Thermoviscoplasticity at small strains

Sören Bartels[1,*] and Tomáš Roubíček[3,4,†]

[1] Institut für Numerische Simulation, Rheinische Friedrich-Wilhelms-Universität Bonn, Wegelerstraße 6, D-53115 Bonn, Germany

[2] Mathematical Institute, Charles University, Sokolovská 83, CZ-186 75 Praha 8, Czech Republic

[3] Institute of Information Theory and Automation, Academy of Sciences, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic

[*] Supported by Deutsche Forschungsgemeinschaft through DFG research center Matheon in Berlin

[†] Supported by grants 201/06/0352 (GA ČR), LC 06052, and MSM 21620839 (MŠMT ČR)

Abstract. A viscoelastic solid in Kelvin-Voight rheology involving plasticity coupled with a heat-transfer equation through a temperature-dependent yield stress is investigated. No hardening is studied but the evolution of the plastic strain is considered to be rate-dependent. A numerical scheme which is semi-implicit in time and employs lowest order finite elements on weakly acute triangulations in space is devised and its convergence is proved by careful subsequent limit passage. Computational studies underline robustness and efficiency of the method and illustrate physical effects such as the softening of a material due to dissipated energy that causes a rise in temperature and a local decrease of the yield stress.

Key Words. Thermodynamics of plasticity, temperature dependence of yield stress, Kelvin-Voight rheology, thermal expansion, adiabatic effects.

AMS Subject Classification: 35K85, 49S05, 65M60, 74C10, 80A17.

1. Introduction

Slip plasticity in metals is an *activated inelastic process* that *dissipates mechanical energy to heat* and, if hardening effects are considered, also to changes of the internal structure of the material. If not infinitesimally slow or the body is not perfectly isolated, the mentioned heat production leads to temperature changes that may conversely *influence the activation threshold (=yield stress) for the plastic material behaviour*. In such a way, heat transfer/production and mechnical processes are intimately *coupled*. We remark that variations of such an activation threshold may be very significant, e.g., in steel products manufacturing it may easily vary by a ratio of 5:1 or more when the temperature ranges about 1000°C.

There is an extensive engineering literature addressing thermoplasticity and advocating computationally sophisticated models, see, e.g., [2, 6, 13, 20, 23, 30]. Mathematically supported theories seem unavailable except for an efficient hysteresis-operator approach for thermoplastic processes of a one-dimensional character [18, 19] and certain models in [12] considering a temperature-independent yield stress. For the mathematically well understood case of isothermal plasticity we refer the reader to [1, 10, 14, 15, 21] and also [22, Sect. 5.2].

The main mathematical difficulties in the analysis of inelastic material behaviour are related to multiplicative plasticity at finite strains, evolution of even only elastic response if kinetic effects are taken into account, and coupling of rate-indendent processes with rate-dependent ones. In fact, each of these issues represents itself a hard open problem, especially in three-dimensional settings and if no regularization (e.g., by capillarity or higher viscosity) is involved. This is why we adopt the following simplifications: *small strains* and *additive plasiticity*, *linear elastic* response (so that, in particular, adiabatic effects are suppresed), and *rate-dependent plasticity*, meaning that fast evolution of plastic strain dissipates (at least slightly) more energy than slow evolution. This additional rate-dependent dissipation is advocated even on microstructural level by rate-dependent evolution of dislocations, cf. [17, Figure 1], and allows us to avoid hardening without creating spatial concentrations of plastic strain as it would be in the case of fully rate-independent plasticity [10] and, most importantly, also awkward interactions of concentrating plastic-strain rates with thermal effects. The mathematical model will be formulated in Section 2 where also its thermodynamics will be addressed.

The main purpose of this paper, performed in Sections 3 and 4, is to develop an implementable numerical scheme for the model and prove its stability, i.e., derive a-priori estimates, and convergence to a suitably defined weak solution of the mathematical model. In Section 5 efficiency of our numerical scheme is demonstrated on a 2D example modeling a practical experiment and exhibiting rate-dependent effects of the thermal coupling known as thermal necking within a fast loading experiment. We will mostly focus on a *temperature-independent* elastic response, which is eligible when a thermal expansion in the mechanical part is employed and corresponding adiabatic effects in the heat equation can be neglected.

2. The mathematical model

We assume that the physical body under consideration occupies the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \le 3$. The state variables are the *displacement* $u : \Omega \to \mathbb{R}^d$, the *plastic strain* $\pi : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym},0}$ and the *temperature* variable $\theta : \Omega \to \mathbb{R}$, where

$$\mathbb{R}^{d\times d}_{\mathrm{sym},0} := \big\{A\in\mathbb{R}^{d\times d}_{\mathrm{sym}};\ \mathrm{tr}(A) = 0\big\} \quad \text{and} \quad \mathbb{R}^{d\times d}_{\mathrm{sym}} := \big\{A\in\mathbb{R}^{d\times d};\ A^\top = A\big\}. \tag{2.1}$$

We consider ideal plastic response determined by a convex, closed neighbourhood $S \subset \mathbb{R}^{d\times d}$ of the origin. The interior of S defines the set of *admissible stresses* while its boundary is called the *yield surface* and determines those stresses that trigger the evolution of the plastic strain. We assume that S depends on temperature, i.e., $S = S(\theta)$ for a set-valued mapping $S : \mathbb{R}^+ \rightrightarrows \mathbb{R}^{d\times d}$ qualified below in (3.5d).

Considering a Kelvin-Voigt-type viscous material, our model consists of the equilibrium equation balancing inertial, viscous, and elastic mechanical forces,

$$\varrho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\Big(\mathbb{D}\frac{\partial e(u)}{\partial t}\Big) - \mathrm{div}\big(\mathbb{C}(e(u) - \pi)\big) = 0, \tag{2.2}$$

the evolution law for the plastic strain π, modeled with the Prandtl-Reuß flow rule through the inclusion

$$\partial\delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big)+\mathbb{B}\frac{\partial\pi}{\partial t}-\mathbb{C}\big(e(u)-\pi\big)\ni 0, \tag{2.3}$$

and the heat equation

$$c_{\rm v}\frac{\partial\theta}{\partial t}-{\rm div}\big(\mathbb{K}\nabla\theta\big)=\delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big)+\mathbb{B}\frac{\partial\pi}{\partial t}:\frac{\partial\pi}{\partial t}+\mathbb{D}\frac{\partial e(u)}{\partial t}:\frac{\partial e(u)}{\partial t}. \tag{2.4}$$

Here, " $:$ " denotes the product of two $d{\times}d$-tensors, $e(u)$ is the small-strain tensor

$$e_{ij}(u):=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big), \tag{2.5}$$

and $\delta_S:\mathbb{R}^{3\times 3}\longrightarrow\{0,+\infty\}$ is the indicator function of S with conjugate δ^*_S defined through the duality pairing $s:e=\sum_{i,j=1}^3 s_{ij}e_{ij}$. The subdifferential (i.e., the normal cone) of δ^*_S is denoted $\partial\delta^*_S$. Moreover, $\mathbb{C}=[\mathbb{C}_{ijkl}]\in\mathbb{R}^{d\times d\times d\times d}$ is a positive-definite 4th-order tensor of elastic moduli satisfying the symmetries $\mathbb{C}_{ijkl}=\mathbb{C}_{klij}=\mathbb{C}_{jikl}$. Similarly, $\mathbb{D}\in\mathbb{R}^{d\times d\times d\times d}$ and $\mathbb{B}\in\mathbb{R}^{d\times d\times d\times d}$ are symmetric, positive-definite fourth-order tensors determining viscous moduli and the rate-dependent part of plastic-strain response, respectively. The symmetric, positive-definite second-order tensor $\mathbb{K}\in\mathbb{R}^{d\times d}$ models heat conductivities while the real numbers ϱ and $c_{\rm v}$ define the mass density and the heat capacity of the material, respectively.

We remark that the physical dimension of the stress/strain pairing $s:e=\sum_{i,j=1}^3 s_{ij}e_{ij}$ is Pa=J/m^3 so that S determines the degree-1 homogeneous "plastic" dissipation potential δ^*_S which acts on the dimensionless tensor π and has the dimension J/m^3. The set $S(\theta)$ need not be bounded but should be the direct sum of a closed convex set $S_0(\theta)\subseteq\mathbb{R}^{d\times d}_{\rm sym,0}$ containing the origin and the orthogonal complement of the $\frac{1}{2}d(d{+}1)-1$ dimensional space $\mathbb{R}^{d\times d}_{\rm sym,0}$. In other words, $S_0(\theta)$ contains all matrices whose deviator belongs to a given convex set. This ensures that $\delta^*_{S(\theta)}$ is finite only on $\mathbb{R}^{d\times d}_{\rm sym,0}$.

We consider Dirichlet boundary conditions on the nonempty, open part Γ_0 of the boundary $\Gamma:=\partial\Omega$ and a mechanical loading defined by a time-varying *surface force* g acting on another open part $\Gamma_1\subset\Gamma$. We assume that $\Gamma_0\cap\Gamma_1=\emptyset$ and that $\Gamma_0\cup\Gamma_1$ covers Γ up to a set of zero surface measure. We allow heat transfer through Γ described by a heat-transfer coefficient $\alpha\geq 0$ and a prescribed *heat flux* f. This corresponds to the boundary conditions

$$u|_{\Gamma_0}=u_{\rm D}\qquad\text{on }\Gamma_0, \tag{2.6a}$$

$$\Big(\mathbb{D}\frac{\partial e(u)}{\partial t}+\mathbb{C}\big(e(u)-\pi\big)\Big)\nu=g\qquad\text{on }\Gamma_1, \tag{2.6b}$$

$$\big(\mathbb{K}\nabla\theta\big)\cdot\nu+\alpha\theta=f\qquad\text{on }\Gamma, \tag{2.6c}$$

where " $\cdot$ " denotes the scalar product of two vectors and ν the outer unit normal on Γ.

The above equations and inclusion (2.2)–(2.5) are to hold on the space-time domain $Q:=(0,T)\times\Omega$ with a fixed time horizon $T>0$. Thus, we consider an initial-boundary-value problem for the system (2.2)–(2.5) and impose the initial conditions

$$u(0,\cdot)=u_0,\qquad\frac{\partial u}{\partial t}(0,\cdot)=\dot u_0,\qquad\pi(0,\cdot)=\pi_0,\qquad\theta(0,\cdot)=\theta_0. \tag{2.7}$$

The energetics of the model problem relates the *stored energy*

$$\Phi(u,\pi):=\frac{1}{2}\int_\Omega\mathbb{C}(e(u)-\pi):(e(u)-\pi)\,{\rm d}x, \tag{2.8}$$

the *kinetic energy*

$$T_{\rm kin}(\dot u) := \frac{1}{2}\int_\Omega \varrho|\dot u|^2\,{\rm d}x, \tag{2.9}$$

the *dissipation rate*

$$\Xi_\theta(\dot u,\dot\pi) := \int_\Omega \delta^*_{S(\theta)}\big(\dot\pi\big) + \mathbb{B}\dot\pi : \dot\pi + \mathbb{D}e(\dot u) : e(\dot u)\,{\rm d}x, \tag{2.10}$$

the *internal energy*

$$E(\theta) := \int_\Omega c_{\rm v}\theta\,{\rm d}x, \tag{2.11}$$

and the *power of external mechanical forces and heating*

$$P(t,\dot u,\sigma^{\rm nor},\theta) := \Big\langle \sigma^{\rm nor},\frac{\partial u_{\rm D}}{\partial t}\Big\rangle_{\Gamma_0} + \int_{\Gamma_1} g(t,x)\cdot\dot u(x)\,{\rm d}S + \int_\Gamma \big(f-\alpha\theta\big)\,{\rm d}S, \tag{2.12}$$

where $\sigma^{\rm nor}$ denotes the *normal stress* which is in duality with $\frac{\partial u_{\rm D}}{\partial t}$ and defined by

$$\begin{aligned}\big\langle\sigma^{\rm nor},v\big\rangle_{\Gamma_0} &= \Big\langle\varrho\frac{\partial^2 u}{\partial t^2},\overline v\Big\rangle + \int_\Omega \mathbb{D}e(u):e(\overline v){\rm d}x\\ &\quad+\int_\Omega \mathbb{C}(e(u)-\pi):e(\overline v){\rm d}x - \int_{\Gamma_1} g(t,\cdot)\cdot\overline v\,{\rm d}S\end{aligned} \tag{2.13}$$

for every $v\in W^{1/2,2}(\Gamma_0;\mathbb{R}^d)$ with (arbitrary) extension $\overline v\in W^{1,2}(\Omega;\mathbb{R}^d)$. The *energy balance* of the model (2.2)–(2.4) is then obtained by testing (2.2), (2.3), and (2.4) respectively by the velocity $\frac{\partial(u-\overline u_{\rm D})}{\partial t}$, by the plastic strain rate $\frac{\partial\pi}{\partial t}$, and by 1, which gives, after application of Green's formula for both (2.2) and (2.4) together with the boundary conditions (2.6) and the identity $\frac{\partial\pi}{\partial t}:\partial\delta^*_{S(\theta)}(\dot\pi) = \delta_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big)$ (cf. (3.3) below),

$$\frac{\rm d}{{\rm d}t}\Big(T_{\rm kin}\Big(\frac{\partial u}{\partial t}\Big) + \Phi(u,\pi) + E(\theta)\Big) = P\Big(t,\frac{\partial u}{\partial t},\sigma^{\rm nor}\Big). \tag{2.14}$$

Integrating over $(0,t)$ and introducing the total energy $T_{\rm kin}(\dot u)+\Phi(u,\pi)+E(\theta) =: E_{\rm tot}(u,\pi,\theta,\dot u)$, the energy balance can be written as

$$\begin{aligned}E_{\rm tot}\Big(u(t,\cdot),\pi(t,\cdot),\theta(t,\cdot),\frac{\partial u}{\partial t}(t,\cdot)\Big) - E_{\rm tot}\big(u_0,\pi_0,\theta_0,\dot u_0\big)\\ = \int_0^t P\Big(r,\frac{\partial u}{\partial t}(r,\cdot),\sigma(r,\cdot)\Big)\,{\rm d}r,\end{aligned}$$

which says that the difference of total energies at time t and at the initial time equals the work done by external forces within the time interval $(0,t)$.

Remark 2.1. (*Thermodynamics of the model.*) One can derive the above model from the *free energy*

$$\psi(u,\pi,\theta) = \frac{1}{2}\mathbb{C}(e(u)-\pi):(e(u)-\pi) - c_{\rm v}\theta\ln\Big(\frac{\theta}{\theta_0}\Big) \tag{2.15}$$

in which θ_0 plays the role of a reference temperature. Then, *entropy* is given by $s := -\frac{\partial\psi}{\partial\theta} = c_{\rm v}(1+\ln(\theta/\theta_0))$ and (2.4) can be written in the form of the *entropy equation* as

$$\theta\frac{\partial s}{\partial t} - {\rm div}(\mathbb{K}\nabla\theta) = \xi \quad \text{with}\ \ \xi := \delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big) + \mathbb{B}\frac{\partial\pi}{\partial t}:\frac{\partial\pi}{\partial t} + \mathbb{D}\frac{\partial e(u)}{\partial t}:\frac{\partial e(u)}{\partial t}. \tag{2.16}$$

Note that $\int_\Omega \xi \,\mathrm{d}x = \Xi_\theta(\frac{\partial u}{\partial t}, \frac{\partial \pi}{\partial t})$ with Ξ_θ from (2.10). At least formally, assuming positivity of temperature and realizing that $\xi \geq 0$, we deduce that the *Clausius-Duhem inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s \,\mathrm{d}x = \int_\Omega \Big(\mathrm{div}\Big(\mathbb{K}\frac{\nabla\theta}{\theta}\Big) + \frac{\mathbb{K}\nabla\theta\cdot\nabla\theta}{\theta^2} + \frac{\xi}{\theta}\Big)\,\mathrm{d}x = \int_\Omega \frac{\mathbb{K}\nabla\theta\cdot\nabla\theta}{\theta^2} + \frac{\xi}{\theta}\,\mathrm{d}x + \int_\Gamma \frac{f}{\theta}\,\mathrm{d}S \geq 0$$

holds provided $f \geq 0$ and the system is thermally isolated corresponding to $\alpha = 0$.

Remark 2.2. The mathematical model describes a scenario in which the entire dissipated mechanical energy is transferred into heat. In reality, even without considerable hardening effects, only about 90% goes into heat, e.g., in metals, while in polymers it may be only 60%, cf. [25]. The remaining energy dissipates presumably into the change of structure, acoustic emission, etc. These aspects can partially be incorporated in our model by multiplying the right-hand side of the heat equation (2.4) by a non-negative factor not greater than 1. This would make no significant difference in the analysis and implementation below.

3. Weak formulation of the model, data qualification

Throughout, we abbreviate $I := (0,T)$, $Q := I\times\Omega$, $\Sigma := I\times\Gamma$, $\Sigma_0 := I\times\Gamma_0$, and $\Sigma_1 := I\times\Gamma_1$. We use the standard notation $C^\infty(\cdot;\mathbb{R}^d)$ for the space of smooth $\mathbb{R}^d$-valued functions, $L^p(\cdot;\mathbb{R}^d)$ for pth-power Lebesgue integrable functions and $W^{k,p}(\cdot;\mathbb{R}^d)$ for the Sobolev spaces of functions whose kth weak derivatives are in $L^p(\cdot;\mathbb{R}^d)$ on the domain indicated. If values range over a Banach space X, then $L^p(I;X)$ refers to the L^p-Bochner space of X-valued functions and $W^{k,p}(I;X)$ is the corresponding Sobolev-Bochner space.

To introduce a suitable definition of a weak solution, we rewrite (2.3) as the system

$$\omega + \mathbb{B}\frac{\partial\pi}{\partial t} - \mathbb{C}(e(u)-\pi) = 0, \tag{3.1a}$$

$$\omega \in \partial\delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big). \tag{3.1b}$$

The meaning of ω is the *driving stress* for the evolution of the plastic strain. Then, (2.4) can equally be written in the form

$$c_\mathrm{v}\frac{\partial\theta}{\partial t} - \mathrm{div}(\mathbb{K}\nabla\theta) = \omega : \frac{\partial\pi}{\partial t} + \mathbb{B}\frac{\partial\pi}{\partial t} : \frac{\partial\pi}{\partial t} + \mathbb{D}\frac{\partial e(u)}{\partial t} : \frac{\partial e(u)}{\partial t}. \tag{3.2}$$

Here we used that for every $\omega \in \partial\delta^*_{S(\theta)}(\frac{\partial\pi}{\partial t})$ we have

$$\omega : \frac{\partial\pi}{\partial t} = \delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big). \tag{3.3}$$

For a proof of (3.3) notice first that $\delta^*_{S(\theta)}(0) = 0$ and the definition of $\partial\delta^*_{S(\theta)}\big(\frac{\partial\pi}{\partial t}\big)$ imply $\delta^*_{S(\theta)}(\frac{\partial\pi}{\partial t}) \leq \omega : \frac{\partial\pi}{\partial t}$. The converse inequality follows from $\delta^*_{S(\theta)}(0) = 0$ and the definition of $\partial\delta^*_{S(\theta)}(0)$, i.e., for every $\xi \in \partial\delta^*_{S(\theta)}(0) = S(\theta)$ we have

$$\xi : \frac{\partial\pi}{\partial t} = \delta^*_{S(\theta)}(0) + \xi : \Big(\frac{\partial\pi}{\partial t} - 0\Big) \leq \delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big) \tag{3.4}$$

Since $\partial\delta^*_{S(\theta)}(\frac{\partial\pi}{\partial t}) \subseteq S(\theta)$ we deduce that (3.4) also holds with $\xi = \omega$.

To introduce the notion of a very weak solution of the model, we impose the following data qualification:

(3.5a) Ω a bounded Lipschitz domain in $\mathbb{R}^d$ such that there exists an acute triangulation of Ω,

(3.5b) Γ_0, Γ_1 polyhedral subsets of $\Gamma = \partial\Omega$,

(3.5c) $\mathbb{B}, \mathbb{C}, \mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}$ and $\mathbb{K} \in \mathbb{R}^{d\times d}$ positive definite tensors satisfying $\mathbb{B}_{ijkl} = \mathbb{B}_{jikl}, \quad \mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}, \quad \mathbb{D}_{ijkl} = \mathbb{D}_{jikl}, \quad \mathbb{K}_{ij} = \mathbb{K}_{ji}$,

(3.5d) $S : \mathbb{R}^+ \rightrightarrows \mathbb{R}^{d\times d}$ such that $S(\theta) = \Upsilon(\theta)S_0$ for $S_0 \subset \mathbb{R}^{d\times d}$ such that $S_0 = S_{0,0} + \{\alpha \mathbf{I}_{d\times d} : \alpha \in \mathbb{R}\}$ with $0 \in S_{0,0} \subseteq \mathbb{R}^{d\times d}_{\mathrm{sym},0}$ closed and convex and $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, bounded, and $\inf \Upsilon(\cdot) > 0$,

(3.5e) $\varrho, c_{\mathrm{v}}, \kappa > 0, \quad \alpha \in W^{1-1/d-\epsilon,d+\epsilon}(\Gamma), \quad \epsilon > 0, \quad \alpha \geq 0$,

(3.5f) $f \in L^1(\Sigma), \quad f \geq 0$,

(3.5g) $g \in L^2(I; L^q(\Gamma_1; \mathbb{R}^d))$ with $q > 2 - \dfrac{2}{d}$ for $d \geq 2$ or $q = 1$ for $d = 1$,

(3.5h) $u_{\mathrm{D}} = \overline{u}_{\mathrm{D}}|_{\Sigma_0} \in W^{1,1}(I; W^{1/2,2}(\Gamma_0; \mathbb{R}^d))$ for $\overline{u}_{\mathrm{D}} \in W^{2,1}(I; L^2(\Omega; \mathbb{R}^d))$,

(3.5i) $u_0 \in W^{1,2}(\Omega; \mathbb{R}^d)$,

(3.5j) $\dot{u}_0 \in L^2(\Omega; \mathbb{R}^d)$,

(3.5k) $\pi_0 \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym},0})$,

(3.5l) $\theta_0 \in L^1(\Omega)$.

While every bounded Lipschitz domain in $\mathbb{R}^2$ with polygonal boundary can be triangulated with triangles whose inner angles are bounded by 90°, it is not clear that every bounded Lipschitz domain in $\mathbb{R}^3$ with polyhedral boundary admits triangulations into tetrahedra such that every angle between two faces that belong to the same tetrahedron is bounded by 90°. The results in [16] allow to refine an acute triangulation in such a way that the resulting triangulation is acute, too. Therefore, assumption (3.5a) guarantees that there exists a sequence of acute triangulations with arbitrarily small mesh-size. The assumption (3.5d) simplifies our convergence analysis below: if $\theta_k \to \theta$ pointwise almost everywhere in Ω then any $w \in L^p(\Omega; \mathbb{R}^{d\times d})$ satisfying $w \in \partial\delta^*_{S(\theta)}(z)$ can be approximated by a sequence $\{w_k\}_k$ such that $w_k \in \partial\delta^*_{S(\theta_k)}(z)$, e.g., for every $1 \leq q < p$ we have

$$w_k := \frac{\Upsilon(\theta_k)}{\Upsilon(\theta)} w \to w \quad \text{in } L^q(\Omega; \mathbb{R}^{d\times d}). \tag{3.6}$$

For a proof of (3.6) notice that $w \in \partial\delta^*_{S(\theta)}(z)$ and $\partial\delta^*_{\Upsilon(\theta)S_0}(z) = \Upsilon(\theta)\partial\delta^*_{S_0}(z)$ so that

$$w_k \in \frac{\Upsilon(\theta_k)}{\Upsilon(\theta)} \partial\delta^*_{S(\theta)}(z) = \frac{\Upsilon(\theta_k)}{\Upsilon(\theta)} \partial\delta^*_{\Upsilon(\theta)S_0}(z) = \Upsilon(\theta_k)\partial\delta^*_{S_0}(z) = \partial\delta^*_{S(\theta_k)}(z) \tag{3.7}$$

and $\Upsilon(\theta_k) \to \Upsilon(\theta)$ in $L^r(\Omega)$ for every $1 \leq r < \infty$. Such a simple approximation would not hold if, e.g., we had assumed merely $S(\theta_k) \to S(\theta)$ in a Hausdorff metric on $\mathbb{R}^{d\times d}$. Of course, (3.5d) restricts generality because the yield surface has always the same shape varying only with temperature. Other plastic material laws would require more sophisticated approximation arguments.

The qualification (3.5e) implies that is α continuous but, in fact, modifications for α only piecewise continuous with possible jumps that are compatible with employed triangulations would also be possible.

The very weak formulation of (2.2)–(2.4) is obtained by standard procedure: we multiply (2.2), (2.4), and (3.1a) by test functions and use Green's formula and by-part integration in time for both (2.2) and (3.2) together with the boundary and initial conditions (2.6)-(2.7) and the identity $\sigma : \nabla z = \sigma : e(z)$ if σ is symmetric.

Definition 3.1. ***(A very weak formulation.)*** *We call the triple* (u, π, θ) *with*

$$u \in W^{1,2}(I; W^{1,2}(\Omega; \mathbb{R}^d)) \tag{3.8a}$$

$$\pi \in W^{1,2}(I; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym},0})) \tag{3.8b}$$

$$\theta \in L^{\frac{d+2}{d+1}-\delta}(I; W^{1,\frac{d+2}{d+1}-\delta}(\Omega)) \cap L^\infty(I; L^1(\Omega)) \quad \text{with any } 0 < \delta \le \tfrac{1}{d+1}, \tag{3.8c}$$

a very weak solution *of (2.2) and (3.1)–(3.2) with the initial conditions (2.7) and the boundary conditions (2.6) if* $u(0,\cdot) = u_0$, $\pi(0,\cdot) = \pi_0$, $u|_{\Sigma_0} = u_{\mathrm{D}}$, *and if, for* ω *satisfying (3.1a) in the sense of* $L^2(Q; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, *i.e.,*

$$\omega = \mathbb{C}(e(u) - \pi) - \mathbb{B}\frac{\partial \pi}{\partial t} \in L^2(Q; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{3.9}$$

we have

$$\int_Q \bigg(\Big(\mathbb{D}\frac{\partial e(u)}{\partial t} + \mathbb{C}(e(u) - \pi) \Big) : e(z) \ - \varrho \frac{\partial u}{\partial t} \cdot \frac{\partial z}{\partial t} \bigg) \,\mathrm{d}x\mathrm{d}t \tag{3.10}$$

$$= \int_{\Sigma_1} g \cdot z \,\mathrm{d}S\mathrm{d}t + \int_\Omega \varrho \dot{u}_0(x) \cdot z(0,x) \,\mathrm{d}x \tag{3.11}$$

for all $z \in C^\infty(Q; \mathbb{R}^d)$ *with* $z(T,\cdot) = 0$ *and* $z|_{\Sigma_0} = 0$, *and*

$$\int_Q (\omega - w) : \Big(\frac{\partial \pi}{\partial t} - z \Big) \mathrm{d}x\mathrm{d}t \ge 0 \tag{3.12}$$

for all $z, w \in L^2(Q; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ *such that* $z \in N_{S(\theta)}(w)$ *almost everywhere on* Q, *and*

$$\int_Q \bigg(\mathbb{K}\nabla\theta \cdot \nabla z - \Big(\omega : \frac{\partial \pi}{\partial t} + \mathbb{B}\frac{\partial \pi}{\partial t} : \frac{\partial \pi}{\partial t} + \mathbb{D}\frac{\partial e(u)}{\partial t} : \frac{\partial e(u)}{\partial t} \Big) z - c_{\mathrm{v}}\theta \frac{\partial z}{\partial t} \bigg) \mathrm{d}x\mathrm{d}t \tag{3.13}$$

$$+ \int_\Sigma \alpha\theta z \,\mathrm{d}S\mathrm{d}t = \int_\Sigma f\, z \,\mathrm{d}S\mathrm{d}t + \int_\Omega c_{\mathrm{v}}\theta_0(x)\, z(0,x) \,\mathrm{d}x$$

for all $z \in C^\infty(Q)$ *with* $z(T,\cdot) = 0$.

We remark that since $\delta^*_{S(\theta(t,x))}$ is a proper convex function on $\mathbb{R}^{d\times d}$, its subdifferential has a maximal monotone graph in $\mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d}$ whose inversion $[\partial\delta^*_{S(\theta(t,x))}]^{-1} = \partial\delta_{S(\theta(t,x))} = N_{S(\theta(t,x))}$ is the normal-cone set-valued mapping for almost every $(t,x) \in Q$. The maximal monotonicity is inherited by the graph induced on $L^2(Q; \mathbb{R}^{d\times d}) \times L^2(Q; \mathbb{R}^{d\times d})$, and thus (3.12) is an equivalent formulation of the inclusion (3.1b), i.e., $\omega \in \partial\delta^*_{S(\theta)}(\frac{\partial}{\partial t}\pi)$, cf., e.g., [26].

4. Existence of solutions and their numerical approximation

In this section we devise a numerical scheme to approximate solutions of the model problem and show that these approximations converge to very weak solutions in the sense of Definition 3.1. To cope with the fact that the right-hand side in the heat equation (2.4) only belongs to $L^1(\Omega)$, we need to employ two different spatial meshes for the discretization of (3.10) and (3.13) and perform a successive limit passage.

4.1. **Triangulations and data approximation.** We assume that we are given sequences of regular triangulations $\{\mathscr{T}^1_{h_1}\}_{h_1>0}$ and $\{\mathscr{T}^2_{h_2}\}_{h_2>0}$ of the polyhedral domain Ω. We suppose that for each $h_1 > 0$ the parts Γ_0 and Γ_1 of Γ are matched exactly by edges (or faces) of elements in $\mathscr{T}^1_{h_1}$, that $h_1, h_2 > 0$ range over countable sets of positive real numbers with accumulation points at 0, and that $\max_{K\in\mathscr{T}^j_{h_j}} \operatorname{diam}(K) \le h_j$ for $j = 1, 2$. Each triangulation $\mathscr{T}^2_{h_2}$ is assumed to allow for a discrete maximum principle of the implicitly discretized linear heat equation. If numerical integration is used, then sufficient for this is that $\mathscr{T}^2_{h_2}$ is *weakly acute* if $d = 2$ (i.e., the sum of every pair of angles opposite to an inner edge does not exceed 90° whereas angles opposite to boundary edges are bounded by 45°) or *strongly acute* if $d = 3$ (i.e., all angles between faces of tetrahedra are bounded by 45°). We refer the reader to [8, 11, 16, 31] for further details.

We consider C^0-conforming P1-elements for the approximation of u and θ and P0-elements for the approximation of π and ω. The finite-dimensional subspaces of $L^2(\Omega)$ and $W^{1,2}(\Omega)$ related to P0- and P1-elements and subordinate to the triangulation $\mathscr{T}^\ell_{h_\ell}$ respectively by V_{0,h_ℓ} and V_{1,h_ℓ}. For $j = 0, 1$, the L^2 orthogonal projection onto V_{j,h_ℓ} is denoted by P_{j,h_ℓ}. We employ numerical integration in Ω and on Γ defined through the discrete inner products

$$(\theta, z)_{h_2} := \int_\Omega \mathscr{I}_{h_2}\big(\theta z\big)\,\mathrm{d}x \qquad \text{and} \qquad (\theta, z)_{\Gamma,h_2} := \int_\Gamma \mathscr{I}_{h_2}\big(\theta z\big)\,\mathrm{d}S \tag{4.1}$$

for $\theta, v \in V_{1,h_2}$ and the nodal interpolation operator $\mathscr{I}_{h_2} : C(\overline{\Omega}) \to V_{1,h_2}$. Standard results on nodal interpolation imply that for $\theta, v \in V_{1,h_2}$ we have

$$\Big|(\theta, z)_{h_2} - \int_\Omega \theta z\,\mathrm{d}x\Big| + \Big|(\theta, z)_{\Gamma,h_2} - \int_\Gamma \theta z\,\mathrm{d}S\Big| \le C_0 h_2 \big\|\theta\big\|_{W^{1,2}(\Omega)}\big\|z\big\|_{L^2(\Omega)}. \tag{4.2}$$

We let $\tau > 0$ denote a time-step size satisfying $T/\tau \in \mathbb{N}$ and define the *backward difference operator*

$$d_t\phi^k := \tau^{-1}\big(\phi^k - \phi^{k-1}\big) \tag{4.3}$$

for any sequence $\{\phi^k\}_{k\ge0}$. We set $t_k := k\tau$ and

$$X_{h_1h_2} := V^d_{1,h_1} \times V^{d\times d}_{0,h_1} \times V^{d\times d}_{0,h_1} \times V_{1,h_2} \tag{4.4}$$

and choose the set of discrete initial data

$$(u^0_{\tau h_1h_2}, \pi^0_{\tau h_1h_2}, \omega^0_{\tau h_1h_2}, \theta^0_{\tau h_1h_2}) := \big(P_{1,h_1}u_0, P_{0,h}\pi_0, 0, \mathscr{I}_{h_2}\tilde{\mathscr{J}}_{1,h_1}\theta_0\big) \in X_{h_1h_2}, \tag{4.5}$$

where $\tilde{\mathscr{J}}_{1,h_1} : L^1(\Omega) \to V_{1,h_1}$ is a non-negativity preserving weak interpolation operator satisfying $\|\theta_0 - \tilde{\mathscr{J}}_{1,h_1}\theta_0\|_{L^1(\Omega)} \to 0$ as $h_1 \to 0$, cf., e.g., [24]. We also define $\dot{u}_{0,h_1} := P_{1,h_1}\dot{u}_0 \in V_{1,h_1}$ and set $u^{-1}_{\tau h_1h_2} := u_{0,h_1} - \tau\dot{u}_{0,h_1}$. Discrete approximations of the given data $u_{\mathrm{D}}(t_k,\cdot)$, $f_{h_1}(t_k,\cdot)$ and $g(t_k,\cdot)$ at time level t_k are defined through

$$u^k_{\mathrm{D},\tau h_1} := P_{1,h_1}\overline{u}_{\mathrm{D}}(t_k,\cdot) \qquad\qquad f^k_{\tau,h_1} := \big[f_{h_1}\big]^k_\tau, \qquad\qquad g^k_\tau := \big[g\big]^k_\tau, \tag{4.6}$$

where $\overline{u}_{\mathrm{D}}$ refers to (3.5h), $[v]^k_\tau(x) := \int_{t_{k-1}}^{t_k} v(t,x)\,\mathrm{d}t$, and $f_{h_1}(t,\cdot) := P^\Gamma_{0,h_1}f(t,\cdot)$ is the L^2 orthogonal projection of $f(t,\cdot)$ onto the trace space $V_{0,h_1}|_\Gamma$.

4.2. **Semi-implicit approximation scheme.** With the definitions of Subsection 4.1 we define the following *semi-implicit in time and finite element in space discretization* of (2.2), (3.1)–(3.2).

Algorithm ($\mathscr{A}$)**.** Given $k \geq 1$ and the approximations $(u^{k-1}_{\tau h_1h_2}, \pi^{k-1}_{\tau h_1h_2}, \omega^{k-1}_{\tau h_1h_2}, \theta^{k-1}_{\tau h_1h_2}) \in X_{h_1h_2}$ at level $k-1$ and $u^{k-2}_{\tau h_1h_2} \in V^d_{1,h_1}$ at level $k-2$, compute $(u^{k}_{\tau h_1h_2}, \pi^{k}_{\tau h_1h_2}, \omega^{k}_{\tau h_1h_2}, \theta^{k}_{\tau h_1h_2}) \in X_{h_1h_2}$ such that $u^k_{\tau h_1h_2}|_{\Gamma_0} = u^k_{\mathrm{D},\tau h_1}$ and

$$\int_\Omega \big(d_t^2 u^k_{\tau h_1h_2}\cdot z + \mathbb{D} d_t e(u^k_{\tau h_1h_2}) : e(z) + \mathbb{C}(e(u^k_{\tau h_1h_2}) - \pi^k_{\tau h_1h_2}) : e(z)\big)\,\mathrm{d}x = \int_{\Gamma_1} g^k_\tau \cdot z\,\mathrm{d}S \tag{4.7}$$

for all $z \in V^d_{1,h_1}$ with $z|_{\Gamma_0} = 0$,

$$\omega^k_{\tau h_1h_2} + \mathbb{B} d_t \pi^k_{\tau h_1h_2} - \mathbb{C}(e(u^k_{\tau h_1h_2}) - \pi^k_{\tau h_1h_2}) = 0, \tag{4.8}$$

$$\omega^k_{\tau h_1h_2} \in \partial\delta^*_{S(P_{0,h_1}\theta^{k-1}_{\tau h_1h_2})}\big(d_t\pi^k_{\tau h_1h_2}\big), \tag{4.9}$$

and

$$\begin{aligned} c_{\mathrm{v}}\big(d_t\theta^k_{\tau h_1h_2}, z\big)_{h_2} + \int_\Omega \mathbb{K}\nabla\theta^k_{\tau h_1h_2}\cdot\nabla z\,\mathrm{d}x + \big(\alpha\theta^k_{\tau h_1h_2}, z\big)_{\Gamma,h_2} &= \int_\Gamma f^k_{\tau,h_1} z\,\mathrm{d}S \\ + \int_\Omega \big(\omega^k_{\tau h_1h_2} : d_t\pi^k_{\tau h_1h_2} + \mathbb{B} d_t\pi^k_{\tau h_1h_2} : d_t\pi^k_{\tau h_1h_2} + \mathbb{D} d_t e(u^k_{\tau h_1h_2}) &: d_t e(u^k_{\tau h_1h_2})\big) z\,\mathrm{d}x \end{aligned} \tag{4.10}$$

for all $z \in V_{1,h_2}$.

Employing two independent sequences of triangulations allows us to use simple triangulations for the nonlinear system (4.7)–(4.9) while the presumably more complicated (weakly) acute triangulation is only required for the linear heat equation (4.10). Algorithm ($\mathscr{A}$) is unconditionally well posed as the following proposition shows.

Proposition 4.1. *(**Existence of the approximate solution.**) For each $k = 1, 2, ..., T/\tau$, there exists a unique solution $(u^{k}_{\tau h_1h_2}, \pi^{k}_{\tau h_1h_2}, \omega^{k}_{\tau h_1h_2}, \theta^{k}_{\tau h_1h_2}) \in X_{\tau h_1h_2}$ of the system (4.7)–(4.10) satisfying $\pi^k_{\tau h_1h_2} \in \mathbb{R}^{d\times d}_{\mathrm{sym},0}$ almost everywhere in Ω.*

Proof. Given $k \geq 1$, $(u^{k-1}_{\tau h_1h_2}, \pi^{k-1}_{\tau h_1h_2}, \omega^{k-1}_{\tau h_1h_2}, \theta^{k-1}_{\tau h_1h_2}) \in X_{h_1h_2}$, and $u^{k-2}_{\tau h_1h_2} \in V^d_{1,h_1}$ we first consider the minimization problem

$$\left\{\begin{aligned} &\text{minimize} && \int_\Omega \varrho\big|u - 2u^{k-1}_{\tau h_1h_2} + u^{k-2}_{\tau h_1h_2}\big|^2 + 2\tau\delta^*_{S(P_{0,h_1}\theta^{k-1}_{\tau h_1h_2})}(\pi - \pi^{k-1}_{\tau h_1h_2}) \\ &&& \quad + \tau\mathbb{D}e(u - u^{k-1}_{\tau h_1h_2}) : e(u - u^{k-1}_{\tau h_1h_2}) \\ &&& \quad + \tau\mathbb{B}(\pi - \pi^{k-1}_{\tau h_1h_2}) : (\pi - \pi^{k-1}_{\tau h_1h_2})\,\mathrm{d}x + 2\tau^2\Phi(u,\pi) - 2\tau^2\int_{\Gamma_1} g^k_\tau\cdot u\,\mathrm{d}S \\ &\text{subject to} && (u,\pi) \in V^d_{1,h_1}\times V^{d\times d}_{0,h_1} \text{ and } u|_{\Gamma_0} = u^k_{\mathrm{D},\tau h_1}, \end{aligned}\right. \tag{4.11}$$

where Φ is defined by (2.8). The existence of a unique solution, denoted $(u^k_{\tau h_1h_2}, \pi^k_{\tau h_1h_2})$, to (4.11) follows from the coercivity and strict convexity of the involved functionals. The corresponding optimality conditions yield (4.7)–(4.9). Noting that $\pi^0_{\tau h_1h_2} \in \mathbb{R}^{d\times d}_{\mathrm{sym},0}$ almost everywhere in Ω and that $\delta^*_{S(P_{0,h_1}\theta^{k-1}_{\tau h_1h_2})}$ is finite only on $\mathbb{R}^{d\times d}_{\mathrm{sym},0}$ we argue by induction to deduce that $\pi^k_{\tau h_1h_2} \in \mathbb{R}^{d\times d}_{\mathrm{sym},0}$ almost everywhere in Ω. The solution $(u^k_{\tau h_1h_2}, \pi^k_{\tau h_1h_2})$ defines $\omega^k_{\tau h_1h_2}$ as in (4.9) and we introduce the discrete dissipation term

$$\xi^k_{\tau h_1h_2} := \omega^k_{\tau h_1h_2} : d_t\pi^k_{\tau h_1h_2} + \mathbb{B} d_t\pi^k_{\tau h_1h_2} : d_t\pi^k_{\tau h_1h_2} + \mathbb{D} d_t e(u^k_{\tau h_1h_2}) : d_t e(u^k_{\tau h_1h_2}), \tag{4.12}$$

cf. (2.16). We then consider the minimization problem

$$\left\{\begin{aligned} &\text{minimize} && c_{\mathrm{v}}\big(\theta - \theta^{k-1}_{\tau h_1h_2}, \theta - \theta^{k-1}_{\tau h_1h_2}\big)_{h_2} + \tau\int_\Omega \mathbb{K}\nabla\theta\cdot\nabla\theta - 2\xi^k_{\tau h_1h_2}\theta\,\mathrm{d}x \\ &&& \qquad + \tau\big(\alpha\theta\,,\,\theta\big)_{\Gamma,h_2} - 2\tau\int_\Gamma f^k_{\tau,h_1}\theta\,\mathrm{d}S \\ &\text{subject to} && \theta \in V_{1,h_2}. \end{aligned}\right. \tag{4.13}$$

Since (4.13) defines a convex optimization problem which involves a quadratic functional there exists a unique solution, denoted $\theta^k_{\tau h_1 h_2}$, which solves (4.10). $\square$

Approximations provided by Algoritm ($\mathscr{A}$) obey the following discrete energy balance.

Proposition 4.2. *(**Discrete energy balance.**) For iterates of Algorithm ($\mathscr{A}$) and every $K = 1, 2, ..., T/\tau$ we have*

$$(4.14)\quad E_{\rm tot}\big(u^K_{\tau h_1 h_2}, \pi^K_{\tau h_1 h_2}, \theta^K_{\tau h_1 h_2}, d_t u^K_{\tau h_1 h_2}\big) - E_{\rm tot}\big(u^0_{\tau h_1 h_2}, \pi^0_{\tau h_1 h_2}, \theta^0_{\tau h_1 h_2}, d_t u^0_{\tau h_1 h_2}\big)$$
$$+ \frac{\tau^2}{2}\sum_{k=1}^K \int_\Omega \big|d_t^2 u^k_{\tau h_1 h_2}\big|^2 \,{\rm d}x + \frac{\tau^2}{2}\sum_{k=1}^K \int_\Omega \big|d_t \mathbb{C}(e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2}) : (e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2})\big|^2 \,{\rm d}x$$
$$= \tau \sum_{k=1}^K P^k_{\tau h_1 h_2}(d_t u^k_{\tau h_1 h_2}, \sigma^{{\rm nor},k}_{\tau h_1 h_2}, \theta^k_{\tau h_1 h_2}),$$

where

$$(4.15)\ P^k_{\tau h_1 h_2}(\dot u, \sigma, \theta) := \langle \sigma, d_t u^k_{{\rm D},\tau h_1}\rangle_{\Gamma_0} + \int_{\Gamma_1} g^k_\tau \cdot \dot u \,{\rm d}S + \int_\Gamma f^k_{\tau, h_1}\, dS - (\alpha\theta, 1)_{\Gamma, h_2}$$

and with $\sigma^{{\rm nor},k}_{\tau h_1 h_2}$ defined for every $v_{h_1} \in V_{1,h_1}$ through

$$(4.16)\ \langle \sigma^{{\rm nor},k}_{\tau h_1 h_2}, v_{h_1}\rangle_{\Gamma_0} := \int_\Omega d_t^2 u^k_{\tau h_1 h_2} \cdot v_{h_1}\,{\rm d}x + \int_\Omega \mathbb{D} d_t u^k_{\tau h_1 h_2} : v_{h_1}\,{\rm d}x$$
$$+ \int_\Omega \mathbb{C}(e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2}) : v_{h_1}\,{\rm d}x - \int_{\Gamma_1} g^k_\tau \cdot v_{h_1}\,{\rm d}S.$$

Proof. We substitute $z := d_t(u^k_{\tau h_1 h_2} - u^k_{{\rm D},\tau h_1}) \in V_{1,h_1}$ into (4.7), test (4.8) by $d_t \pi^k_{\tau h_1 h_2}$, and use the identity

$$(4.17)\ \mathbb{C}(e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2}) : d_t\big(e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2}\big)$$
$$= \frac{1}{2} d_t \Phi(u^k_{\tau h_1 h_2}, \pi^k_{\tau h_1 h_2}) + \frac{\tau}{2}\big|d_t \mathbb{C}(e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2}) : (e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2})\big|^2$$

with Φ as in (2.8). We then add resulting identities, multiply by τ, sum over $k = 1, 2, ..., K$, and employ discrete integration (or summation) by parts, i.e.,

$$(4.18)\ \tau \sum_{k=1}^K d_t^2 u^k_{\tau h_1 h_2} \cdot d_t u^k_{\tau h_1 h_2} = \frac{1}{2}\big|d_t u^K_{\tau h_1 h_2}\big|^2 - \frac{1}{2}\big|d_t u^0_{\tau h_1 h_2}\big|^2 + \frac{\tau^2}{2}\sum_{k=1}^K \big|d_t^2 u^k_{\tau h_1 h_2}\big|^2,$$

to deduce that

$$T_{\rm kin}\big(d_t u^K_{\tau h_1 h_2}\big) + \frac{1}{2}\Phi(u^K_{\tau h_1 h_2}, \pi^K_{\tau h_1 h_2}) + \tau \sum_{k=1}^K \int_\Omega \omega^k_{\tau h_1 h_2} : d_t \pi^k_{\tau h_1 h_2}\,{\rm d}x$$
$$+ \tau \sum_{k=1}^K \int_\Omega \mathbb{D} d_t e(u^k_{\tau h_1 h_2}) : d_t e(u^k_{\tau h_1 h_2})\,{\rm d}x + \tau \sum_{k=1}^K \int_\Omega \mathbb{B} d_t \pi^k_{\tau h_1 h_2} : d_t \pi^k_{\tau h_1 h_2}\,{\rm d}x$$
$$+ \frac{\tau^2}{2}\sum_{k=1}^K \int_\Omega \big|d_t^2 u^k_{\tau h_1 h_2}\big|^2 \,{\rm d}x + \frac{\tau^2}{2}\sum_{k=1}^K \int_\Omega \big|d_t \mathbb{C}(e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2}) : (e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2})\big|^2 \,{\rm d}x$$
$$= T_{\rm kin}\big(d_t u^0_{\tau h_1 h_2}\big) + \frac{1}{2}\Phi(u^0_{\tau h_1 h_2}, \pi^0_{\tau h_1 h_2}) + \tau \sum_{k=1}^K \int_{\Gamma_1} g^k_\tau \cdot d_t u^k_{\tau h_1 h_2}\,{\rm d}S + \tau \sum_{k=1}^K \langle \sigma^{{\rm nor},k}_{\tau h_1 h_2}, d_t u^k_{{\rm D},\tau h_1}\rangle_{\Gamma_0}.$$

Choosing $z = 1$ in (4.10) to replace the third to fifth term on the left-hand side implies (4.14). $\square$

4.3. **Passage to a semi-discrete scheme as $h_2 \to 0$.** Given the sequence of approximations provided by Algorithm ($\mathscr{A}$), we define piecewise constant approximations $\overline{u}_{\tau h_1 h_2}$, $\overline{\pi}_{\tau h_1 h_2}$, $\overline{\omega}_{\tau h_1 h_2}$, $\overline{\theta}_{\tau h_1 h_2}$, $\overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2}$, and $\overline{\xi}_{\tau h_1 h_2}$ by

$$\overline{u}_{\tau h_1 h_2}(t,\cdot) := u^k_{\tau h_1 h_2}, \quad \overline{\pi}_{\tau h_1 h_2}(t,\cdot) := \pi^k_{\tau h_1 h_2}, \quad \overline{\omega}_{\tau h_1 h_2}(t,\cdot) := \omega^k_{\tau h_1 h_2}, \tag{4.19}$$

$$\overline{\theta}_{\tau h_1 h_2}(t,\cdot) := \theta^k_{\tau h_1 h_2}, \quad \overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2}(t,\cdot) := \theta^{k-1}_{\tau h_1 h_2}, \quad \overline{\xi}_{\tau h_1 h_2}(t,\cdot) := \xi^k_{\tau h_1 h_2}, \tag{4.20}$$

with $\xi^k_{\tau h_1 h_2}$ as in (4.12), as well as piecewise affine interpolants $u_{\tau h_1 h_2}$, $\pi_{\tau h_1 h_2}$, and $\theta_{\tau h_1 h_2}$ by

$$\begin{aligned} u_{\tau h_1 h_2}(t,\cdot) &:= u^{k-1}_{\tau h_1 h_2} + (t-t_{k-1})\, d_t u^k_{\tau h_1 h_2}, \\ \pi_{\tau h_1 h_2}(t,\cdot) &:= \pi^{k-1}_{\tau h_1 h_2} + (t-t_{k-1})\, d_t \pi^k_{\tau h_1 h_2} \\ \theta_{\tau h_1 h_2}(t,\cdot) &:= \theta^{k-1}_{\tau h_1 h_2} + (t-t_{k-1})\, d_t \theta^k_{\tau h_1 h_2} \end{aligned} \tag{4.21}$$

for $t_{k-1} < t \le t_k$, $k = 1, ..., T/\tau$.

We denote by $W^{1,2}(\Omega)^*_{\mathrm{LCS}}$ the Hausdorff locally convex space consisting of $W^{1,2}(\Omega)^*$ equipped by the topology induced by countable collection of seminorms $|\cdot|_{h_2}$ induced by test with the countable collection of finite-dimensional spaces V_{h_2}, i.e. $|\xi|_{h_2} = \sup_{\|z\|_{W^{1,2}(\Omega)} \le 1,\, z \in V_{h_2}} \int_\Omega \xi z \, \mathrm{d}x$ for all $h_2 > 0$.

Proposition 4.3. ***(First a-priori estimates.)*** *For all $\tau, h_1, h_2, h_2^* > 0$ with $h_2 \le h_2^*$ we have $\theta_{\tau h_1 h_2} \ge 0$ almost everywhere in Q and*

$$\big\|u_{\tau h_1 h_2}\big\|_{W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^d))} \le C, \tag{4.22a}$$

$$\big\|\pi_{\tau h_1 h_2}\big\|_{W^{1,2}(I;L^2(\Omega;\mathbb{R}^{d\times d}))} \le C, \tag{4.22b}$$

$$\big\|\overline{\omega}_{\tau h_1 h_2}\big\|_{L^2(Q;\mathbb{R}^{d\times d})} \le C, \tag{4.22c}$$

$$\big\|\overline{\theta}_{\tau h_1 h_2}\big\|_{L^\infty(I;L^1(\Omega))} \le C, \tag{4.22d}$$

$$\big\|\theta_{\tau h_1 h_2}\big\|_{L^2(I;W^{1,2}(\Omega))} \le C_{h_1}, \tag{4.22e}$$

$$\int_0^T \left|\frac{\partial \theta_{\tau h_1 h_2}}{\partial t}\right|^2_{h_2^*} \mathrm{d}t \le C_{h_1}. \tag{4.22f}$$

Proof. We argue as in the proof of Proposition 4.2 and notice that in the last identity we have

$$\omega^k_{\tau h_1 h_2} : d_t \pi^k_{\tau h_1 h_2} \ge 0. \tag{4.23}$$

To bound the terms on the right-hand side of the last identity in the proof of Proposition 4.2 we use

$$\tau \sum_{k=1}^K d_t^2 u^k_{\tau h_1 h_2} \cdot d_t u^k_{\mathrm{D},\tau h_1} = d_t u^K_{\tau h_1 h_2} \cdot d_t u^K_{\mathrm{D},\tau h_1} - \tau \sum_{k=1}^K d_t u^{k-1}_{\tau h_1 h_2} \cdot d_t^2 u^k_{\mathrm{D},\tau h_1} \tag{4.24}$$

and owing to assumption (3.5h) and the definition (4.6) we have that $\tau \|d_t^2 u^k_{\mathrm{D},\tau h_1}\|_{L^2(\Omega;\mathbb{R}^d)}$ is summable over $k = 1, ..., T/\tau$ uniformly with respect to $\tau > 0$. We also use estimate

$$\int_{\Gamma_1} g^k_\tau \cdot d_t u^k_{\tau h_1 h_2} \, \mathrm{d}x \le C_N \big\|g^k_\tau\big\|_{L^{q/(q-1)}(\Gamma_1;\mathbb{R}^d)} \big\|d_t u^k_{\tau h_1 h_2}\big\|_{W^{1,2}(\Omega;\mathbb{R}^d)} \tag{4.25}$$

with C_N denoting the norm of the trace operator $W^{1,2}(\Omega) \to L^q(\Gamma_1)$. After adding and subtracting $d_t u^k_{\mathrm{D},\tau h_1}$ and employing Korn's inequality we may use Young's inequalities to absorb terms in the left-hand side and deduce (4.22a,b). The estimate (4.22c) follows from (4.22a,b) via (4.8). By assumption on the triangulation $\mathscr{T}^2_{h_2}$ and employed numerical integration we have $\theta^k_{\tau h_1 h_2} \geq 0$. Arguing as above to bound the right-hand side of (4.14) and using $(\alpha\theta^k_{\tau h_1 h_2}, 1)_{\Gamma,h_2} \geq 0$ we then deduce (4.22d). Now, for $h_1 > 0$ fixed, all norms on V_{0,h_1} and V_{1,h_1} are equivalent, e.g., with the norm induced from $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$, respectively, which shows that the piecewise constant interpolant $\overline{\xi}_{\tau h_1 h_2}$ of $\{\xi^k_{\tau h_1 h_2}\}_{k=1}^{T/\tau}$ (as defined in (4.12)) is uniformly bounded in $L^1(I; L^\infty(\Omega))$, hence also in $L^1(I; L^2(\Omega))$. Also $f_{h_1} \in L^2(I; L^2(\Gamma))$ and $\theta_{0,h_1} \in L^2(\Omega)$, which allows for usage of the standard L^2-theory for the heat equation to verify (4.22e,f). $\square$

In terms of the interpolated functions, we may write system (4.7)–(4.10) in the form

$$\int_\Omega \varrho \frac{\partial}{\partial t}\Big[\frac{\partial u_{\tau h_1 h_2}}{\partial t}\Big]^{\mathrm{i}} z + \mathbb{D}\frac{\partial e(u_{\tau h_1 h_2})}{\partial t} : e(z) + \mathbb{C}(e(\overline{u}_{\tau h_1 h_2}) - \overline{\pi}_{\tau h_1 h_2}) : e(z)\,\mathrm{d}x = \int_{\Gamma_1} \overline{g}_\tau \cdot z\,\mathrm{d}S, \tag{4.26}$$

$$\overline{\omega}_{\tau h_1 h_2} + \mathbb{B}\frac{\partial \pi_{\tau h_1 h_2}}{\partial t} - \mathbb{C}(e(\overline{u}_{\tau h_1 h_2}) - \overline{\pi}_{\tau h_1 h_2}) = 0, \tag{4.27}$$

$$\overline{\omega}_{\tau h_1 h_2} \in \partial\delta^*_{S(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2})}\Big(\frac{\partial \pi_{\tau h_1 h_2}}{\partial t}\Big), \tag{4.28}$$

$$\int_\Omega c_{\mathrm{v}} \frac{\partial \theta_{\tau h_1 h_2}}{\partial t} w + \mathbb{K}\nabla\overline{\theta}_{\tau h_1 h_2} \cdot \nabla w - \overline{\xi}_{\tau h_1 h_2} w\,\mathrm{d}x + \int_\Gamma \alpha\theta_{\tau h_1 h_2} w\,\mathrm{d}S = \int_\Gamma \overline{f}_{\tau h_1} w\,\mathrm{d}S + \overline{R}_{\tau h_1 h_2}(w), \tag{4.29}$$

for every $(z, w) \in V_{1,h_1} \times V_{1,h_2}$ and where $[\cdot]^{\mathrm{i}}$ denotes piece-wise affine interpolation in time. The functional $\overline{R}_{\tau h_1 h_2}$ involves error contributions coming from the employed numerical integration according to (4.1), cf. (4.32) below.

Proposition 4.4. *(**Convergence for** $h_2 \to 0$.) For fixed $\tau, h_1 > 0$ we have as $h_2 \to 0$, in terms of subsequences, $u_{\tau h_1 h_2} \to u_{\tau h_1}$ in $W^{1,2}(I; W^{1,2}(\Omega; \mathbb{R}^d))$, $\pi_{\tau h_1 h_2} \to \pi_{\tau h_1}$ in $W^{1,2}(I; L^2(\Omega; \mathbb{R}^{d\times d}))$, $\overline{\omega}_{\tau h_1 h_2} \to \overline{\omega}_{\tau h_1}$ in $L^\infty(Q; \mathbb{R}^{d\times d})$, $\theta_{\tau h_1 h_2} \rightharpoonup \theta_{\tau h_1}$ in $L^2(I; W^{1,2}(\Omega))$, and $(u_{\tau h_1}, \pi_{\tau h_1}, \overline{\omega}_{\tau h_1}, \theta_{\tau h_1})$ solves (4.26)–(4.28) with h_2 omitted and*

$$\int_Q \mathbb{K}\nabla\overline{\theta}_{\tau h_1} \cdot \nabla w - \overline{\xi}_{\tau h_1} w - c_{\mathrm{v}}\theta_{\tau h_1}\frac{\partial w}{\partial t}\,\mathrm{d}x\mathrm{d}t + \int_\Sigma \alpha\overline{\theta}_{\tau h_1} w\,\mathrm{d}S\mathrm{d}t = \int_\Sigma \overline{f}_{\tau h_1} w\,\mathrm{d}S + \int_\Omega c_{\mathrm{v}}\theta_{0,h_1}(x) w(0,x)\,\mathrm{d}x \tag{4.30}$$

for all $w \in C^\infty(\overline{Q})$ with $w(T,\cdot) = 0$ and with $\theta_{0,h_1} := \mathscr{I}^+_{1,h_1}\theta_0$ and

$$\overline{\xi}_{\tau h_1} = \overline{\omega}_{\tau h_1} : \frac{\partial \pi_{\tau h_1}}{\partial t} + \mathbb{B}\frac{\partial \pi_{\tau h_1}}{\partial t} : \frac{\partial \pi_{\tau h_1}}{\partial t} + \mathbb{D}\frac{\partial e(u_{\tau h_1})}{\partial t} : \frac{\partial e(u_{\tau h_1})}{\partial t}. \tag{4.31}$$

Proof. The selection of weakly convergent subsequences follows from the estimates (4.22a-c,e). By the Aubin-Lions theorem generalized for Hausdorff locally convex spaces as used in the estimate (4.22e), cf. [26, Lemma 7.7], we have $\theta_{\tau h_1 h_2} \to \theta_{\tau h_1}$ in $L^2(Q)$.

As τ and h_1 are fixed, the sequences $\{u_{\tau h_1 h_2}\}_{h_2>0}$, $\{\pi_{\tau h_1 h_2}\}_{h_2>0}$, and $\{\omega_{\tau h_1 h_2}\}_{h_2>0}$ belong to finite-dimensional subspaces and thus, the subsequences converge in fact strongly, as claimed. Moreover, also the piecewise constant interpolant $\overline{u}_{\tau h_1 h_2}$ converges to $\overline{u}_{\tau h_1}$ which is indeed the piecewise constant interpolant related to $u_{\tau h_1}$, and similarly $\overline{\pi}_{\tau h_1 h_2} \to \overline{\pi}_{\tau h_1}$ and $\overline{\theta}_{\tau h_1 h_2} \to \overline{\theta}_{\tau h_1}$. The limit passage in (4.26) and (4.27) is then straightforward. As to (4.29), the only peculiarity is in the numerical integration contained in residual functional

$\overline{R}_{\tau h_1 h_2}$. For any $w \in C^\infty(\overline{Q})$ with $w(T) = 0$ and $w_h(t,\cdot) := \mathscr{I}_{h_2} w(t,\cdot) \in V_{1,h_2}$ we have

$$(4.32)\int_0^T \overline{R}_{\tau h_1 h_2}(w_{h_2})\,\mathrm{d}t = \int_0^T \int_\Omega \frac{\partial \theta_{\tau h_1 h_2}}{\partial t} w_{h_2}\,\mathrm{d}x - \Big(\frac{\partial \theta_{\tau h_1 h_2}}{\partial t}, w_{h_2}\Big)_{h_2}\,\mathrm{d}t$$
$$+ \int_0^T \big(\alpha \overline{\theta}_{\tau h_1 h_2}, w_{h_2}\big)_{h_2} - \int_\Gamma \alpha \overline{\theta}_{\tau h_1 h_2} w_{h_2}\,\mathrm{d}S\,\mathrm{d}t =: E_1 + E_2.$$

For E_1 we verify, using by-part integration in time and (4.2),

$$(4.33)\, E_1 = -\int_0^T \int_\Omega \theta_{\tau h_1 h_2} \frac{\partial w_{h_2}}{\partial t}\,\mathrm{d}x - \Big(\theta_{\tau h_1 h_2}, \frac{\partial w_{h_2}}{\partial t}\Big)_{h_2}\,\mathrm{d}t$$
$$+ \int_\Omega \theta_{\tau h_1 h_2}(0,\cdot) w_{h_2}(0,\cdot)\,\mathrm{d}x - \Big(\theta_{\tau h_1 h_2}(0,\cdot), w_{h_2}(0,\cdot)\Big)_{h_2}$$
$$\le C_0 h_2 \Big(\|\theta_{\tau h_1 h_2}\|_{L^2(I;W^{1,2}(\Omega))} \Big\| \frac{\partial w_{h_2}}{\partial t} \Big\|_{L^2(I;L^2(\Omega))} + \|\theta_{\tau h_1 h_2}(0,\cdot)\|_{L^2(\Omega)} \|w_{h_2}(0,\cdot)\|_{W^{1,2}(\Omega)} \Big).$$

Notice that $\theta_{\tau h_1 h_2}(0,\cdot) = \mathscr{I}_{h_2}\theta_{0,h_1}$ and since $\theta_{0,h_1} \in V_{1,h_1}$ we have $\theta_{\tau h_1 h_2}(0,\cdot) \to \theta_{0,h_1}$ in $L^2(\Omega)$ as $h_2 \to 0$. Employing (4.2) and a nodal interpolation estimate we verify for E_2 that

$$(4.34)\qquad E_2 = \int_0^T \big(\mathscr{I}_{h_2}\big[\widetilde{\alpha}\overline{\theta}_{\tau h_1 h_2}\big], w_{h_2}\big)_{h_2} - \int_\Gamma \mathscr{I}_{h_2}\big[\widetilde{\alpha}\overline{\theta}_{\tau h_1 h_2}\big] w_{h_2}\,\mathrm{d}S\,\mathrm{d}t$$
$$+ \int_0^T \int_\Gamma \big(\mathscr{I}_{h_2}\big[\widetilde{\alpha}\overline{\theta}_{\tau h_1 h_2}\big] - \widetilde{\alpha}\overline{\theta}_{\tau h_1 h_2}\big) w_{h_2}\,\mathrm{d}S\,\mathrm{d}t$$
$$\le C_0 h_2 \big\|\widetilde{\alpha}\overline{\theta}_{\tau h_1 h_2}\big\|_{L^2(I;W^{1,2}(\Omega))} \big\|w_{h_2}\big\|_{L^2(I;W^{1,2}(\Omega))} \to 0,$$

where $\widetilde{\alpha}$ is an extension of α qualified in (3.5e). It is a routine calculation to show that this qualification allows for $\widetilde{\alpha} \in W^{1,d+\epsilon}(\Omega) \subset W^{1,d}(\Omega) \cap L^\infty(\Omega)$ which just guarantees that $\theta \mapsto \widetilde{\alpha}\theta$ is a bounded operator $W^{1,2}(\Omega) \to W^{1,2}(\Omega)$. On combining the estimates we deduce that the left-hand side of (4.32) tends to zero as $h_2 \to 0$.

As to (4.28), we first realize that $\overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2} \to \overline{\theta}^{\mathrm{R}}_{\tau h_1}$ in $L^2(Q)$ as $h_2 \to 0$ with $\overline{\theta}^{\mathrm{R}}_{\tau h_1}$ piecewise constant in time. Taking $w, z \in V_{0,h_1}$ such that $w \in \partial\delta^*_{S(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1}(t,\cdot))}(z)$, we need to pass for almost every $t \in I$ to the limit in

$$(4.35)\qquad \int_\Omega (\overline{\varpi}_{\tau h_1 h_2}(t,\cdot) - w_{h_2}) : \Big(\frac{\partial \pi_{\tau h_1 h_2}}{\partial t}(t,\cdot) - z\Big)\,\mathrm{d}x \ge 0$$

for suitable $w_{h_2} \in V_{0,h_2}$ satisfying $w_{h_2} \in \partial\delta^*_{S(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2}(t,\cdot))}(z)$. As in (3.6), we employ

$$(4.36)\qquad w_{h_2} = \Upsilon(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2}(t,\cdot))w / \Upsilon(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1}(t,\cdot)).$$

Since $\overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2}(t,\cdot) \to \overline{\theta}^{\mathrm{R}}_{\tau h_1}(t,\cdot)$ in $L^2(\Omega)$, we have also $P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2}(t,\cdot) \to P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1}(t,\cdot)$ in $L^2(\Omega)$ and, owing to dominated convergence theorem, $\Upsilon(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1 h_2}) \to \Upsilon(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1})$ in any $L^p(Q)$ with $p < \infty$. Therefore, we have $w_{h_2} \to w$ in $L^p(Q;\mathbb{R}^{d\times d})$ even for any $p < \infty$. Recalling the strong convergence of $\{\overline{\varpi}_{\tau h_1 h_2}\}_{h_2>0}$ and $\{\frac{\partial}{\partial t}\pi_{\tau h_1 h_2}\}_{h_2>0}$ in the finite-dimensional subspaces the limit pasage in (4.35) is straightforward. $\square$

4.4. **Passage to the very weak formulation.** Having passed to the semi-discrete scheme (4.26)–(4.28) (with h_2 omitted) and (4.30) we may employ various nonlinearities of $\theta_{\tau h_1}$ as test functions in (4.30). This allows to derive bounds for $\nabla\theta_{\tau h_1}$ that are uniform in τ, h_1 which would not have been available in the fully discrete heat equation.

Proposition 4.5. ***(Further a-priori estimates.)*** *For all $\tau, h_1 > 0$ and $0 < \delta \le \frac{1}{d+1}$ we have*

$$\big\|\nabla\theta_{\tau h_1}\big\|_{L^{\frac{d+2}{d+1}-\delta}(Q;\mathbb{R}^d)} \le C_\delta, \tag{4.37a}$$

$$\Big\|\frac{\partial\theta_{\tau h_1}}{\partial t}\Big\|_{L^1(I;W^{-1-d/2,2}(\Omega))} \le C. \tag{4.37b}$$

Comments to the proof. The estimate (4.37a) can be obtained from (4.22d) for any "semi-discrete-in-time" solution to (4.30) with the technique developed by Boccardo, Gallouët et al. [4, 5], see also [28]. The technique uses tests of the heat equation by $\varphi(\overline{\theta}_{\tau h_1})$ with various bounded nondecreasing Lipschitz-continuous nonlinearities $\varphi : \mathbb{R} \to \mathbb{R}$, and then employs Hölder inequalities and Gagliardo-Nirenberg interpolation with (4.22d). For the time-discrete case, it is important that φ is nondecreasing and bounded, hence its potential, denoted by $\widehat{\varphi}$, is convex and has at most linear growth. Therefore, we have

$$\big(d_t\theta^k_{\tau h_1}\big)\varphi(\theta^k_{\tau h_1}) \ge d_t\widehat{\varphi}(\theta^k_{\tau h_1}), \tag{4.38}$$

which, after summation over $k = 1, ..., T/\tau$ and integration over Ω is bounded since $\widehat{\varphi}(\theta^0_{\tau h_1}) \in L^1(\Omega)$. Thus, the estimation reduces to the spatial gradient term which is done exactly in the same way as in the time-continuous case in [4, 5].

Finally, the estimate (4.37b) follows from (4.22d) with h_2 omitted and (4.37a) by testing (4.30) with functions $w \in L^\infty(I; W_0^{1+d/2,2}(\Omega))$ which have essentially bounded gradients. $\square$

Proposition 4.6. ***(Convergence for*** $(\tau, h_1) \to (0, 0)$***.)*** *As $(\tau, h_1) \to (0, 0)$, in terms of subsequences, we have $u_{\tau h_1} \rightharpoonup u$ in $W^{1,2}(I; W^{1,2}(\Omega;\mathbb{R}^d))$, $\pi_{\tau h_1} \rightharpoonup \pi$ in $W^{1,2}(I; L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym},0}))$, $\overline{\omega}_{\tau h_1} \overset{*}{\rightharpoonup} \omega$ in $L^2(Q;\mathbb{R}^{d\times d})$,*

$$\theta_{\tau h_1} \overset{*}{\rightharpoonup} \theta \qquad in \qquad L^{\frac{d+2}{d+1}-\delta}(I; W^{1,\frac{d+2}{d+1}-\delta}(\Omega)) \cap L^\infty(I; L^1(\Omega)) \tag{4.39}$$

with any $0 < \delta \le \frac{1}{d+1}$, and (u, π, ω, θ) is a very weak solution of (2.2) and (3.1)–(3.2) in the sense of Definition 3.1.

Proof. The selection of weakly convergent subsequences follows by the standard Banach selection principle from the estimates (4.22a-d) which are inherited also for the approximate solution $(u_{\tau h_1}, \pi_{\tau h_1}, \overline{\omega}_{\tau h_1}, \theta_{\tau h_1})$, and by the estimates (4.37) and

$$\int_0^T \Big|\frac{\partial}{\partial t}\Big[\frac{\partial u_{\tau h_1}}{\partial t}\Big]^{\mathrm{i}}\Big|^2_{\Gamma_0,h_1^*} \mathrm{d}t \le C \qquad \text{for any } h_1 \le h_1^*, \tag{4.40a}$$

where the seminorm $|\cdot|_{\Gamma_0,h_1^*}$ is defined as

$$|\xi|_{\Gamma_0,h_1^*} := \sup_{\substack{\|v\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^d))}\le 1\\ v\in L^2(I;V^d_{1,h_1^*}),\ v|_{\Sigma_0}=0}} \int_Q \xi\cdot v\,\mathrm{d}x\mathrm{d}t. \tag{4.41}$$

The estimate (4.40) can be obtained by testing (4.26), limitted by $h_2 \to 0$, by test functions from $L^2(I; V^d_{1,h_1^*})$. Using (4.37b), arguing with the Aubin-Lions theorem, and interpolating with the estimate (4.22d) for $\theta_{\tau h_1}$, cf. [26, Lemmas 7.7 and 7.8], we have also $\theta_{\tau h_1} \to \theta$ in $L^{\frac{d+2}{d}-\varepsilon}(Q)$, $0 < \varepsilon \le 2/d$.

We want to prove that $\omega \in \partial\delta^*_{S(\theta)}(\frac{\partial}{\partial t}\pi)$. As in the proof of Proposition 4.4, we consider $w, z \in L^2(Q;\mathbb{R}^{d\times d})$ such that $w \in \partial\delta^*_{S(\theta)}(z)$ almost everywhere in Q. We know that $\overline{\omega}_{\tau h_1} \in$

$\partial\delta^*_{S(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1})}(\frac{\partial}{\partial t}\pi_{\tau h_1})$, and, owing to the maximal monotonicity of $\partial\delta^*_{S(\theta)}(\cdot):\mathbb{R}^{d\times d}\to\mathbb{R}^{d\times d}$, it suffices to pass to the limit in the inequality

$$\int_Q(\overline{\varpi}_{\tau h_1}-w_{\tau h_1}):\Big(\frac{\partial\pi_{\tau h_1}}{\partial t}-z\Big)\,\mathrm{d}x\mathrm{d}t\ge 0 \tag{4.42}$$

for suitable $w_{\tau h_1}\in\partial\delta^*_{S(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1})}(z)$. Like in (3.6), we take $w_{\tau h_1}=\Upsilon(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1})w/\Upsilon(\theta)$ which satisfies $w_{\tau h_1}\to w$ in $L^q(Q;\mathbb{R}^{d\times d})$ for every $1\le q<\infty$ and $w_{\tau h_1}\in\partial\delta^*_{S(P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1})}(z)$ almost everywhere in Q. Following [9] (see also [26, Prop. 11.5]), the critical limit passage of the product $\overline{\varpi}_{\tau h_1}:\frac{\partial\pi_{\tau h_1}}{\partial t}$ can be performed by using the information from the equations themselves. Namely, we use (4.27) (with h_2 omitted) tested by $\frac{\partial}{\partial t}\pi_{\tau h_1}$ and add (4.26) (with h_2 omitted) tested by $\frac{\partial}{\partial t}(u_{\tau h_1}-u_{\mathrm{D},\tau h_1})$, and estimate the limit superior as follows:

$$\begin{aligned}
&\limsup_{(\tau,h_1)\to(0,0)}\int_Q\overline{\varpi}_{\tau h_1}:\frac{\partial\pi_{\tau h_1}}{\partial t}\,\mathrm{d}x\mathrm{d}t\\
&\quad=\limsup_{(\tau,h_1)\to(0,0)}\int_Q\mathbb{C}(e(\overline{u}_{\tau h_1})-\overline{\pi}_{\tau h_1}):\frac{\partial\pi_{\tau h_1}}{\partial t}-\mathbb{B}\frac{\partial\pi_{\tau h_1}}{\partial t}:\frac{\partial\pi_{\tau h_1}}{\partial t}\,\mathrm{d}x\mathrm{d}t\\
&\quad=\limsup_{(\tau,h_1)\to(0,0)}\bigg(\int_Q-\mathbb{C}(e(\overline{u}_{\tau h_1})-\overline{\pi}_{\tau h_1}):e\big(\frac{\partial u_{\tau h_1}}{\partial t}\big)-\varrho\frac{\partial}{\partial t}\Big[\frac{\partial u_{\tau h_1}}{\partial t}\Big]^{\mathrm{i}}\frac{\partial u_{\tau h_1}}{\partial t}\\
&\qquad-\mathbb{D}e\big(\frac{\partial u_{\tau h_1}}{\partial t}\big):e\big(\frac{\partial u_{\tau h_1}}{\partial t}\big)\,\mathrm{d}x\mathrm{d}t+\int_{\Sigma_0}\sigma_{\tau h_1}\cdot\frac{\partial u_{\mathrm{D},\tau h_1}}{\partial t}\,\mathrm{d}S\mathrm{d}t+\int_{\Sigma_1}\overline{g}_\tau\cdot\frac{\partial u_{\tau h_1}}{\partial t}\,\mathrm{d}S\mathrm{d}t\\
&\qquad+\int_Q\mathbb{C}(e(\overline{u}_{\tau h_1})-\overline{\pi}_{\tau h_1}):\frac{\partial\pi_{\tau h_1}}{\partial t}-\mathbb{B}\frac{\partial\pi_{\tau h_1}}{\partial t}:\frac{\partial\pi_{\tau h_1}}{\partial t}\,\mathrm{d}x\mathrm{d}t\bigg)\\
&\quad\le\lim_{h_1\to 0}\int_\Omega\frac12\mathbb{C}(e(u_{0,h_1}(x))-\pi_{0,h_1}(x)):(e(u_{0,h_1}(x))-\pi_{0,h_1}(x))+\frac{\varrho}{2}\big|\dot{u}_{0,h_1}(x)\big|^2\mathrm{d}x\\
&\qquad-\liminf_{(\tau,h_1)\to(0,0)}\bigg(\int_\Omega\frac12\mathbb{C}(e(u_{\tau h_1}(T,x))-\pi_{\tau h_1}(T,x)):(e(u_{\tau h_1}(T,x))-\pi_{\tau h_1}(T,x))\\
&\qquad+\frac{\varrho}{2}\Big|\frac{\partial u_{\tau h_1}}{\partial t}(T,x)\Big|^2\mathrm{d}x+\int_Q\mathbb{B}\frac{\partial\pi_{\tau h_1}}{\partial t}:\frac{\partial\pi_{\tau h_1}}{\partial t}+\mathbb{D}e\big(\frac{\partial u_{\tau h_1}}{\partial t}\big):e\big(\frac{\partial u_{\tau h_1}}{\partial t}\big)\,\mathrm{d}x\mathrm{d}t\bigg)\\
&\qquad+\lim_{(\tau,h_1)\to(0,0)}\bigg(\int_{\Sigma_0}\sigma_{\tau h_1}\cdot\frac{\partial u_{\mathrm{D},\tau h_1}}{\partial t}\,\mathrm{d}S\mathrm{d}t+\int_{\Sigma_1}\overline{g}_\tau\cdot\frac{\partial u_{\tau h_1}}{\partial t}\,\mathrm{d}S\mathrm{d}t\bigg).
\end{aligned}\tag{4.43}$$

The first inequality in (4.43) is due to a numerical time-integration error, cf. the last identity in the proof of Proposition 4.2. Since initial and boundary data converge strongly and since $u_{\tau h_1}(T,\cdot)\rightharpoonup u(T,\cdot)$ in $W^{1,2}(\Omega;\mathbb{R}^d)$ and $\pi_{\tau h_1}(T,\cdot))\rightharpoonup\pi(T,\cdot)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ by the a-priori estimates (4.22a,b) we may argue with weakly lower semi-continuity to verify that

$$\begin{aligned}
&\limsup_{(\tau,h_1)\to(0,0)}\int_Q\overline{\varpi}_{\tau h_1}:\frac{\partial\pi_{\tau h_1}}{\partial t}\,\mathrm{d}x\mathrm{d}t\\
&\le\int_\Omega\frac12\mathbb{C}(e(u_0(x))-\pi_0(x)):(e(u_0(x))-\pi_0(x))+\frac{\varrho}{2}\big|\dot{u}_0(x)\big|^2\mathrm{d}x\\
&\quad-\int_\Omega\frac12\mathbb{C}(e(u(T,x))-\pi(T,x)):(e(u(T,x))-\pi(T,x))+\frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T,x)\Big|^2\mathrm{d}x\\
&\quad-\int_Q\mathbb{B}\frac{\partial\pi}{\partial t}:\frac{\partial\pi}{\partial t}+\mathbb{D}e\big(\frac{\partial u}{\partial t}\big):e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t+\int_0^T\Big\langle\sigma,\frac{\partial u_{\mathrm{D}}}{\partial t}\Big\rangle_{\Gamma_0}\mathrm{d}t+\int_{\Sigma_1}\overline{g}_\tau\cdot\frac{\partial u}{\partial t}\,\mathrm{d}S\mathrm{d}t.
\end{aligned}\tag{4.44}$$

By the above convergence properties of the approximations we may employ (2.2) and (3.1a), which are obtained in the limit from (4.26) and (4.27) (with h_2 omitted) as $(\tau, h_1) \to (0,0)$, to conclude

$$
\begin{aligned}
(4.45)\quad & \limsup_{(\tau,h_1)\to(0,0)} \int_Q \overline{\omega}_{\tau h_1} : \frac{\partial \pi_{\tau h_1}}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
& \le -\int_0^T \Big\langle \varrho \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\Big\rangle \mathrm{d}t - \int_Q \mathbb{C}(e(u)-\pi) : e\Big(\frac{\partial u}{\partial t}\Big) + \mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t \\
& \quad + \int_0^T \Big\langle \sigma, \frac{\partial u_{\mathrm{D}}}{\partial t}\Big\rangle_{\Gamma_0} \mathrm{d}t + \int_{\Sigma_1} g \cdot \frac{\partial u}{\partial t}\,\mathrm{d}S\mathrm{d}t + \int_Q \mathbb{C}(e(u){-}\pi) : \frac{\partial \pi}{\partial t} - \mathbb{B}\frac{\partial \pi}{\partial t} : \frac{\partial \pi}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
& = \int_Q \omega : \frac{\partial \pi}{\partial t}\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where $\sigma_{\tau h_1}$ is the approximate normal stress on Σ_0 defined, like in (2.13), by

$$
\begin{aligned}
(4.46)\quad \int_{\Sigma_0} \sigma_{\tau h_1} \cdot v\,\mathrm{d}S\mathrm{d}t = & \int_Q \varrho \frac{\partial}{\partial t}\Big[\frac{\partial u_{\tau h_1}}{\partial t}\Big]^{\mathrm{i}} \frac{\partial u_{\tau h_1}}{\partial t} \cdot \overline{v} + \mathbb{C}(e(\overline{u}_{\tau h_1}){-}\overline{\pi}_{\tau h_1}) : e(\overline{v})\mathrm{d}x\mathrm{d}t \\
& - \int_{\Sigma_1} \overline{g}_\tau(t,\cdot) \cdot \overline{v}\,\mathrm{d}S\mathrm{d}t
\end{aligned}
$$

where $\overline{v} \in V^d_{1,h_1}$ is (any) extention of v, and σ and $\langle \sigma, \cdot \rangle_{\Gamma_0}$ is as in (2.13).

Eventually, we estimate the limit superior in (4.42) to verify that $\omega \in \partial \delta^*_{S(\theta)}(\frac{\partial}{\partial t}\pi)$.

Limit passage in the heat equation requires still the (at least weak) convergence of the disipation heat $\overline{\xi}_{\tau h_1} \to \xi$ in $L^1(Q)$ with ξ satisfying (2.16). We know that $\{\delta^*_{S(P_{0,h_1}\overline{\theta}_{\tau h_1})}(\frac{\partial}{\partial t}\pi_{\tau h_1})\}_{\tau,h_1>0}$ is bounded in $L^2(Q)$, hence as a subsequence it must converge to some ξ_1 weakly in $L^2(Q)$. As $P_{0,h_1}\overline{\theta}^{\mathrm{R}}_{\tau h_1} \to \theta$ in $L^{\frac{d+2}{d}-\epsilon}(Q)$ with $\epsilon > 0$ and the integrand $(\vartheta, \dot{\pi}) \mapsto \delta^*_{S(\vartheta)}(\dot{\pi})$ is continous in both variables and convex in $\dot{\pi}$-variable, we have by (norm×weak)-lower semicontinuity argument (cf. also [22, Lemma 5.1]), we have $\xi_1 \ge \delta^*_{S(\theta)}(\frac{\partial \pi}{\partial t})$. In fact, in our special case (3.5d), we have simply $\delta^*_{S(\vartheta)}(\dot{\pi}) = \Upsilon(\vartheta)\delta^*_{S_0}(\dot{\pi})$.

Also $\{\mathbb{D}e(\frac{\partial}{\partial t}u_{\tau h_1}) : e(\frac{\partial}{\partial t}u_{\tau h_1})\}_{\tau,h_1>0}$ and $\{\mathbb{B}\frac{\partial}{\partial t}\pi_{\tau h_1} : \frac{\partial}{\partial t}\pi_{\tau h_1}\}_{\tau,h_1>0}$ are bounded in $L^1(Q)$, hence as subsequences they must converge weakly* in measures on $\overline{Q}$ to some ξ_2 and ξ_3, respectively. By weak* lower semicontinuity, again $\xi_2 \ge \mathbb{D}e(\frac{\partial}{\partial t}u) : e(\frac{\partial}{\partial t}u)$ and $\xi_3 \ge \mathbb{B}\frac{\partial}{\partial t}\pi : \frac{\partial}{\partial t}\pi$. We put $\xi := \xi_1 + \xi_2 + \xi_3$.

Arguing as in (4.43)-(4.45), we have

$$
\begin{aligned}
(4.47)\quad & \int_Q \delta^*_{S(\theta)}\Big(\frac{\partial \pi}{\partial t}\Big) + \mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) : e\Big(\frac{\partial u}{\partial t}\Big) + \mathbb{B}\frac{\partial \pi}{\partial t} : \frac{\partial \pi}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
& \le \int_{\overline{Q}} \xi(\mathrm{d}x\mathrm{d}t) = \int_Q \xi_1\,\mathrm{d}x\mathrm{d}t + \int_{\overline{Q}} \xi_2(\mathrm{d}x\mathrm{d}t) + \int_{\overline{Q}} \xi_3(\mathrm{d}x\mathrm{d}t) \\
& \le \liminf_{(\tau,h_1)\to(0,0)} \int_Q \xi_{\tau h_1}\,\mathrm{d}x\mathrm{d}t \le \limsup_{(\tau,h_1)\to(0,0)} \int_Q \xi_{\tau h_1}\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

and

$$\int_Q \delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big) + \mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) : e\Big(\frac{\partial u}{\partial t}\Big) + \mathbb{B}\frac{\partial\pi}{\partial t} : \frac{\partial\pi}{\partial t}\,\mathrm{d}x\mathrm{d}t \tag{4.48}$$

$$\leq \limsup_{(\tau,h_1)\to(0,0)} \bigg(\int_Q -\mathbb{C}(e(\overline{u}_{\tau h_1}) - \overline{\pi}_{\tau h_1}) : e\Big(\frac{\partial u_{\tau h_1}}{\partial t}\Big) - \varrho\frac{\partial}{\partial t}\Big[\frac{\partial u_{\tau h_1}}{\partial t}\Big]^{\mathrm{i}}\frac{\partial u_{\tau h_1}}{\partial t}\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_{\Sigma_0} \sigma_{\tau h_1}\cdot\frac{\partial u_{\mathrm{D},\tau h_1}}{\partial t}\,\mathrm{d}S\mathrm{d}t + \int_{\Sigma_1} \overline{g}_\tau\cdot\frac{\partial u_{\tau h_1}}{\partial t}\,\mathrm{d}S\mathrm{d}t$$

$$+ \int_Q \mathbb{C}(e(\overline{u}_{\tau h_1}) - \overline{\pi}_{\tau h_1}) : \frac{\partial\pi_{\tau h_1}}{\partial t}\,\mathrm{d}x\mathrm{d}t\bigg)$$

$$\leq -\int_0^T \Big\langle\varrho\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\Big\rangle\mathrm{d}t - \int_Q \mathbb{C}(e(u)-\pi) : e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^T \Big\langle\sigma, \frac{\partial u_{\mathrm{D}}}{\partial t}\Big\rangle_{\Gamma_0}\mathrm{d}t + \int_{\Sigma_1} g\cdot\frac{\partial u}{\partial t}\,\mathrm{d}S\mathrm{d}t + \int_Q \mathbb{C}(e(u)-\pi) : \frac{\partial\pi}{\partial t}\,\mathrm{d}x\mathrm{d}t$$

$$= \int_Q \delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big) + \mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) : e\Big(\frac{\partial u}{\partial t}\Big) + \mathbb{B}\frac{\partial\pi}{\partial t} : \frac{\partial\pi}{\partial t}\,\mathrm{d}x\mathrm{d}t.$$

The last equality follows by testing the limit equations (2.2) and (2.3) by $\frac{\partial(u-u_{\mathrm{D}})}{\partial t}$ and by $\frac{\partial\pi}{\partial t}$, respectively, and by using also (3.3) and the fact that $\frac{\partial^2 u}{\partial t^2}$ is in duality with $\frac{\partial u}{\partial t}$. Therefore, all inequalities in (4.48) are, in fact, equalities and

$$\lim_{(\tau,h_1)\to(0,0)} \int_Q \xi_{\tau h_1}\,\mathrm{d}x\mathrm{d}t = \int_Q \delta^*_{S(\theta)}\Big(\frac{\partial\pi}{\partial t}\Big) + \mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) : e\Big(\frac{\partial u}{\partial t}\Big) + \mathbb{B}\frac{\partial\pi}{\partial t} : \frac{\partial\pi}{\partial t}\,\mathrm{d}x\mathrm{d}t. \tag{4.49}$$

Hence $\xi_{\tau h_1}$ converges to $\delta^*_{S(\theta)}(\frac{\partial\pi}{\partial t}) + \mathbb{D}e(\frac{\partial u}{\partial t}) : e(\frac{\partial u}{\partial t}) + \mathbb{B}\frac{\partial\pi}{\partial t} : \frac{\partial\pi}{\partial t}$ weakly* in measures on $\overline{Q}$, and because of the absolute continuity of the limit, even weakly in $L^1(Q)$. Then the limit passage in the discrete heat equation is straightforward. $\square$

Corollary 4.7. *Under the data qualification (3.5) a very weak solution in the sense of Definition 3.1 exists.*

5. Implementation of the algorithm and illustrative simulations

In our implementation we solved the variational inclusion exactly, making use of the equivalence

$$\omega^k_{\tau h_1 h_2} \in \partial\delta^*_{S(P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2})}(d_t\pi^k_{\tau h_1 h_2}) \quad\Longleftrightarrow\quad d_t\pi^k_{\tau h_1 h_2} \in \partial\delta_{S(P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2})}(\omega^k_{\tau h_1 h_2}). \tag{5.1}$$

We make the simplification $\mathbb{B} = 0$ and introduce $A^k_{\tau h_1 h_2} := d_t e(u^k_{\tau h_1 h_2}) - \tau^{-1}\mathbb{C}^{-1}\omega^{k-1}_{\tau h_1 h_2}$. Noting the identity $\pi^k_{\tau h_1 h_2} = e(u^k_{\tau h_1 h_2}) - \mathbb{C}^{-1}\omega^k_{\tau h_1 h_2}$ the flow rule then reads

$$A^k_{\tau h_1 h_2} - \tau^{-1}\mathbb{C}^{-1}\omega^k_{\tau h_1 h_2} \in \partial\delta_{S(P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2})}(\omega^k_{\tau h_1 h_2}). \tag{5.2}$$

For certain material laws and stress-strain relations it is possible to derive an explicit formula for the unique solution $\omega^k_{\tau h_1 h_2}$ of (5.2) in terms of (given) $A^k_{\tau h_1 h_2}$, $\theta^{k-1}_{\tau h_1 h_2}$, and τ. We confine ourselves to the linear stress-strain relation

$$\omega^k_{\tau h_1 h_2} = \mathbb{C}\,\varepsilon^k_{\tau h_1 h_2} = \lambda\,\mathrm{tr}\,\varepsilon^k_{\tau h_1 h_2}\mathbf{I}_{d\times d} + 2\mu\,\varepsilon^k_{\tau h_1 h_2} \tag{5.3}$$

for Lamé coefficients $\lambda \geq 0$ and $\mu > 0$ and the elastic strain tensor $\varepsilon^k_{\tau h_1 h_2} = e(u^k_{\tau h_1 h_2}) - \pi^k_{\tau h_1 h_2}$. Moreover, we consider perfect plasticity defined through the von-Mises yield function $\Phi(\omega) := |\mathrm{dev}\,\omega| - \omega_{y,0}$ and the corresponding set of admissible elastic stresses

$$S_0 := \{\omega \in \mathbb{R}^{d\times d}_{\mathrm{sym}};\ |\mathrm{dev}\,\omega| \leq \omega_{y,0}\}, \tag{5.4}$$

where $\omega_{y,0}$ is the yield stress and "dev" denotes the trace free part of a tensor. Temperature dependence is included through the bounded, continuous function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ which describes a decrease of the yield stress for large temperatures; we have

$$S(\theta) = \Upsilon(\theta) S_0 = \{\omega \in \mathbb{R}^{d\times d}_{\mathrm{sym}};\ |\mathrm{dev}\,\omega| \le \omega_{y,0}\Upsilon(\theta)\}.$$

With these definitions we are in the setting of [7, Theorem 3.1] and may deduce that for given $A^k_{\tau h_1 h_2}$, $\theta^{k-1}_{\tau h_1 h_2}$, and $\tau > 0$ there exists a unique solution $\omega^k_{\tau h_1 h_2}$ of (5.2) which is given by

$$\begin{aligned}\omega^k_{\tau h_1 h_2} &= \Sigma(A^k_{\tau h_1 h_2}, P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2}, \tau)\\ &:= (\lambda + 2\mu/d)\mathrm{tr}\,\tau A^k_{\tau h_1 h_2}\,\mathbf{I}_{d\times d} + F(A^k_{\tau h_1 h_2}, P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2}, \tau)\mathrm{dev}\,\tau A^k_{\tau h_1 h_2}\end{aligned} \tag{5.5}$$

where

$$F(A^k_{\tau h_1 h_2}, \theta^{k-1}_{\tau h_1 h_2}, \tau) = \begin{cases} \frac{\omega_{y,0}\Upsilon(P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2})}{|\mathrm{dev}\,\tau A^k_{\tau h_1 h_2}|} & \text{for } |\mathrm{dev}\,\tau A^k_{\tau h_1 h_2}| \ge \frac{\omega_{y,0}\Upsilon(P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2})}{2\mu},\\ 2\mu & \text{for } |\mathrm{dev}\,\tau A^k_{\tau h_1 h_2}| \le \frac{\omega_{y,0}\Upsilon(P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2})}{2\mu}.\end{cases} \tag{5.6}$$

In particular, the plastic phase occurs for $|\mathrm{dev}\,\tau A^k_{\tau h_1 h_2}| \ge \omega_{y,0}\Upsilon(P_{0,h_1}\theta^{k-1}_{\tau h_1 h_2})/(2\mu)$. For explicit formulas in case of other plastic material behavior such as plasticity with isotropic or linear kinematic hardening we refer the reader to [7].

In addition to the simplification $\mathbb{B} = 0$ we set $\varrho := 0$ and $\mathbb{D} := 0$ in the numerical experiments reported below. The discrete scheme (4.7)-(4.10) then reduces to the following quasi-stationary, pure (nonlinear) displacement and temperature formulation: Given $(u^{k-1}_{\tau h_1 h_2}, \pi^{k-1}_{\tau h_1 h_2}, \omega^{k-1}_{\tau h_1 h_2}, \theta^{k-1}_{\tau h_1 h_2}) \in V^d_{1,h1} \times V^{d\times d}_{0,h_1} \times V^{d\times d}_{0,h_1} \times V_{1,h_2}$ find $u^k_{\tau h_1 h_2} \in V^d_{1,h1}$ such that $u^k_{\tau h_1 h_2}|_{\Gamma_0} = u_{\mathrm{D},\tau h_1}$ and

$$\int_\Omega \Sigma\big(A^k_{\tau h_1 h_2}[u^k_{\tau h_1 h_2}], \theta^{k-1}_{\tau h_1 h_2}, \tau\big) : e(z)\,\mathrm{d}x = \int_{\Gamma_1} g^k_\tau \cdot z\,\mathrm{d}S \tag{5.7}$$

for all $z \in V^d_{1,h_1}$ with $z|_{\Gamma_0} = 0$. Subsequently, set $\omega^k_{\tau h_1 h_2} := \Sigma(A^k_{\tau h_1 h_2}, \theta^{k-1}_{\tau h_1 h_2}, \tau)$ and $\pi^k_{\tau h_1 h_2} := e(u^k_{\tau h_1 h_2}) - \mathbb{C}^{-1}\omega^k_{\tau h_1 h_2}$ and compute $\theta^k_{\tau h_1 h_2} \in V_{1,h2}$ such that

$$\begin{aligned} c_{\mathrm{v}}\big(d_t\theta^k_{\tau h_1 h_2}, z\big)_{h_2} + \int_\Omega \mathbb{K}\nabla\theta^k_{\tau h_1 h_2}\cdot\nabla z\,\mathrm{d}x + \big(\alpha\theta^k_{\tau h_1 h_2}, z\big)_{\Gamma,h_2}\\ = \int_\Omega \omega^k_{\tau h_1 h_2} : d_t\pi^k_{\tau h_1 h_2}\,\mathrm{d}x + \int_\Gamma f^k_{\tau,h_1} z\,\mathrm{d}S\end{aligned} \tag{5.8}$$

for all $z \in V_{1,h_2}$.

The implementation of the approximation scheme was done in MATLAB in the spirit of [7, 3]. In this implementation, the nonlinear system of equations (5.7) is approximated with a Newton iteration and all occurring systems of linear equations are solved using MATLAB's backslash operator. In our experiments the Newton scheme always terminated within at most 7 iterations to achieve an ℓ^2 norm of the residual vector (defined through nodal basis functions) less than 10^{-2}.

We used the scheme (5.7)–(5.8) to simulate the hard device loading of a thermally isolated body occupying the domain Ω depicted in Figure 1 and specified in the following example. The problem leads to the phenomenon of so-called "thermal necking" which has also been observed in [6, 13, 20].

Setting. Let $d := 2$,

$$\begin{aligned}\Omega := \big((-a/2, a/2) \times (-b/2, b/2)\big)\\ \setminus \big(\mathrm{conv}\{(0, c/2), (d/2, b/2), (-d/2, b/2)\} \cup \mathrm{conv}\{(0, -c/2), (d/2, -b/2), (-d/2, -b/2)\}\big)\end{aligned}$$

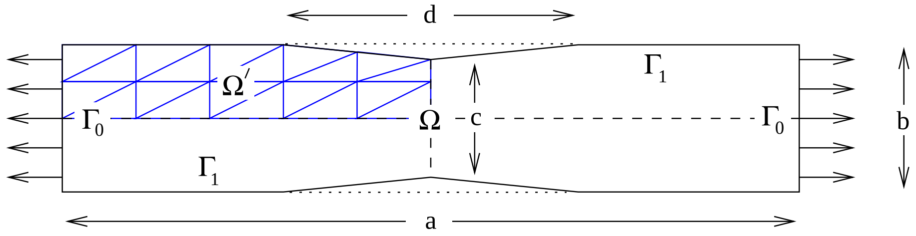

FIGURE 1. Graphical description of the model problem and coarse triangulation of the computational domain. The body is initially "hot" and thermally isolated on all sides.

$\Gamma_0 := \{-a/2\} \times (-b/2, b/2) \cup \{a/2\} \times (-b/2, b/2)$, and $\Gamma_1 := \partial\Omega \setminus \Gamma_0$ with $a = 40$mm, $b = 2$mm, $c = 1.8$mm, and $d = 16$mm. The material constants determing heat transfer are $\tilde{c}_{\mathrm{v}} = 400\mathrm{J/kg\,K}$ and $\mathbb{K} = \kappa \mathbf{I}_{2\times 2}$ for $\kappa = 80\mathrm{W/m\,K}$. The mass density is $\varrho = 8 \cdot 10^3\mathrm{kg/m^3}$ so that $c_{\mathrm{v}} = \varrho \tilde{c}_{\mathrm{v}} = 32 \cdot 10^5\mathrm{J/m^3\,K}$. The Lamé coefficients are defined with the Young's modulus $E = 137$GPa and the Poisson ratio $\nu = 0.3$ through $\lambda = \nu E/((1+\nu)(1-2\nu))$ and $\mu = E/(2(1+\nu))$. The temperature-dependent set of admissible stresses is defined through $\omega_{y,0} := 450$MPa and a smooth function Υ satisfying $\Upsilon(\theta) = 1$ for $\theta \le 800$K and $\Upsilon(\theta) = 0.2$ for $\theta \ge 820$K.

With these definitions we try two different sets of data functions.

Example 5.1 (Fast loading.). *Set* $\alpha := 0$, $f := 0$, $g := 0$, $T := 2 \cdot 10^{-3}\mathrm{s}$, *and*

$$u_{\mathrm{D}}(t,x) := 10^2 t\,\mathrm{mm/s}\,\nu(x)$$

for $t \in [0,T]$ *and* $x \in \Gamma_0$. *As initial data we employ* $u_0 := 0$, $\pi_0 := 0$, *and* $\theta_0 := 800\mathrm{K}$.

Example 5.2 (Slow loading.). *Set* $\alpha := 0$, $f := 0$, $g := 0$, $T := 2\mathrm{s}$, *and*

$$u_{\mathrm{D}}(t,x) := 10^{-1} t\,\mathrm{mm/s}\,\nu(x)$$

for $t \in [0,T]$ *and* $x \in \Gamma_0$. *As initial data we employ* $u_0 := 0$, $\pi_0 := 0$, *and* $\theta_0 := 800\mathrm{K}$.

We simplified this model problem by assuming that there exists a solution that reflects the symmetry of the problem (indicated by the dashed lines in Figure 1) and restricting to the part

$$\Omega' := \big((-a/2, 0) \times (0, b/2)\big) \setminus \mathrm{conv}\{(0, c/2), (0, b/2), (-d/2, b/2)\}$$

of the domain. This enforces us to implement gliding boundary conditions along the sides $(-a/2, 0) \times \{0\}$ and $\{0\} \times (0, c/2)$, i.e., to impose (homogeneous) Dirichlet conditions on one of the two components of the displacement field u and a (homogeneous) Neumann condition on the remaining component.

For a triangulation of Ω' into 1280 triangles obtained from three uniform refinements of the coarse triangulation of Ω' into 20 triangles shown in Figure 1 and used for both equations (5.7) and (5.8), we employed the time-step size $\tau = 10^{-2}T/2$. In Figures 3 and 4 we plotted respectively the evolution of the modulus of the plastic strain and the temperature on the deformed body defined by the deformations $u_{\tau h_1 h_2}(t, \cdot)$ corresponding to the loading $|u_{\mathrm{D}}(t)| = 4, 8, 12, 16, 20 \cdot 10^{-2}$mm (i.e., for $t = 4, 8, 12, 16, 20 \cdot 10^{-4}$s) obtained with the scheme (5.7)–(5.8) and the loading defined in Example 5.1. For a better visualization we magnified the displacement by a factor 5. We observe that the occurrence of plastic material behavior is accompanied by a local rise of the temperature leading in turn to a softening of the material. This softening effect due to an increasing temperature is also illustrated in the plots of Figure 2 where we plotted the discrete power of external mechanical forces

$$(5.9) \qquad P^k_{\tau h_1 h_2}(d_t u^k_{\tau h_1 h_2}, \sigma^{\mathrm{nor},k}_{\tau h_1 h_2}, \theta^k_{\tau h_1 h_2}) = \langle \sigma^{\mathrm{nor},k}_{\tau h_1 h_2}, d_t u^k_{\tau h_1 h_2} \rangle_{\Gamma_0}$$

defined in (4.15) as a function of the loadings defined in Example 5.1 and 5.2. For the slow process, diffusion of heat within the isolated body is much faster and hence the (non-uniform) yield stresses decrease slower in the plastic region leading to a significantly reduced softening behavior. The evolution of the modulus of the plastic strain and the temperature for the slow loading experiment defined in Example 5.2 is displayed in Figures 5 and 6 and we observe that owing to the more equally distributed temperature, the deformation of the right end $\{0\} \times (0, c/2)$ of the body is less significant than for the fast process defined in Example 5.1.

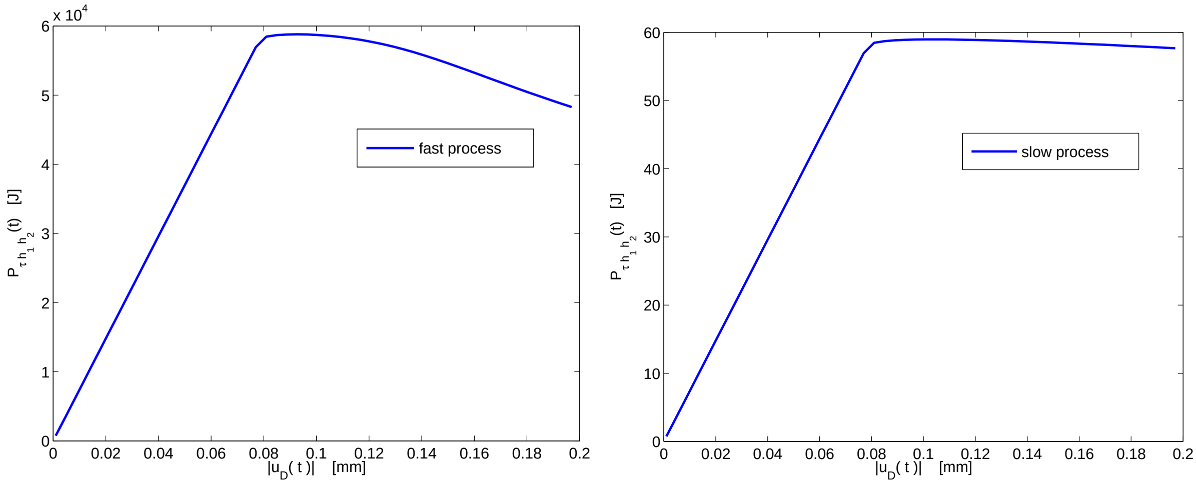

FIGURE 2. Power of external mechanical forces versus applied load for fast (left plot) and slow (right plot) loading. A softening effect due to the increase in temperature can be observed for $|u_{\mathrm{D}}(t)| \geq 0.1$mm. The softening effect is more significant in the case of fast loading.

Finally, to study the accuracy of our numerical scheme, we plotted in Figure 7 the effect of numerical dissipation represented by

$$(5.10) \qquad \xi_{\tau h_1 h_2}(t_K) := \frac{\left| E^K_{\mathrm{tot}} - E^0_{\mathrm{tot}} - \tau \sum_{k=1}^K P^k_{\tau h_1 h_2}(d_t u^k_{\tau h_1 h_2}, \sigma^{\mathrm{nor},k}_{\tau h_1 h_2}, \theta^k_{\tau h_1 h_2}) \right|}{\left| \tau \sum_{k=1}^K P^k_{\tau h_1 h_2}(d_t u^k_{\tau h_1 h_2}, \sigma^{\mathrm{nor},k}_{\tau h_1 h_2}, \theta^k_{\tau h_1 h_2}) \right|}$$

where $E^k_{\mathrm{tot}} := \int_\Omega c_{\mathrm{v}} \theta^k_{\tau h_1 h_2} \, \mathrm{d}x + \frac{1}{2} \mathbb{C}^{-1} \sigma^k_{\tau h_1 h_2} : \sigma^k_{\tau h_1 h_2} \, \mathrm{d}x$. The quantity measures the failure of a discrete version of the continuous energy balance. Owing to the implicit discretization of the heat equation and the explicit treatment of the temperature dependence in the variational inclusion, numerical dissipation occurs, cf. Proposition 4.2. The results shown in Figure 7 indicate that this effect is comparatively small and that $\max_{t \in (0,2)} |\xi_{\tau h_1 h_2}(t)|$ decays linearly to 0 with h as $h \to 0$ in our model problem and for the employed time-step size $\tau = h\, 20 \cdot 10^{-6}$s/mm. This rate is in correct agreement with the assertion of Proposition 4.2.

Acknowledgements: The authors are thankful to professors Alexander Mielke and Patrizio Neff for fruitful discussions. The support from a collaboration project between Humboldt University Berlin and Charles University Prague as well as hospitality of these

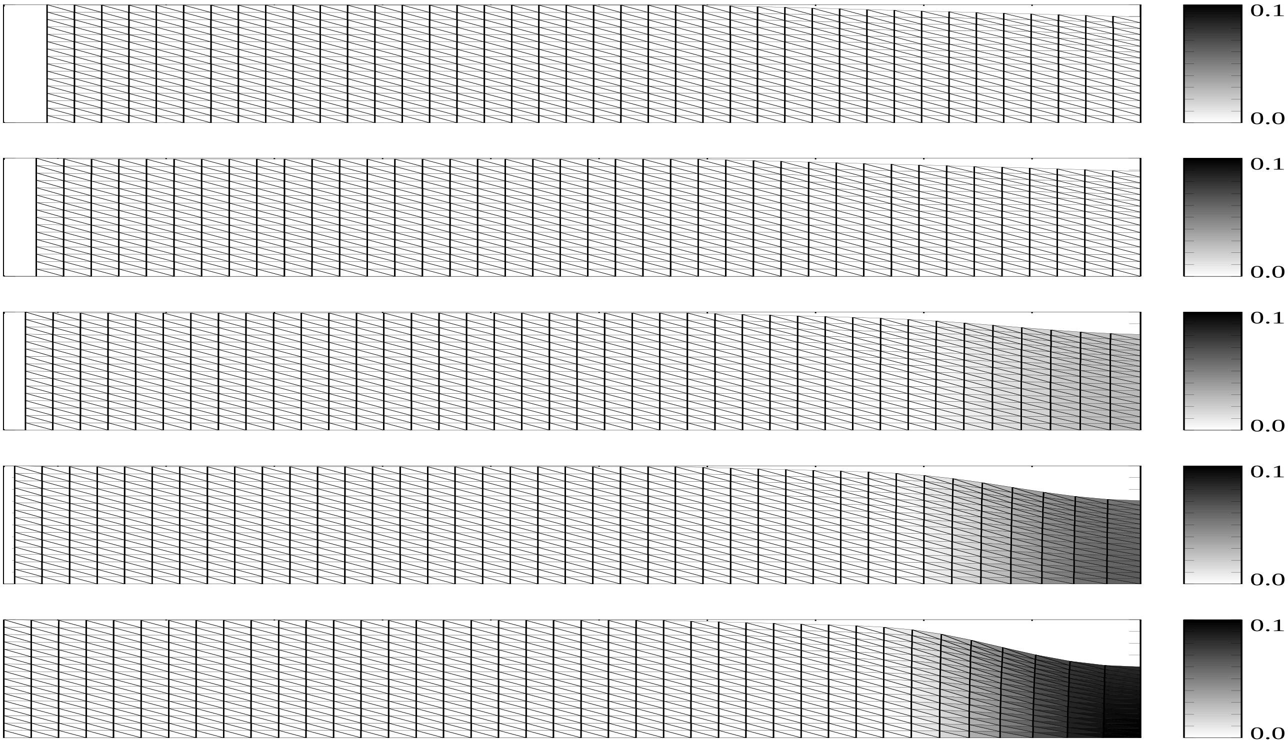

FIGURE 3. Deformation and modulus of plastic strain for $|u_{\mathrm{D}}(t)| = 4$, 8, 12, 16, $20 \cdot 10^{-2}$mm (displacement is multiplied by factor 5) in the fast loading experiment defined in Example 5.1.

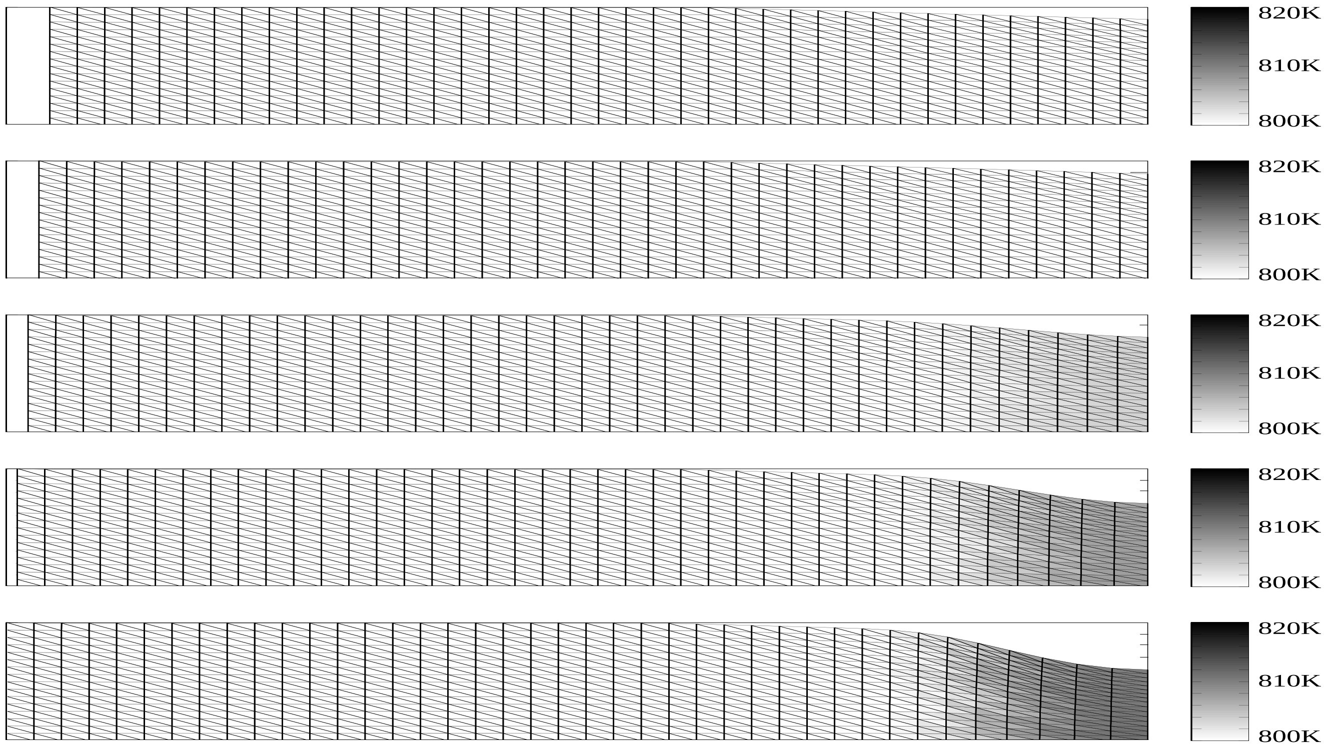

FIGURE 4. Deformation and temperature for $|u_{\mathrm{D}}(t)| = 4$, 8, 12, 16, $20 \cdot 10^{-2}$mm (displacement is multiplied by factor 5) in the fast loading experiment defined in Example 5.1.

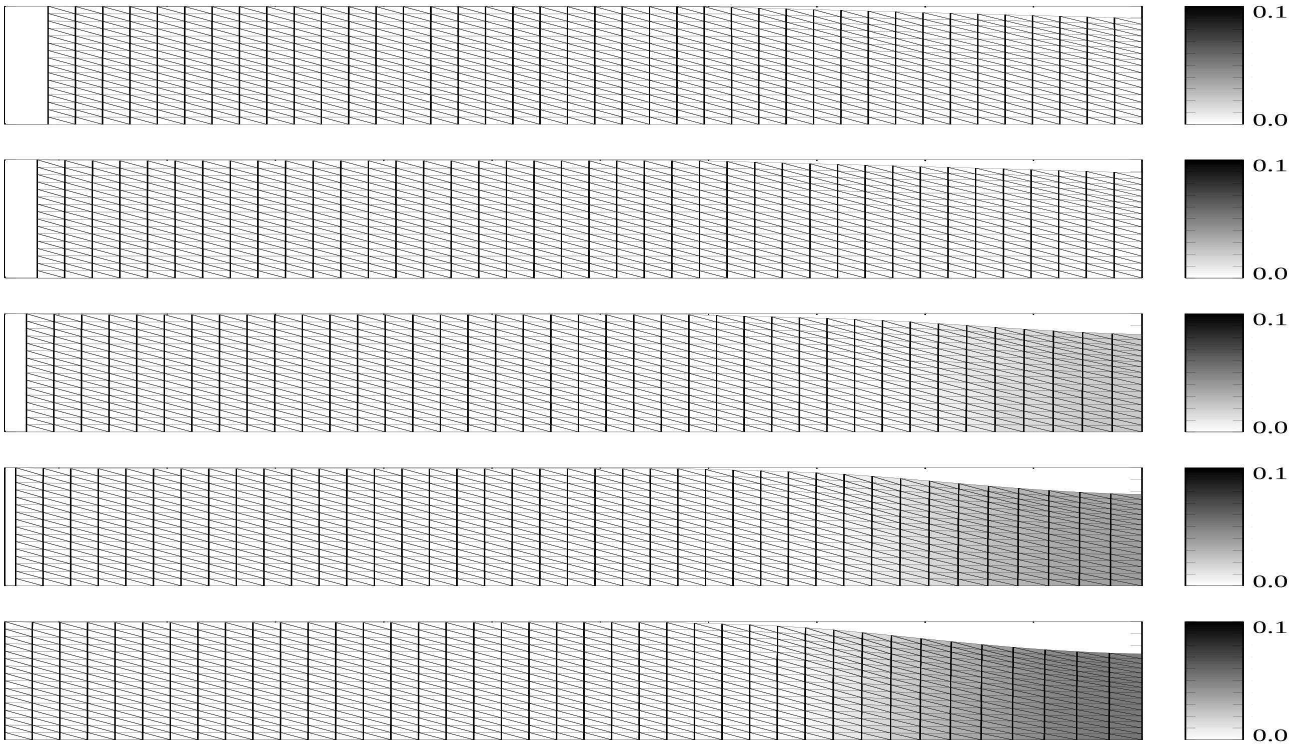

FIGURE 5. Deformation and modulus of plastic strain for $|u_{\mathrm{D}}(t)| = 4$, 8, 12, 16, $20 \cdot 10^{-2}$mm (displacement is multiplied by factor 5) in the slow loading experiment defined in Example 5.2.

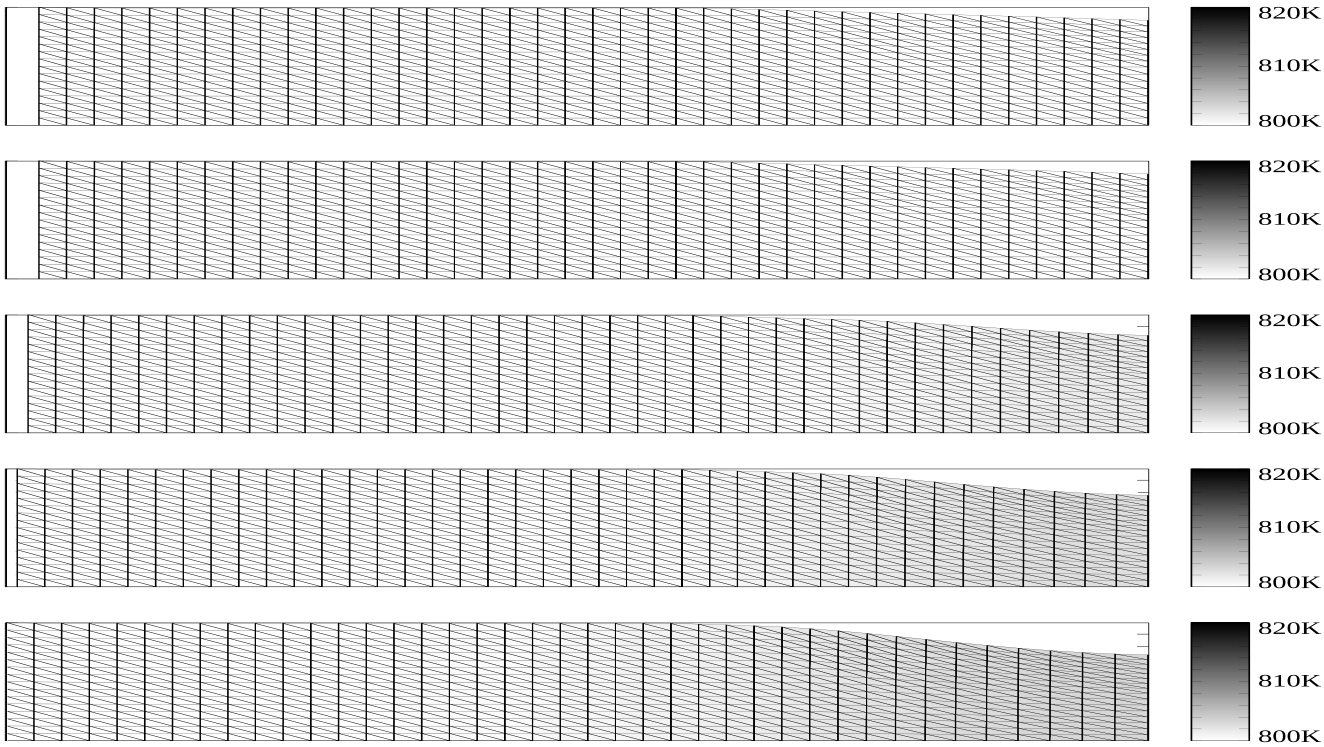

FIGURE 6. Deformation and temperature for $|u_{\mathrm{D}}(t)| = 4$, 8, 12, 16, $20 \cdot 10^{-2}$mm (displacement is multiplied by factor 5) the slow loading experiment defined in Example 5.2.

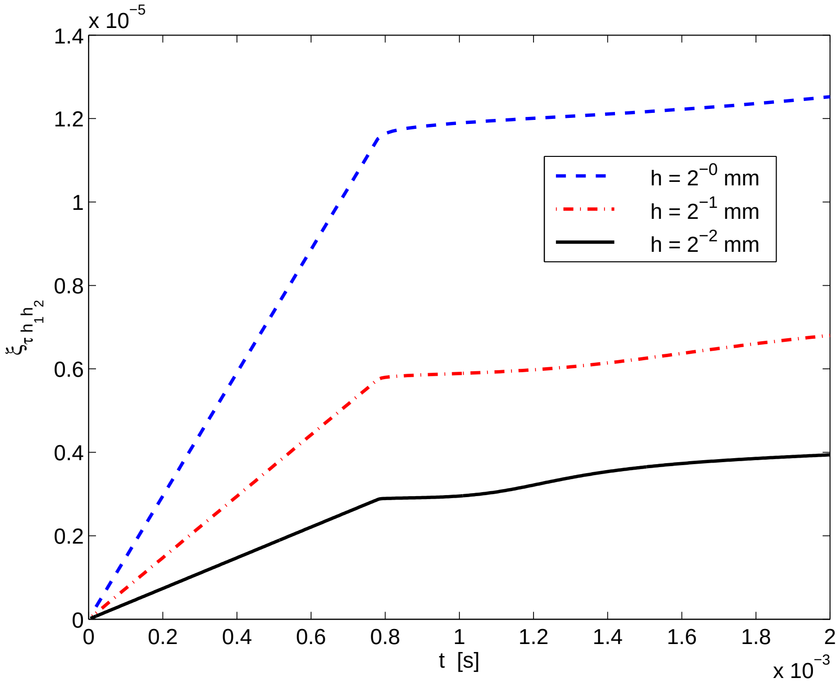

FIGURE 7. Numerical dissipation for mesh-sizes $h = 2^{-0}$, 2^{-1}, 2^{-2}mm and $\tau = h\,20 \cdot 10^{-6}$s/mm in the model example with loading defined by (a).

institutions is gratefully acknowledged. The support and hospitality of the *Nečas center for mathematical modeling* (LC 06052, MŠMT ČR) is also gratefully acknowledged.

REFERENCES

[1] H.-D. Alber: *Materials with Memory.* Lect. Notes in Math. **1682**, Springer, Berlin, 1998.
[2] C. Agelet de Saracibar, M. Cervera, M. Chiumenti: On the formulation of coupled thermoplastic problems with phase-change. *Int. J. Plasticity* **15** (1999), 1–34.
[3] J. Alberty, C. Carstensen, S.A. Funken: Remarks around 50 lines of Matlab: short finite element implementation. *Numer. Algorithms* **20** (1999), 117–137.
[4] L. Boccardo, T. Gallouët: Nonlinear elliptic and parabolic equations with measure data. *J. of Funct. Anal.* **87** (1989), 149–169.
[5] L. Boccardo, A. Dall'aglio, T. Gallouët, L. Orsina: Nonlinear parabolic equations with measure data. *J. of Funct. Anal.* **147** (1997), 237–258.
[6] M. Canadija, J. Brnic: Associative coupled thermoplasticity at finite strain with temperature-dependent material parameters. *Int. J. Plasticity* **20** (2004), 1851–1874.
[7] C. Carstensen, R. Klose: Elastoviscoplastic finite element analysis in 100 lines of Matlab. *J. Numer. Math.* **10** (2002), 157–192.
[8] P. G. Ciarlet: *The finite element method for elliptic problems.* SIAM, Philadelphia, PA, 2002.
[9] P. Colli, A. Visintin: On a class of doubly nonlinear evolution equations. *Comm. P.D.E.* **15** (1990), 737–756.
[10] G. DalMaso, A. DeSimone, M.G. Mora: Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Rat. Mech. Anal.* **180** (2006), 237–291.
[11] I. Faragó, R. Horváth, S. Korotov: Discrete maximum principle for linear parabolic problems solved on hybrid meshes. *Appl. Numer. Math.* **53** (2005), 249–264.
[12] K. Gröger, R. Hünlich: An existence-uniqueness results for some models of thermoplasticity. *Zeitschrift angew. Math. Mech.* **60** (1980), 169-171.
[13] P. Hakansson, M. Wallin, M. Ristinmaa: Comparison of isotropic hardening and kinematic hardening in thermoplasticity. *Int. J. Plasticity* **21** (2005), 1435–1460.
[14] W. Han, D.B. Reddy: *Plasticity. Mathematical theory and numerical analysis.* Springer, New York, 1999.
[15] W. Han, D.B. Reddy: Convergence of approximations to the primal problem in plasticity under conditions of minimal regularity. *Numer. Math.* **87** (2000), 283–315.

[16] S. Korotov, M. Křížek: Acute type refinements of tetrahedral partitions of polyhedral domains. *SIAM J. Numer. Anal.* **39** (2001), 724–733.

[17] J.Kratochvíl, O.W.Dillon, Jr.: Thermodynamics of elastic-plastic materials as a theory with internal state variables. *J. Appl. Physics* **40** (1969), 3207–3218.

[18] P. Krejčí, J. Sprekels: On a system of nonlinear PDEs with temperature-dependent hysteresis in one-dimensional thermoplasticity. *J. Math. Anal. Appl.* **209**, 25–46.

[19] P. Krejčí, J. Sprekels: Temperature-dependent hysteresis in one-dimensional thermovisco-elastoplasticity. *Appl. Math.* **43** (1998), 173–205.

[20] C. Miehe: A theory of large-strain isotropic thermoplasticity based on metric transformation tensor. *Archive Appl. Mech.* **66** (1995), 45–64.

[21] A. Mielke: Evolution of rate-independent systems. In: *Handbook of Differential Equations: Evolutionary Diff. Eqs.* (Eds. C.Dafermos, E.Feireisl), pp. 461–559, Elsevier, Amsterdam, 2005.

[22] A. Mielke, T. Roubíček: Numerical approaches to rate-independent processes and applications in inelasticity. (Preprint no.1169, WIAS, Berlin.) Submitted.

[23] D.W. Nicholson: Large defomration theory of coupled thermoplasticity including kinematic hardening. *Acta Mechanica* **142** (2000), 207–222.

[24] R. H. Nochetto, L. Wahlbin: Positivity preserving finite element approximation. *Math. Comp.* **71** (2002), 1405–1419

[25] D. Rittel: Thermomechanical aspects of dynamic crack initiation. *Int. J. Fracture* **99** (1999), 199–209.

[26] T. Roubíček: *Nonlinear partial differential equations with applications.* Birkhäuser, Basel, 2005.

[27] A. Quarteroni, A. Valli: *Numerical Approximation of Partial Differential Equations.* Springer, Berlin, 1994.

[28] T. Roubíček: Thermo-visco-elasticity at small strains with L^1-data. *Quart. Appl. Math.*, accepted.

[29] T. Roubíček, M. Kružík: Microstructure evolution model in micromagnetics. *Zeit. für angew. Math. und Physik*, **55** (2004), 159-182.

[30] A. Srikanth, N. Zabaras: A computational model for the finite element analysis of thermoplasticity coupled with ductile damage at fonite strains. *Int. J. Numer. Methods Engr.* **45** (1999), 1569–1605.

[31] V. Thomée: *Galerkin finite element methods for parabolic problems.* Springer, Berlin, 2006.