We observe that at $t \approx 0.5$ the vector at the origin points in another direction than the surrounding vectors resulting in a large (maximal) $W^{1,\infty}$ norm. The results thus indicate occurence of singular solutions for $p < 2$ which is not understood analytically. A similar behaviour cannot be expected for $p > 2$ since exact solutions are continuous with respect to the spatial variable. Indeed, the numerical results for (c) displayed in Figure 4 show that the $W^{1,\infty}$ seminorm is uniformly bounded and thus indicate that no discrete blow up occurs for $p = 5/2$.

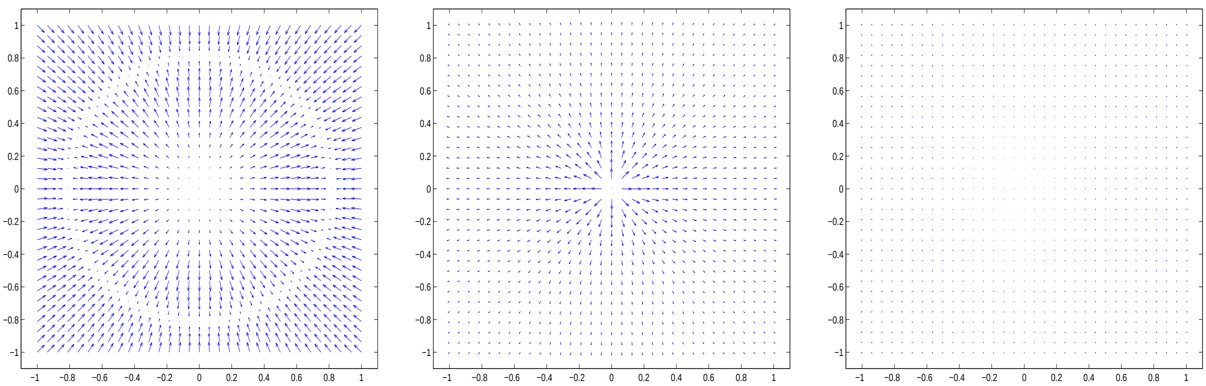

FIGURE 1. Nodal interpolant of initial data $\mathbf{u}_0$ on $\mathcal{T}_h$, and numerical solution $\boldsymbol{\mathcal{U}}(t,\cdot)$ for $t \approx 0.4$ (before blow up) and for $t \approx 0.7$ (after blow up) in Example 5.1 (a).

6. Conclusion

Construction of an unconditionally convergent finite element discretization of the 2-harmonic heat flow into spheres requires to overcome stiffness of used (lowest order) finite elements and competition of local and nonlocal discretizations; tools to obtain a discrete energy law and to satisfy the sphere constraint at nodal points as key properties for convergence are reduced spatial integration, (temporal) midpoint rule, and projected discrete Laplacian, cf. [5]. In this paper, the general case $1 < p < \infty$ is considered, where convexity arguments allow for corresponding inequality version of the energy law; the sphere constraint for solutions to Algorithm 1.1 is conserved; a compactness argument (Lemma 2.3) is used to conclude convergence for $\frac{2N}{2+N} < p < \infty$ (Theorem 1.1), partly with a mild mesh constraint to hold. A simple fixed-point method, together with a stopping criterion is presented (Algorithm 4.1), where iterates stay on the sphere, together with necessary criteria for overall convergence towards weak solutions of (1.1)–(1.2) (Theorem 4.1). Computational studies of qualitative phenomena underline the theoretical results.

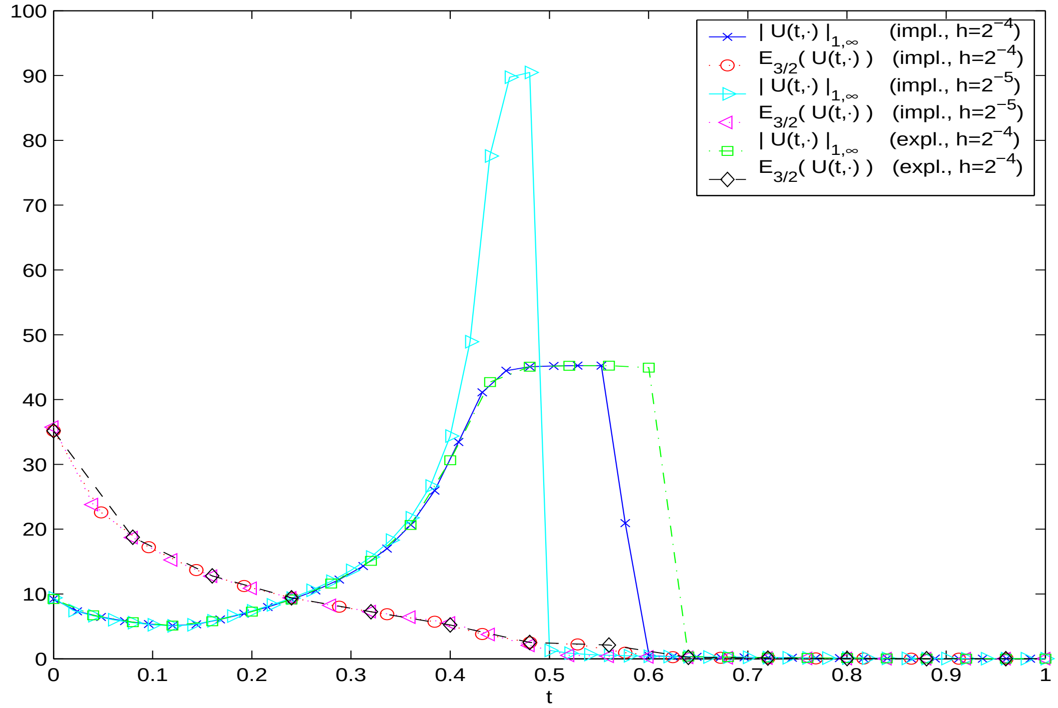

FIGURE 2. Energy $E_{3/2}\big(\boldsymbol{\mathcal{U}}(t,\cdot)\big)$ and $W^{1,\infty}(\Omega)$ semi-norm $\big|\boldsymbol{\mathcal{U}}(t,\cdot)\big|_{1,\infty}$ for implicit and explicit approximation scheme in Example 5.1 (a).

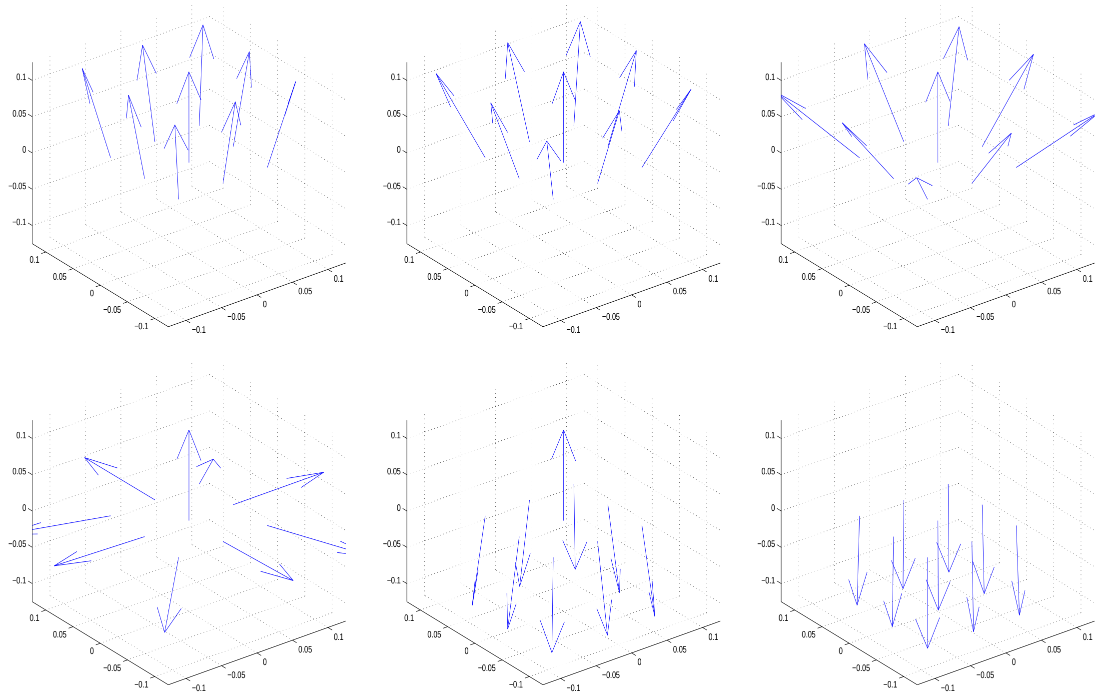

FIGURE 3. Numerical solution in a neighborhood of the origin at $t \approx 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6 in Example 5.1 (a).

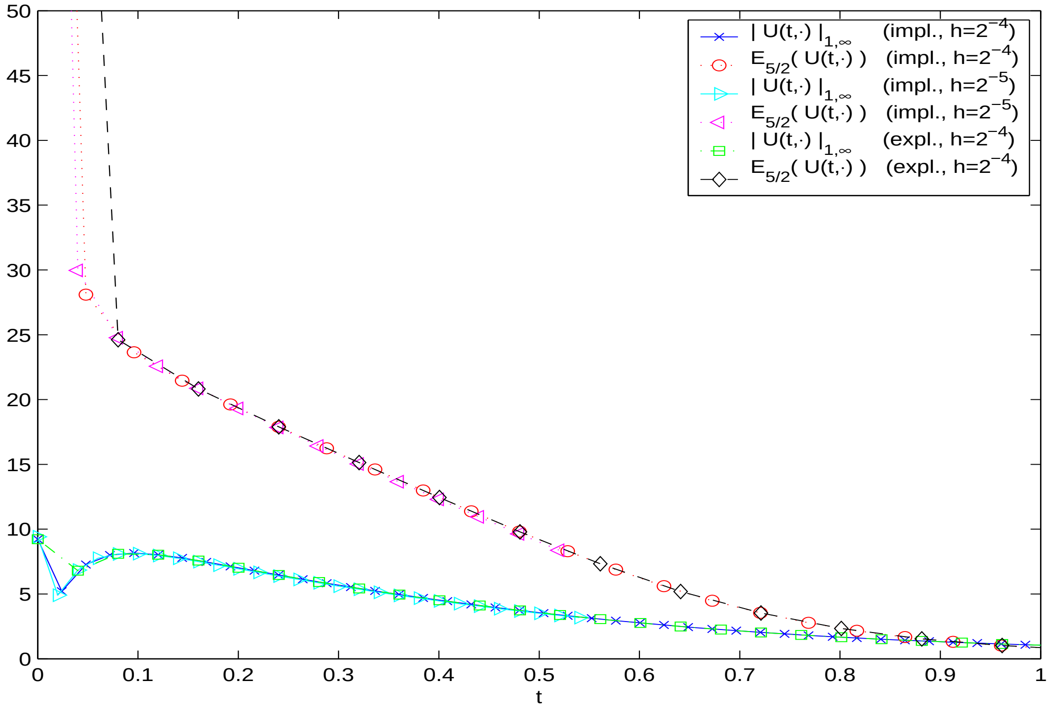

FIGURE 4. Energy $E_{5/2}\big(\boldsymbol{\mathcal{U}}(t,\cdot)\big)$ and $W^{1,\infty}(\Omega)$ semi-norm $\big|\boldsymbol{\mathcal{U}}(t,\cdot)\big|_{1,\infty}$ for implicit and explicit approximation scheme in Example 5.1 (b).

Acknowledgment: Part of the work was written when S.B. visited 'Forschungsinstitut für Mathematik' (ETH Zürich) in December 2005 and February 2006.

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