Thermo-visco-elasticity with rate-independent plasticity in isotropic materials undergoing thermal expansion

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Abstract. We consider a viscoelastic solid in Kelvin-Voigt rheology exhibiting also plasticity with hardening and coupled with heat-transfer through dissipative heat production by viscoplastic effects and through thermal expansion and corresponding adiabatic effects. Numerical discretization of the thermodynamically consistent model is proposed by implicit time discretization, suitable regularization, and finite elements in space. Fine a-priori estimates are derived, and convergence is proved by careful successive limit passage. Computational 3D simulations illustrate an implementation of the method as well as physical effects of residual stresses substantially depending on rate of heat treatment.

Key Words. Thermodynamics of plasticity, Kelvin-Voigt rheology, hardening, thermal expansion, adiabatic effects, finite element method, implicit time discretization, convergence.

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1 Introduction

Thermal expansion in metallic bodies may create enormous elastic stresses if the temperature profile varies considerably. It occurs both within manufacturing processes (especially heat treatment of large bulks) and sometimes in working regimes, too. These "thermo-elastic" stresses may trigger *activated inelastic processes*, typically *slip plasticity* or even damage. Here we focus on plasticity and consider also *hardening* effects. Mechanical *energy dissipated* during the plastic deformation is converted to changes of an internal structure of the material due to hardening but also to heat, which ultimately couples the mechanical and heat parts. Moreover, thermal expansion leads to heat production/consumption due to *adiabatic effects*.

There is an extensive engineering literature addressing thermoplasticity in thermally expanding materials, employing computationally sophisticated models, e.g. [1, 7, 8, 14, 21, 25], sometimes even at large strains [9, 13, 15, 26, 27] but lacking a rigorous mathematical justification.

Mathematically supported theories seem, however, nearly missing for a long time. This was mainly because the relevant L^1-theory for the heat equation was developed only in the 1990s [4, 5], and the mathematical theory for rate-independent processes is even more recent, cf. [11, 12, 16, 17, 19, 20], as well as the interpolation technique of the adiabatic-heat term in three-dimensional case [22], and the coupling with rate-independent processes with viscous/inertial effects [23] and thermal effects [24].

The main mathematical difficulties are related to finite strain and multiplicative plasticity at finite strains, in evolution driven even by mere elastic response if kinetic effects are counted, and coupling of rate-independent processes with rate-dependent ones. In fact, each of the above mentioned three difficulties represents itself a hard open problem, especially in a three-dimensional setting and if no regularization (e.g. by capillarity or higher viscosity is involved). This is why we adopt the following simplifications: *small strains* and strain-driven *linearized additive plasticity*, and a *linear viscoelastic* response. On the other hand, we allow for a fully *rate-independent plastic* flow rule although, of course, the whole system is necessarily rate dependent due to the heat transfer, and here also due to considered kinetic and viscous effects. As mentioned above, we also consider hardening to avoid spatial concentration of plastic strains as has been studied in the isothermal case in [11] which, in general, would lead to awkward interactions of concentrating plastic-strain rate with thermal effects. Recently, rigorous mathematical studies for thermoviscoplasticity at small strains had been performed in [3] (considering, however, a rate-dependent plastic flow rule and no thermal expansion effects) and, as mentioned above, in [24] (considering general analytical scheme for a slightly different class of generalized standard materials with gradient theories for internal parameters but without numerical analysis and the modification for the linearized non-gradient plasticity only outlined in [24, Remark 4.5 with Example 5.1]).

The model will be formulated in Section 2 where also its thermodynamics will be exposed. We confine ourselves to trace-free plastic strain and to isotropic materials as far as both elastic response and thermal expansion are concerned. This implies a mathematically important cancellation effect owing to the fact that the thermal-expansion strain is diagonal and thus orthogonal to the "plastic stress", cf. (2.22) below. Although this restricts generality and excludes e.g. single-crystal plasticity which is remarkably anisotropic, important applications in engineering which standardly treat polycrystalline (and thus isotropic) metals are allowed by the presented theory.

The main purpose of this paper, performed in Sections 3 and 4, is to develop an implementable numerical scheme for this model and prove its stability, i.e. a-priori estimates, and also convergence to a suitably defined weak solution of the model. In Section 5, the efficiency of the proposed numerical strategy is demonstrated on a 3D example. We illustrate it on a physically motivated example, exhibiting rate-dependent effects of thermal coupling producing residual elastic stresses and plastic deformation depending on rate of heat treatment of a steel workpiece.

2 The model within thermodynamics

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$. The state variables will be the *displacement* $u : \Omega \to \mathbb{R}^d$, the *plastic strain* $\pi \in \mathbb{R}^{d\times d}_{\rm dev}$, possibly a scalar isotropic *hardening* parameter η, and the *temperature* $\theta : \Omega \to \mathbb{R}$, where

$$\mathbb{R}^{d\times d}_{\rm dev} := \big\{A\in\mathbb{R}^{d\times d}_{\rm sym};\ {\rm tr}(A) = 0\big\} \qquad \text{and} \qquad \mathbb{R}^{d\times d}_{\rm sym} := \big\{A\in\mathbb{R}^{d\times d};\ A^\top = A\big\}. \tag{2.1}$$

The variables $(\pi, \eta) =: z$ play the role of *internal parameters.* We consider plastic response determined by a convex closed neighbourhood of the origin, say $S \subset \mathbb{R}^{d\times d}_{\rm sym} \times \mathbb{R}$, defining an *elasticity domain*, while its boundary is called the *yield surface* and has the meaning of the stress that triggers the evolution of plastic strains; we refer to Section 5 for a specific example. Let δ_S denote its indicator function and δ_S^* the Fenchel-Legendre conjugate functional to δ_S with respect to the inner product $\sigma : e = \sum_{i,j=1}^d \sigma_{ij} e_{ij}$. Note that the physical dimension of $\sigma : e$ is Pa=J/m^3 so that S determining the degree-1 positively homogeneous "plastic" dissipation potential δ_S^*, acting on the dimensionless tensor π and on the dimensionless internal hardening variable η, has indeed the dimension J/m^3. For a simpler notation, we write

$$\zeta_1(\dot\pi, \dot\eta) := \delta_S^*(\dot\pi, \dot\eta). \tag{2.2}$$

We remark that the condition $0 \in \mathrm{int}(S)$ implies that $\zeta_1 = \delta_S^*$ is coercive. The set S must be unbounded. More specifically, we assume that

$$S = S_0 \oplus S_1,\ \ S_0 \subset \mathbb{R}^{d\times d}_{\rm dev} \times \mathbb{R} \text{ convex},\ \ S_1 \text{ the orthogonal complement of } \mathbb{R}^{d\times d}_{\rm dev} \times \mathbb{R}. \tag{2.3}$$

This implies that ζ_1 is finite only on $\mathbb{R}^{d\times d}_{\rm dev} \times \mathbb{R}$ and that $\pi(t) \in \mathbb{R}^{d\times d}_{\rm dev}$ a.e. in Ω provided that $\pi_0 \in \mathbb{R}^{d\times d}_{\rm dev}$ a.e. in Ω.

Considering a Kelvin-Voigt-type viscous material, our model will consist of the *equilibrium equation* balancing inertial, viscous, and elastic mechanical forces,

$$\varrho\frac{\partial^2 u}{\partial t^2} - {\rm div}\Big(\mathbb{D}\frac{\partial e(u)}{\partial t}\Big) - {\rm div}\big(\mathbb{C}(e(u){-}\pi{-}\mathbb{E}\theta)\big) = 0, \tag{2.4}$$

where ϱ is the mass density, $\mathbb{D}$ the tensor determining the viscous-type response, $\mathbb{C}$ the tensor determining the elastic response, and $\mathbb{E}$ the thermal expansion tensor, while the *evolution of the internal parameters* π and η are governed by the inclusion

$$\partial\zeta_1\Big(\frac{\partial\pi}{\partial t}, \frac{\partial\eta}{\partial t}\Big) + \begin{pmatrix}\mathbb{C}\pi{+}\mathbb{H}\pi\\ b\eta\end{pmatrix} \ni \begin{pmatrix}\mathbb{C}e(u)\\ 0\end{pmatrix} \tag{2.5}$$

where $b > 0$ is an *isotropic hardening* coefficient and $\mathbb{H}$ is a symmetric positive semidefinite fourth-order tensor determining the *kinematic hardening*, and the *heat transfer/production* is governed by the equation

$$c_{\rm v}(\theta)\frac{\partial\theta}{\partial t} - {\rm div}\big(\mathbb{K}(\theta)\nabla\theta\big) = \zeta_1\Big(\frac{\partial\pi}{\partial t}, \frac{\partial\eta}{\partial t}\Big) + 2\zeta_2\big(\frac{\partial e(u)}{\partial t}\big) - \theta\mathbb{E} : \mathbb{C}\frac{\partial e(u)}{\partial t} \tag{2.6}$$

where $c_{\rm v} = c_{\rm v}(\theta)$ is the *heat capacity* and $\mathbb{K} = \mathbb{K}(\theta)$ is the *thermal conductivity* tensor,

$$\zeta_2(\dot e) = \frac12\mathbb{D}\dot e : \dot e \tag{2.7}$$

is the *pseudopotential of viscous-dissipative forces*, " : " denotes the product of two $(d{\times}d)$-tensors, and $e(u)$ is the small-strain tensor defined as

$$e_{ij}(u) = \frac12\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{2.8}$$

Throughout this paper, we assume *isotropic thermal expansion*, i.e.

$$\mathbb{E} = \alpha\mathbb{I} \tag{2.9}$$

with α a single thermal-expansion coefficient. It is important that (2.9) together with the further isotropy (2.19) below will imply that (2.5) does not depend explicitly on temperature (since the driving forces $\frac{\partial}{\partial\pi}\psi$ and $\frac{\partial}{\partial\eta}\psi$ with ψ from (2.25) below are independent of θ) and ensures the orthogonality (2.22) below.

Using the identity $[\partial\zeta_1]^{-1} = \partial\zeta_1^*$, the inclusion (2.5) can equivalently be written in a form which is more standard in engineering literature, namely

$$\begin{pmatrix}\partial\pi/\partial t\\ \partial\eta/\partial t\end{pmatrix} \in \partial\zeta_1^*\begin{pmatrix}\sigma - \mathbb{H}\pi\\ -b\eta\end{pmatrix} \qquad \text{with} \quad \sigma = \mathbb{C}\big(e(u){-}\pi\big), \tag{2.10}$$

which reveals that $\mathbb{H}\pi$ is in the position of the *back stress* to the *elastic stress* σ. This is also known as Ziegler's type model [28].

The above equations/inclusion (2.4)–(2.8) are to hold on the space/time domain $Q := (0,T)\times\Omega$ with $T>0$ a fixed time horizon.

As we focus on processes in the bulk, we consider only the simplest boundary conditions, namely a prescribed normal stress and heat flux on $\Gamma := \partial\Omega$:

$$\Big(\mathbb{D}\frac{\partial e(u)}{\partial t} + \mathbb{C}\big(e(u){-}\pi{-}\mathbb{E}\theta\big)\Big)\nu = g \qquad \text{on } \Gamma, \tag{2.11a}$$

$$\big(\mathbb{K}(\theta)\nabla\theta\big)\cdot\nu = f \qquad \text{on } \Gamma, \tag{2.11b}$$

where "$\cdot$" denotes the scalar product of two vectors and ν is the outward normal to Γ.

The energetics of the model plays with the *mechanical part of the internal energy*

$$\Phi(u,\pi,\eta) := \frac{1}{2}\int_\Omega \mathbb{C}(e(u){-}\pi) : (e(u){-}\pi) + \pi\mathbb{H}\pi + b\eta^2\,\mathrm{d}x, \tag{2.12}$$

the *kinetic energy*

$$T_{\rm kin}(\dot u) := \frac{1}{2}\int_\Omega \varrho|\dot u|^2\,\mathrm{d}x, \tag{2.13}$$

the *dissipation energy rate*

$$\Xi(\dot u,\dot\pi,\dot\eta) := \int_\Omega \zeta_1\big(\dot\pi,\dot\eta\big) + 2\zeta_2\big(e(\dot u)\big)\,\mathrm{d}x \tag{2.14}$$

with ζ_1 from (2.2) and ζ_2 from (2.7), the *thermal part of the internal energy*

$$E(\theta) := \int_\Omega h(\theta)\,\mathrm{d}x, \qquad \text{where} \qquad h(\theta) := \int_0^\theta c_{\rm v}(w)\mathrm{d}w, \tag{2.15}$$

and the *power of external mechanical forces and heating*

$$F(t,\dot u) = \int_\Gamma g(t,x){\cdot}\dot u(x) + f(t,x)\,\mathrm{d}S, \tag{2.16}$$

The energetics of the model (2.4)–(2.6) can be obtained by testing (2.4), (2.5), and (2.6) respectively by the velocity $\frac{\partial u}{\partial t}$, by the plastic strain rate $\frac{\partial \pi}{\partial t}$, and by 1, which gives after using Green's formula for both (2.4) and (2.6) together with the boundary conditions (2.11) and eventually by summation the *energy balance*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(T_{\rm kin}\Big(\frac{\partial u}{\partial t}\Big) + \Phi(u,\pi,\eta) + E(\theta)\Big) = F\Big(t,\frac{\partial u}{\partial t}\Big); \tag{2.17}$$

cf. (3.7d) below.

We consider an initial-boundary-value problem for the system (2.4)–(2.8). Hence, we take the initial conditions

$$u(0,\cdot) = u_0, \quad \frac{\partial u}{\partial t}(0,\cdot) = \dot u_0, \quad \pi(0,\cdot) = \pi_0, \quad \eta(0,\cdot) = \eta_0, \quad \theta(0,\cdot) = \theta_0. \tag{2.18}$$

Also, the *non-negativity of temperature* is ensured provided $\theta_0 \ge 0$ and provided the boundary heat flux f is non-negative, cf. also the proof of Proposition 4.4 below.

Throughout the article, we will rely on the following data qualification. The positive definite fourth order tensors $\mathbb{C} = [\mathbb{C}_{ijkl}]$, $\mathbb{D} = [\mathbb{D}_{ijkl}]$ are assumed to be symmetric and isotropic so that in fact

$$
\begin{aligned}
&\mathbb{C}_{ijkl} = \lambda_{\rm e}\delta_{ij}\delta_{kl} + \mu_{\rm e}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big),\\
&\mathbb{D}_{ijkl} = \lambda_{\rm v}\delta_{ij}\delta_{kl} + \mu_{\rm v}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big),\\
&\qquad\qquad \text{with } \mu_{\rm e},\mu_{\rm v} > 0, \quad \lambda_{\rm e} > -\frac{2}{d}\mu_{\rm e}, \quad \lambda_{\rm v} > -\frac{2}{d}\mu_{\rm v},
\end{aligned}
\tag{2.19}
$$

with δ denoting here the Kronecker symbol, λ'a and μ's are the Lamé constants. Thus the elastic stress is $\mathbb{C}e = \lambda_{\rm e}{\rm tr}(e)\mathbb{I} + 2\mu_{\rm e}e$ with $\mathbb{I} = [\delta_{ij}]$ denoting the unit matrix, and corresponding energy is $\frac{1}{2}\mathbb{C}e : e = \frac{1}{2}\lambda_{\rm e}|{\rm tr}(e)|^2 + \mu_{\rm e}|e|^2$ and, as a quadratic form of e, it is positive definite if $d \le 3$, and similarly also the quadratic form $e \mapsto \frac{1}{2}\mathbb{D}e : e$ is positive definite. In fact, for the analysis presented below, we need the isotropy only for $\mathbb{C}$ but it would be physically inconsistent to have $\mathbb{D}$ anisotropic.

One can derive *thermodynamics* of the above model by postulating the Helmholtz *free energy* as

$$
\psi(e,\pi,\eta,\theta) = \frac{1}{2}\mathbb{C}\big(e-\pi-\mathbb{E}\theta\big):\big(e-\pi-\mathbb{E}\theta\big) + \frac{1}{2}\mathbb{H}\pi{:}\pi + \frac{b}{2}\eta^2 - \frac{\theta^2}{2}\mathbb{C}\mathbb{E}{:}\mathbb{E} - \phi_0(\theta). \tag{2.20}
$$

Then *entropy* is given by

$$
s = s(e,\theta) := -\frac{\partial}{\partial\theta}\psi = \phi_0'(\theta) - \mathbb{E}{:}\mathbb{C}e. \tag{2.21}
$$

Note that the mechanical variables separate from temperature in (2.21) and thus $c_{\rm v} = c_{\rm v}(e,\theta) = \theta\frac{\partial}{\partial\theta}s(e,\theta) = \theta\phi_0''(\theta)$ in (2.6) does not depend on these mechanical variables, which facilitates the analysis very considerably. This separation effect is due to the *orthogonality*

$$
\mathbb{C}\pi : \mathbb{E} = \alpha\big(\lambda_{\rm e}{\rm tr}(\pi)\mathbb{I} + 2\mu_{\rm e}\pi\big) : \mathbb{I} = \alpha\big(d\lambda_{\rm e}{+}2\mu_{\rm e}\big){\rm tr}(\pi) = 0, \tag{2.22}
$$

which is owing to (2.3) together with (2.9) and (2.19) provided also ${\rm tr}(\pi_0) = 0$.

The equation (2.6) itself can be written in the form of the *entropy equation* as

$$
\theta\frac{\partial s}{\partial t} - {\rm div}(\mathbb{K}\nabla\theta) = \xi \quad \text{ with dissipation rate } \ \xi := \zeta_1\Big(\frac{\partial\pi}{\partial t},\frac{\partial\eta}{\partial t}\Big) + 2\zeta_2\Big(\frac{\partial e(u)}{\partial t}\Big). \tag{2.23}
$$

Note that $\int_\Omega \xi \,{\rm d}x = \Xi(\frac{\partial u}{\partial t},\frac{\partial\pi}{\partial t},\frac{\partial\eta}{\partial t})$ with Ξ from (2.14). At least formally, assuming positivity of temperature and $f \ge 0$, and realizing that always $\xi \ge 0$, from (2.23) we can see the *Clausius-Duhem inequality*

$$
\frac{\rm d}{{\rm d}t}\int_\Omega s\,{\rm d}x = \int_\Omega {\rm div}\Big(\mathbb{K}\frac{\nabla\theta}{\theta}\Big) + \frac{\mathbb{K}\nabla\theta\cdot\nabla\theta}{\theta^2} + \frac{\xi}{\theta}\,{\rm d}x = \int_\Omega \frac{\mathbb{K}\nabla\theta\cdot\nabla\theta}{\theta^2} + \frac{\xi}{\theta}\,{\rm d}x + \int_\Gamma \frac{f}{\theta}\,{\rm d}S \ge 0; \tag{2.24}
$$

obviously, ξ/θ is the *entropy-production rate*. Note also that a combination of (2.9), (2.19), and (2.22) allows us to write ψ from (2.20) in the more specific form

$$
\psi(e,\pi,\eta,\theta) = \frac{\lambda_{\rm e}}{2}\big|{\rm tr}(e{-}\pi)\big|^2 + \mu_{\rm e}\big|e{-}\pi\big|^2 - \alpha\big(d\lambda_{\rm e}{+}2\mu_{\rm e}\big)\theta\,{\rm tr}(e) + \frac{1}{2}\mathbb{H}\pi{:}\pi + \frac{b}{2}\eta^2 - \phi_0(\theta) \tag{2.25}
$$

where (2.22) was used, too. In fact, (2.24) in the form $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s\,\mathrm{d}x \ge \int_\Omega \frac{\mathbb{K}\nabla\theta\cdot\nabla\theta}{\theta^2} + \frac{\xi}{\theta}\,\mathrm{d}x + \int_\Gamma \frac{f}{\theta}\,\mathrm{d}S$ can conversely serve as the origin of the constitutional relation for the entropy $s = -\frac{\partial}{\partial\theta}\psi$, cf. (2.21), and the elastic stress $\sigma = \frac{\partial}{\partial e}\psi$, cf. (2.10), as well as the driving force for the flow rule $\frac{\partial}{\partial(\pi,\eta)}\psi = (\mathbb{H}\pi - \sigma, b\zeta)$.

3 Enthalpy transformation and weak formulation

It is desirable to allow for a certain growth of $c_\mathrm{v}(\cdot)$ if we have the viscosity in the form $\mathbb{D}e(\frac{\partial u}{\partial t})$ in order to be able to treat the adiabatic term, cf. [22]. On the other hand, the technique from [22] specifically relies on Galerkin's method and does not seem directly transferable if also the time discretization is involved, which is in turn needed both for designing a fully discrete scheme and for efficient treatment of the rate-independent flow rule. The particular difficulty is in limiting a time-discretization of the nonlinear term $c_\mathrm{v}(\theta)\frac{\partial\theta}{\partial t}$. Therefore, we first write the original system (2.4)–(2.6) in terms of enthalpy instead of temperature, using so-called enthalpy transformation

$$w = h_0(\theta) := \int_0^\theta c_\mathrm{v}(r)\,\mathrm{d}r; \tag{3.1}$$

thus h_0 is a primitive function to c_v normalized such that $h_0(0) = 0$. Further, we define

$$\Theta(w) := \begin{cases} h_0^{-1}(w) & \text{if } w \ge 0, \\ 0 & \text{if } w < 0, \end{cases} \qquad\qquad \mathscr{K}(w) := \frac{\mathbb{K}(\Theta(w))}{c_\mathrm{v}(\Theta(w))}, \tag{3.2}$$

where h_0^{-1} here denotes the inverse function to h. This transforms the system (2.4)–(2.6) into the form

$$\varrho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) + \big(\mathbb{C}e(u) - \Theta(w)\mathbb{E}\big)\Big) = 0, \tag{3.3a}$$

$$\partial\zeta_1\Big(\frac{\partial\pi}{\partial t}, \frac{\partial\eta}{\partial t}\Big) + \begin{pmatrix} \mathbb{C}\pi + \mathbb{H}\pi \\ b\eta \end{pmatrix} \ni \begin{pmatrix} \mathbb{C}e(u) \\ 0 \end{pmatrix}, \tag{3.3b}$$

$$\frac{\partial w}{\partial t} - \mathrm{div}\big(\mathscr{K}(w)\nabla w\big) = \zeta_1\Big(\frac{\partial\pi}{\partial t}, \frac{\partial\eta}{\partial t}\Big) + 2\zeta_2\Big(\frac{\partial e(u)}{\partial t}\Big) + \Theta(w)\mathbb{E}{:}\mathbb{C}e\big(\frac{\partial u}{\partial t}\big). \tag{3.3c}$$

We will call (3.3c) shortly the *enthalpy equation* rather than the heat-transfer equation in the enthalpy formulation. The boundary conditions (2.11) transforms to

$$\Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) + \mathbb{C}\big(e(u) - \pi - \mathbb{E}\Theta(w)\big)\Big)\nu = g \qquad\qquad \text{on } \Gamma, \tag{3.4a}$$

$$\big(\mathscr{K}(w)\nabla w\big)\cdot\nu = f \qquad\qquad \text{on } \Gamma, \tag{3.4b}$$

while the initial conditions (2.18) transform into

$$u(0,\cdot) = u_0, \quad \frac{\partial u}{\partial t}(0,\cdot) = \dot{u}_0, \quad \pi(0,\cdot) = \pi_0, \quad \eta(0,\cdot) = \eta_0, \quad w(0,\cdot) = h_0(\theta_0). \tag{3.5}$$

The following definition of a certain sort of a weak solution has been devised in [24], based on the concept of so-called energetic solution invented by Mielke at al. [12, 16, 19, 20] for the theory of rate independent processes and adapted also for coupling with viscous/inertial effects in [23]. We refer to [24, Proposition 3.2] for justification (and not entirely obvious fact) that this definition is indeed selective in the sense that, under an additional absolute continuity of $\frac{\partial\pi}{\partial t}$ and $\frac{\partial\eta}{\partial t}$, it gives indeed a conventional notion of a weak solution. (For an isothermal situation, cf. also [23, Proposition 5.2].) It should be however emphasized that this additional regularity of $\frac{\partial\pi}{\partial t}$ and $\frac{\partial\eta}{\partial t}$ hardly can be expected due to the fully rate-independent flow rule, which just makes the devised concept properly fitted with this problem.

We consider an evolution in the time interval $I := (0,T)$ with a fixed time horizon $T > 0$ and denote $Q := (0,T)\times\Omega$, $\Sigma := (0,T)\times\partial\Omega$, and $\bar I := [0,T]$. We will use a standard notation for function spaces, namely the space of the continuous $\mathbb{R}^k$-valued functions $C(\bar\Omega;\mathbb{R}^k)$, its dual $\mathscr{M}(\bar\Omega;\mathbb{R}^k)$ (i.e., up to an isometric isomorphism, the space of Borel measures), the continuously differentiable functions $C^1(\bar\Omega;\mathbb{R}^k)$, the Lebesgue space $L^p(\Omega;\mathbb{R}^k)$, the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^k)$, and the Bochner space of X-valued Bochner measurable p-integrable functions $L^p(I;X)$. If $X = (X')^*$, the notation $L^\infty_{\mathrm{w}*}(I;X)$ stands for space of weakly* measurable functions $I \to X$; this space is dual to the space $L^1(I;X')$ and, in general, is not equal to $L^\infty(I;X)$. If X is separable reflexive, then $L^\infty(I;X) = L^\infty_{\mathrm{w}*}(I;X)$ by Pettis' theorem, however. Moreover, we denote by $B(\bar I;X)$, $B_{\mathrm{w}*}(\bar I;X)$, $\mathrm{BV}(\bar I;X)$ or $C_{\mathrm{w}}(\bar I;X)$ the Banach space of functions $\bar I \to X$ that are bounded Bochner measurable, bounded weakly* measurable, have a bounded variation or are weakly continuous, respectively; note that all these functions are defined everywhere on $\bar I$. We will use the notation $q' = q/(q-1)$ for the conjugate exponent to q. Instead of $u(t,\cdot)$ or $z(t,\cdot)$ or $w(t,\cdot)$, we will write briefly $u(t)$ or $z(t)$ or $w(t)$, respectively.

Definition 3.1 ***(Energetic solution.)*** *Assuming (2.19)–(3.10), we call a quadruple* (u,π,η,w) *with*

$$u \in C_{\mathrm{w}}(I;W^{1,2}(\Omega;\mathbb{R}^d)), \tag{3.6a}$$

$$\frac{\partial u}{\partial t} \in L^2(I;W^{1,2}(\Omega;\mathbb{R}^d)) \cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^d)^*), \tag{3.6b}$$

$$\pi \in B(\bar I;W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})) \cap \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})), \tag{3.6c}$$

$$\eta \in B(\bar I;W^{1,2}(\Omega)) \cap \mathrm{BV}(\bar I;L^1(\Omega)), \tag{3.6d}$$

$$w \in L^r(I;W^{1,r}(\Omega)) \cap L^\infty(I;L^1(\Omega)) \cap B_{\mathrm{w}*}(\bar I;\mathscr{M}(\bar\Omega)) \quad \textit{with any } 1 \le r < \frac{d+2}{d+1}, \tag{3.6e}$$

$$\frac{\partial w}{\partial t} \in \mathscr{M}(\bar I;W^{1+d,2}(\Omega)^*) \tag{3.6f}$$

an energetic solution to (3.3) with the initial/boundary conditions (3.5) and (4.3) if the following five conditions hold:

(i) *the weakly formulated momentum-equilibrium equation (3.3a) with (4.3a,b) holds, i.e. for all* $v \in C^1(\bar Q;\mathbb{R}^d)$ *such that* $v|_{\Sigma_0}=0$,

$$\int_\Omega \varrho\frac{\partial u}{\partial t}(T)\cdot v(T)\,\mathrm{d}x + \int_Q \Big(\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) + \mathbb{C}e(u) - \pi - \mathbb{E}\Theta(w)\Big) : e(v) \\ - \varrho\frac{\partial u}{\partial t}\cdot\frac{\partial v}{\partial t}\,\mathrm{d}x\mathrm{d}t = \int_\Sigma g\cdot v\,\mathrm{d}x\mathrm{d}t + \int_\Omega \varrho\dot u_0\cdot v(0)\,\mathrm{d}x, \tag{3.7a}$$

(ii) *the weakly formulated enthalpy equation (3.3c) with (4.3c) holds, i.e. for all* $v \in C^1(\bar Q)$ *with* $v(T) = 0$,

$$\int_Q \mathscr{K}(w)\nabla w\cdot\nabla v - w\frac{\partial v}{\partial t} - \Theta(w)\mathbb{E} : \mathbb{C}e\big(\frac{\partial u}{\partial t}\big)v \\ - \mathbb{D}e\big(\frac{\partial u}{\partial t}\big) : e\big(\frac{\partial u}{\partial t}\big)v\,\mathrm{d}x\mathrm{d}t = \int_{\bar Q} v\,\mathfrak{h}_{\pi,\eta}(\mathrm{d}x\mathrm{d}t) + \int_\Omega w_0 v(0)\,\mathrm{d}x + \int_\Sigma f v\,\mathrm{d}S\mathrm{d}t \tag{3.7b}$$

where $w_0 = h_0(\theta_0)$ and $\mathfrak{h}_{\pi,\eta}$ is a measure (=heat produced by rate-independent dissipation) defined by prescribing its values for every closed set of the type $A := [t_1,t_2]{\times}B$ with B a Borel subset of $\bar\Omega$ by

$$\mathfrak{h}_{\pi,\eta}(A) := \mathrm{Var}_{\zeta_1}\big((\pi,\eta)|_B;t_1,t_2\big) \quad \textit{with}$$
$$\mathrm{Var}_{\zeta_1}(z;t_1,t_2) := \sup\sum_{i=1}^{k}\int_\Omega \zeta_1\big(z(s_i,x){-}z(s_{i-1},x)\big)\,\mathrm{d}x \tag{3.7c}$$

where the supremum is taken over all partitions of the type $t_1{\le}s_0{<}...{<}s_k{\le}t_2$, $k{\in}\mathbb{N}$,

(iii) *the total energy equality holds, i.e. with Φ and $T_{\rm kin}$ from (2.12) and (2.13),*

$$T_{\rm kin}\big(\frac{\partial u}{\partial t}(T)\big) + \Phi\big(u(T),\pi(T),\eta(T)\big) + \int_{\bar\Omega} w(T,\mathrm{d}x)$$
$$= T_{\rm kin}\big(\dot u_0\big) + \Phi\big(u_0,\pi_0,\eta_0\big) + \int_\Omega h_0(\theta_0)\,\mathrm{d}x + \int_\Sigma g\cdot\frac{\partial u}{\partial t} + f\mathrm{d}S\mathrm{d}t, \tag{3.7d}$$

(iv) *the "semistability" holds for any $\tilde\pi \in W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ and $\tilde\eta \in W^{1,2}(\Omega)$ and for all $t \in [0,T]$, i.e.*

$$\Phi\big(u(t),\pi(t),\eta(t)\big) \le \Phi\big(u(t),\tilde\pi,\tilde\eta\big) + \int_\Omega \zeta_1(\tilde\pi-\pi(t),\tilde\eta-\eta(t))\,\mathrm{d}x, \tag{3.7e}$$

(v) *the initial conditions $u(0) = u_0$, $\pi(0) = \pi_0$, and $\eta(0) = \eta_0$ hold.*

Note also that (3.6f) makes values of $w(t)$ well defined in the sense of $W^{1+n,2}(\Omega)^*$ and (3.6e) further shows that even $w(t) \in \mathscr{M}(\bar\Omega)$, which has been exploited in (3.7d) for the time $t = T$. It should be emphasized that $t \mapsto w(t)$ cannot be expected to be continuous in any sense because, since ζ_1 is homogeneous degree-1, the measure $\mathfrak{h}_{\pi,\eta}$ may concentrate at particular time instances.

In addition to (2.19) which guarantees that $\mathbb{C}$ and $\mathbb{D}$ are positive definite, we will assume throughout this article that

$$\varrho \ge 0. \tag{3.8}$$

Other assumptions are on nonlinearities $c_{\rm v}$ and $\mathbb{K}$, namely we assume:

$$c_{\rm v} : [0,+\infty) \to \mathbb{R}^+ \ \text{ continuous}, \tag{3.9a}$$
$$\exists\,\omega_1 \ge \omega \ge 1,\ c_1 \ge c_0 > 0\ \forall\theta \in \mathbb{R}^+: \quad c_0(1{+}\theta)^{\omega-1} \le c_{\rm v}(\theta) \le c_1(1{+}\theta)^{\omega_1-1}, \tag{3.9b}$$
$$\mathscr{K} : \mathbb{R} \to \mathbb{R}^{d\times d} \ \text{ bounded, continuous, and } \inf_{(w,\xi)\in\mathbb{R}\times\mathbb{R}^d,\ |\xi|=1} \mathscr{K}(w)\xi{\cdot}\xi > 0 \tag{3.9c}$$

with $\mathscr{K}$ from (3.2) below; later in (4.27) we impose further restrictions on ω. As far as the loading qualification concerns, we assume

$$g \in L^2(I;L^q(\Gamma;\mathbb{R}^d)), \qquad q \ge 2{-}2/d \ \ (\text{or } q > 1 \text{ if } d \le 2), \tag{3.10a}$$
$$f \in L^1(\Sigma), \quad f \ge 0, \tag{3.10b}$$
$$u_0 \in W^{1,2}(\Omega;\mathbb{R}^d), \tag{3.10c}$$
$$\dot u_0 \in L^2(\Omega;\mathbb{R}^d), \tag{3.10d}$$
$$\pi_0 \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}), \tag{3.10e}$$
$$\eta_0 \in L^2(\Omega), \quad \eta_0 > 0, \tag{3.10f}$$
$$\theta_0 \in L^\omega(\Omega), \quad \theta_0 \ge 0, \tag{3.10g}$$

where we denoted $\Sigma := I \times \Gamma$ in (3.10b).

4 Discretization and numerical analysis

An important phenomenon here is that, proving existence of a solution, we need to pass to the limit in the non-linear Nemytskiĭ operators induced by the dissipation heat ξ. Another peculiarity is that, due to degree-1 homogeneity of ζ_1, the heat equation has its right-hand side not only in $L^1(Q)$ (as it would be in case of higher-degree homogeneity of dissipative-force potential) but even in measures. Also, it seems difficult to make spatial discretization of the term $-\mathrm{div}(\mathscr{K}(w)\nabla w)$ compatible with the maximum principle even on acute triangulations if $\mathscr{K}$ is nonconstant (and the qualification (3.9b,d) with (4.27) below exclude constant $\mathscr{K}$ if $d\geq 2$).

Therefore, a design of a convergent numerical scheme is technically rather delicate. Following [24], we will use a *fully implicit time-discretization* with a constant time-step $\tau>0$, assuming $K_\tau=T/\tau\in\mathbb{N}$ and defining the *backward difference operator* by

$$\mathrm{D}_t\phi^k:=\frac{\phi^k-\phi^{k-1}}{\tau} \tag{4.1}$$

for any sequence $\{\phi^k\}_{k\geq 0}$, combined with a *regularization* of the momentum equation and of the flow rule, (using as a parameter just the time-step $\tau>0$) and of the enthalpy equation (using a parameter $\varepsilon=\varepsilon(\tau)>0$ whose dependence on τ will be implicitly specified later in (4.34)). More specifically, we consider the following recursive increment formula

$$\varrho\mathrm{D}_t^2u_\tau^k-\mathrm{div}\Big(\mathbb{D}e\big(\mathrm{D}_tu_\tau^k\big)+\mathbb{C}\big(e(u_\tau^k)-\pi_\tau^k-\mathbb{E}\Theta(w_\tau^k)\big)+\tau\big|e(u_\tau^k)\big|^{\gamma-2}e(u_\tau^k)\Big)=0, \tag{4.2a}$$

$$\partial\zeta_1\big(\mathrm{D}_t\pi_\tau^k,\mathrm{D}_t\eta_\tau^k\big)+\begin{pmatrix}\mathbb{C}\pi_\tau^k+\mathbb{H}\pi_\tau^k\\ b\eta_\tau^k\end{pmatrix}\ni\begin{pmatrix}\mathbb{C}e(u_\tau^k)\\0\end{pmatrix}+\tau\mathscr{S}\begin{pmatrix}\pi_\tau^k\\ \eta_\tau^k\end{pmatrix}, \tag{4.2b}$$

$$\mathrm{D}_tw_\tau^k-\mathrm{div}\big(\mathscr{K}(w_\tau^k)\nabla w_\tau^k\big)+\varepsilon(\tau)|w_\tau^k|^{\beta-2}w_\tau^k \tag{4.2c}$$

$$\qquad=\zeta_1\big(\mathrm{D}_t\pi_\tau^k,\mathrm{D}_t\eta_\tau^k\big)+\mathbb{D}e\big(\mathrm{D}_tu_\tau^k\big){:}e\big(\mathrm{D}_tu_\tau^k\big)+\Theta(w_\tau^k)\mathbb{E}{:}\mathbb{C}e\big(\mathrm{D}_tu_\tau^k\big), \tag{4.2d}$$

for $k=1,...,K_\tau=T/\tau$ with the corresponding boundary conditions

$$\Big(\mathbb{D}e\big(\mathrm{D}_tu_\tau^k\big)+\mathbb{C}\big(e(u_\tau^k)-\pi_\tau^k-\mathbb{E}\Theta(w_\tau^k)\big)+\tau\big|e(u_\tau^k)\big|^{\gamma-2}e(u_\tau^k)\Big)\nu=g_\tau^k, \tag{4.3a}$$

$$\big(\mathscr{K}(w_\tau^k)\nabla w_\tau^k\big)\cdot\nu=f_\tau^k \tag{4.3b}$$

on Γ, starting for $k=1$ by using

$$u_\tau^0=u_{0,\tau},\quad u_\tau^{-1}=u_{0,\tau}-\tau\dot u_0,\quad \pi_\tau^0=\pi_{0,\tau},\quad \eta_\tau^0=\eta_{0,\tau},\quad w_\tau^0=w_0:=h_0(\theta_0), \tag{4.4}$$

where

$$g_\tau^k(t,x):=\frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}g(t,x)\,\mathrm{d}t\quad\text{and}\quad f_\tau^k:=\frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}\tilde f_\tau(t,x)\,\mathrm{d}t. \tag{4.5}$$

Note that, in (4.4) and (4.5), we regularized the initial values and the boundary flux u_0, z_0 and f by $u_{0,\tau}$, $z_{0,\tau}$ and $\tilde f_\tau$, respectively, cf. (4.8) below. Moreover, $\mathscr{S}$ in (4.2b) is a regularizing selfadjoint positive definite linear operator having the quadratic potential

$$\frac12\big|z\big|^2_{W^{a,2}(\Omega)}\quad\text{with some}\quad 0<a<1/2 \tag{4.6}$$

applied component-wise in (4.2b), with $|\cdot|_{W^{a,2}(\Omega)}$ meaning the standard seminorm in the Sobolev-Slobodetskiĭ fractional-derivative space. Later we can also use its square root

$\mathscr{S}^{1/2}$, defined as a selfadjoint positive definite operator such that $\mathscr{S}^{1/2}\circ\mathscr{S}^{1/2}=\mathscr{S}$. Thus

$$\mathscr{S}:W^{a,2}(\Omega)\to W^{a,2}(\Omega)^* \qquad\text{and}\qquad \mathscr{S}^{1/2}:W^{a,2}(\Omega)\to L^2(\Omega). \tag{4.7}$$

Note that we regularized also the initial state for the mechanical part (but not the initial velocity).

Let us comment the purpose of the regularizing terms. The "γ-term" in (4.2a) and the "β-term" in (4.2d) are to compensate the superlinear growth of the right-hand-side terms in the hear equation; the former one has already been used in [24] for a mere time discretization, while the latter one is here needed because, in the spatially discretized scheme, we will not be able to test by nonlinear functions of w, in contrast with the spatially continuous case in [24]. Eventually, the "$\mathscr{S}$-term" in (4.2b) helps to make a limit passage in space discretization without using numerical integration formulae, cf. (4.21) below.

As far as the (regularized) initial and boundary conditions and the loading concerns, we assume

$$u_{0,\tau}{\in}W^{1,\gamma}(\Omega;\mathbb{R}^d),\qquad \lim_{\tau\downarrow0}\sqrt[\gamma]{\tau}\big\|e(u_{0,\tau})\big\|_{L^\gamma(\Omega;\mathbb{R}^{d\times d})}=0,\quad \lim_{\tau\downarrow0}u_{0,\tau}=u_0\ \text{in}\ W^{1,2}(\Omega;\mathbb{R}^d), \tag{4.8a}$$

$$\pi_{0,\tau}{\in}W^{a,2}(\Omega;\mathbb{R}^{d\times d+1}),\quad \lim_{\tau\downarrow0}\sqrt{\tau}\big\|\pi_{0,\tau}\big\|_{W^{a,2}(\Omega;\mathbb{R}^{d\times d+1})}=0,\quad \lim_{\tau\downarrow0}\pi_{0,\tau}=\pi_0\ \text{in}\ L^2(\Omega;\mathbb{R}^{d\times d}), \tag{4.8b}$$

$$\eta_{0,\tau}{\in}W^{a,2}(\Omega;\mathbb{R}^{d\times d+1}),\quad \lim_{\tau\downarrow0}\sqrt{\tau}\big\|\eta_{0,\tau}\big\|_{W^{a,2}(\Omega;\mathbb{R}^{d\times d+1})}=0,\quad \lim_{\tau\downarrow0}\eta_{0,\tau}=\eta_0\ \text{in}\ L^2(\Omega), \tag{4.8c}$$

$$\tilde f_\tau{\in}L^\infty(\Sigma),\qquad \tilde f_\tau\ge0,\qquad \lim_{\tau\downarrow0}\tilde f_\tau=f\ \text{in}\ L^1(\Sigma). \tag{4.8d}$$

We will further make a spacial discretization. For this, we assume that we are given a sequence of triangulations $\{\mathscr{T}_h\}_{h>0}$ of the polyhedral domain Ω without hanging nodes but otherwise entirely general. We suppose that $h>0$ range over countable sets of positive real numbers with accumulation points at 0, and that $\max_{E\in\mathscr{T}_h}\mathrm{diam}(E)\le h$.

We consider C^0-conforming P1-elements for the approximation of u and w and P0-elements for the approximation of π and η. The finite-dimensional subspaces of $L^2(\Omega)$ and $W^{1,2}(\Omega)$ related to P0- and P1-elements and subordinate to the triangulation Θ_h respectively by $V_{0,h}$ and $V_{1,h}$.

For $j=0,1$, the L^2 orthogonal projection onto $V_{j,h}$ is denoted by $P_{j,h}$. We have the following approximation property at our disposal for any $1\le\gamma<\infty$:

$$\forall v\in L^2(\Omega):\qquad P_{0,h}v\to v\qquad \text{in}\ L^2(\Omega), \tag{4.9a}$$
$$\forall v\in W^{1,\gamma}(\Omega):\qquad P_{1,h}v\to v\qquad \text{in}\ W^{1,\gamma}(\Omega). \tag{4.9b}$$

Then we devise the Galerkin scheme as follows. We seek $(u_{\tau h}^k,\pi_{\tau h}^k,\eta_{\tau h}^k,w_{\tau h}^k)\in V_{1,h}^d\times V_{0,h}^{d\times d}\times V_{0,h}\times V_{1,h}$, with $\pi(\cdot)\in\mathbb{R}^{d\times d}_{\rm dev}$ a.e. on Ω, satisfying

$$\int_\Omega\varrho\mathrm{D}_t^2u_{\tau h}^k\cdot v+\Big(\mathbb{D}e\big(\mathrm{D}_tu_{\tau h}^k\big)+\mathbb{C}\big(e(u_{\tau h}^k){-}\pi_{\tau h}^k{-}\mathbb{E}\Theta(w_{\tau h}^k)\big)+\tau\big|e(u_{\tau h}^k)\big|^{\gamma-2}e(u_{\tau h}^k)\Big){:}e(v)\,\mathrm{d}x$$
$$=\int_\Gamma g_\tau^k\cdot v\,\mathrm{d}S\qquad \forall v\in V_{1,h}^d, \tag{4.10a}$$

$$\int_\Omega \zeta_1(\tilde\pi,\tilde\eta) + (\mathbb{C}\pi_{\tau h}^k - \mathbb{C}e(u_{\tau h}^k){+}\mathbb{H}\pi_{\tau h}^k) : (\tilde\pi{-}\mathrm{D}_t\pi_{\tau h}^k) + b\eta_{\tau h}^k(\tilde\eta{-}\mathrm{D}_t\eta_{\tau h}^k)$$
$$+ \tau\mathscr{S}^{1/2}\pi_{\tau h}^k : \mathscr{S}^{1/2}(\tilde\pi{-}\mathrm{D}_t\pi_{\tau h}^k) + \tau\mathscr{S}^{1/2}\eta_{\tau h}^k\mathscr{S}^{1/2}(\tilde\eta{-}\mathrm{D}_t\eta_{\tau h}^k)\,\mathrm{d}x$$
$$\geq \int_\Omega \zeta_1(\mathrm{D}_t\pi_{\tau h}^k, \mathrm{D}_t\eta_{\tau h}^k)\,\mathrm{d}x \qquad \forall(\tilde\pi,\tilde\eta) \in V_{1,h}^{d\times d} \times V_{1,h}, \tag{4.11a}$$
$$\int_\Omega \big(\mathrm{D}_t w_{\tau h}^k + \varepsilon(\tau)|w_{\tau h}^k|^{\beta-2}w_{\tau h}^k\big)v + \mathscr{K}(w_{\tau h}^k)\nabla w_{\tau h}^k\cdot\nabla v - \zeta_1\big(\mathrm{D}_t\pi_{\tau h}^k, \mathrm{D}_t\eta_{\tau h}^k\big)v$$
$$- \mathbb{D}e\big(\mathrm{D}_t u_{\tau h}^k\big) : e\big(\mathrm{D}_t u_{\tau h}^k\big)v\,\mathrm{d}x \tag{4.11b}$$
$$= \int_\Omega \Theta(w_{\tau h}^k)\mathbb{E} : \mathbb{C}e\big(\mathrm{D}_t u_{\tau h}^k\big)v\,\mathrm{d}x + \int_\Gamma f_{\tau,h}^k v\,\mathrm{d}S \qquad \forall v \in V_{1,h}.$$

Let us define the piecewise affine interpolant $(u_{\tau h}, \pi_{\tau h}, \eta_{\tau h}, w_{\tau h})$ by

$$u_{\tau h}(t) := \frac{t-(k{-}1)\tau}{\tau}u_{\tau h}^k + \frac{k\tau - t}{\tau}u_{\tau h}^{k-1} \qquad \text{for } t\in[(k{-}1)\tau, k\tau], \tag{4.12}$$

and similarly $\pi_{\tau h}(t) = \frac{t-(k-1)\tau}{\tau}\pi_{\tau h}^k + \frac{k\tau-t}{\tau}\pi_{\tau h}^{k-1}$, and $\eta_{\tau h}(t) = \frac{t-(k-1)\tau}{\tau}\eta_{\tau h}^k + \frac{k\tau-t}{\tau}\eta_{\tau h}^{k-1}$ and also $w_{\tau h}(t) = \frac{t-(k-1)\tau}{\tau}w_{\tau h}^k + \frac{k\tau-t}{\tau}w_{\tau h}^{k-1}$ for $t \in [(k{-}1)\tau, k\tau]$ with $k = 0, ..., K_\tau := T/\tau$. Besides, we define also the back-ward piecewise constant interpolant $(\bar u_{\tau h}, \bar\pi_{\tau h}, \bar\eta_{\tau h}, \bar w_{\tau h})$ by

$$\bar u_{\tau h}(t) := u_{\tau h}^k, \quad \bar\pi_{\tau h}(t) := \pi_{\tau h}^k, \quad \bar\eta_{\tau h}(t) := \eta_{\tau h}^k, \quad \bar w_{\tau h}(t) := w_{\tau h}^k \tag{4.13}$$

for $(k{-}1)\tau < t \leq k\tau$, $k = 1, ..., K_\tau$. Similarly, we will later use u_τ, $\bar u_\tau$, etc. We will also use the notation $\bar g_\tau$ and $\bar f_\tau$ defined by $\bar g_\tau|_{((k-1)\tau,k\tau]} = g_\tau^k$ and $\bar f_\tau|_{((k-1)\tau,k\tau]} = f_\tau^k$ for $k = 1, ..., K_\tau$.

Lemma 4.1 (Existence and estimates of discrete solutions) *Let (2.19), (3.9), (3.10), and (4.8) hold. Moreover, let*

$$\beta > 2, \qquad \gamma > \max\Big(4, \frac{2\omega}{\omega{-}1}\Big), \qquad \text{and} \qquad \omega > 1. \tag{4.14}$$

Then there exists a solution $(u_{\tau h}^k, \pi_{\tau h}^k, \eta_{\tau h}^k, w_{\tau h}^k) \in V_{1,h}^d{\times}V_{0,h}^{d\times d}{\times}V_{0,h}{\times}V_{1,h}$, with $\pi(\cdot) \in \mathbb{R}_{\mathrm{dev}}^{d\times d}$ a.e. on Ω, for the system (4.10). Moreover,

$$\big\|u_{\tau h}\big\|_{W^{1,\infty}(I;W^{1,\gamma}(\Omega;\mathbb{R}^d))} \leq C_\tau, \tag{4.15a}$$
$$\big\|\pi_{\tau h}\big\|_{W^{1,\infty}(I;W^{a,2}(\Omega;\mathbb{R}_{\mathrm{dev}}^{d\times d}))} \leq C_\tau, \tag{4.15b}$$
$$\big\|\eta_{\tau h}^k\big\|_{W^{1,\infty}(I;W^{a,2}(\Omega))} \leq C_\tau, \tag{4.15c}$$
$$\big\|w_{\tau h}^k\big\|_{W^{1,\infty}(I;W^{1,2}(\Omega))} \leq C_\tau \tag{4.15d}$$

with some C_τ independent of h and with $a \in (0, 1/2)$ referring to (4.6).

Sketch of the proof. We can see existence of a solution to (4.10) by a standard argument for coercive pseudomonotone set-valued operators. The coercivity of the underlying operator can be shown by testing (4.10a), (4.10b), and (4.10c) by $u_{\tau h}^k \in V_{1,h}^d$, $\pi_{\tau h}^k \in V_{0,h}^{d\times d}$, $\eta_{\tau h}^k \in V_{0,h}$, and $w_{\tau h}^k \in V_{1,h}$, respectively. Note that these test-functions live in the corresponding finite-dimensional spaces and are thus legal for this test. It is important that the right-hand sides of (4.10a,c) have the growth that can be dominated by the growth of the coercive terms in the left-hand sides; this is ensured by having taken β

and γ large enough and by the assumption (3.9b) which ensures a sublinear growth of Θ, namely

$$\Theta(w) \le \Big(\frac{w}{\omega c_0}{+}1\Big)^{1/\omega} - 1 \le \Big(\frac{w}{\omega c_0}\Big)^{1/\omega} \tag{4.16}$$

because obviously $h_0(\theta) \ge \omega c_0(1{+}\theta)^\omega - \omega c_0$, cf. the definition (3.2). Realize that the coercivity can be estimated (up to multiplicative constants) as $|e|^\gamma{+}|\pi|^2{+}|\eta|^2{+}|w|^\beta$ which indeed dominates the growth of the "right-hand-side terms" is of the type $|w|^{1/\omega}|e| + (|\pi| + |\eta|)|w| + |e|^2|w| + |w|^{1+1/\omega}|e|$. The term $(|\pi| + |\eta|)|w|$ bears the estimation by $\delta|\pi|^2 + \delta|\eta|^2 + \delta|w|^\beta + C_\delta$ and similarly $|e|^2|w| \le \delta|e|^\gamma + \delta|w|^2 + C_\delta$ with any $\delta > 0$ and some C_δ; here $\beta > 2$ and $\gamma > 4$ have respectively been used. The last term can be estimated as $|w|^{1+1/\omega}|e| \le \frac{1}{\gamma}|e|^\gamma + |w|^{(1+1/\omega)\gamma/(\gamma-1)} \le \frac{1}{\gamma}|e|^\gamma + \frac{1}{2}|w|^2 + C_\gamma$ for some $C_\gamma \in \mathbb{R}$; here the condition $\gamma > 2\omega/(\omega{-}1)$ has originated.

The a-priori estimates (4.15) then follows from the above test by standard procedure, i.e. by using the Hölder, the Young, and the discrete Gronwall inequalities. □

Lemma 4.2 (Convergence for $h{\downarrow}0$) *There is a subsequence of $\{(u_{\tau h}, \pi_{\tau h}, \eta_{\tau h}, w_{\tau h})\}_{h>0}$ converging for $h{\downarrow}0$ weakly* in the topologies indicated in (4.15) to some $(u_\tau, \pi_\tau, \eta_\tau, w_\tau)$ and each quadruple obtained by such way is a weak solution to (4.2)–(4.3), i.e. in term of the interpolants*

$$\varrho \mathrm{D}_t^2 u_\tau - \mathrm{div}\Big(\mathbb{D}e\big(\frac{\partial u_\tau}{\partial t}\big) + \mathbb{C}\big(e(\bar u_\tau){-}\bar\pi_\tau{-}\mathbb{E}\Theta(\bar w_\tau)\big) + \tau\big|e(\bar u_\tau)\big|^{\gamma-2} e(\bar u_\tau)\Big) = 0, \tag{4.17a}$$

$$\partial\zeta_1\Big(\frac{\partial \pi_\tau}{\partial t}, \frac{\partial \eta_\tau}{\partial t}\Big) + \begin{pmatrix} \mathbb{C}\bar\pi_\tau{+}\mathbb{H}\bar\pi_\tau \\ b\bar\eta_\tau \end{pmatrix} \ni \begin{pmatrix} \mathbb{C}e(\bar u_\tau) \\ 0 \end{pmatrix} + \tau\mathscr{S}\begin{pmatrix} \bar\pi_\tau \\ \bar\eta_\tau \end{pmatrix}, \tag{4.17b}$$

$$\begin{aligned} &\frac{\partial w_\tau}{\partial t} - \mathrm{div}\big(\mathscr{K}(\bar w_\tau)\nabla \bar w_\tau\big) + \varepsilon(\tau)|\bar w_\tau|^{\beta-2}\bar w_\tau \\ &\qquad = \zeta_1\Big(\frac{\partial \pi_\tau}{\partial t}, \frac{\partial \eta_\tau}{\partial t}\Big) + \mathbb{D}e\big(\frac{\partial u_\tau}{\partial t}\big){:}e\big(\frac{\partial u_\tau}{\partial t}\big) + \Theta(\bar w_\tau)\mathbb{E}{:}\mathbb{C}e\big(\frac{\partial u_\tau}{\partial t}\big), \end{aligned} \tag{4.17c}$$

with the boundary conditions

$$\Big(\mathbb{D}e\big(\frac{\partial u_\tau}{\partial t}\big) + \mathbb{C}\big(e(\bar u_\tau) - \bar\pi_\tau - \mathbb{E}\Theta(\bar w_\tau)\big) + \tau\big|e(\bar u_\tau)\big|^{\gamma-2} e(\bar u_\tau)\Big)\nu = \bar g_\tau, \tag{4.18a}$$

$$\big(\mathscr{K}(\bar w_\tau)\nabla \bar w_\tau\big)\cdot\nu = \bar f_\tau, \tag{4.18b}$$

and with the initial conditions (4.4); of course, $\mathrm{D}_t^2 u_\tau$ in (4.17a) means the piece-wise constant interpolant in time and $(\bar u_\tau, \bar\pi_\tau, \bar\zeta_\tau, \bar w_\tau)$ is the limit of a subsequence of $\{(\bar u_{\tau h}, \bar\pi_{\tau h}, \bar\eta_{\tau h}, \bar w_{\tau h})\}_{h>0}$ and simultaneously also the piece-wise constant interpolant in time corresponding to $(u_\tau, \pi_\tau, \eta_\tau, w_\tau)$.

Sketch of the proof. By Banach's selection principle, we first select a weakly* convergent subsequence. Due to the construction of $V_{1,h}$, we have the approximation property (4.9b) at our disposal. Hence we can consider also a sequence $\{\widetilde u_{\tau h}\}_{h>0}$ converging strongly to u_τ even in $W^{1,\infty}(I; W^{1,\gamma}(\Omega;\mathbb{R}^d))$ and such that $\widetilde u_{\tau h} : I \to V_{1,h}^d$; here one must take into account that $\tau > 0$ is fixed hence only a finite number of values of u_τ is to be approximated by using (4.9b).

Due to the dissipative-heat term in (4.18a), we need to prove the strong convergence $\frac{\partial}{\partial t}e(u_{\tau h}) \to \frac{\partial}{\partial t}e(u_\tau)$ in $L^2(Q;\mathbb{R}^{d\times d})$. To this goal, we first use the so-called d-monotonicity of $e\mapsto\mathbb{C}e+|e|^{\gamma-2}e$ to prove the strong convergence $e(\bar u_{\tau h})\to e(\bar u_\tau)$ in $L^\gamma(Q;\mathbb{R}^{d\times d})$ by the estimate

$$
\begin{aligned}
&\tau\Big(\big\|e(\bar u_{\tau h})\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d})}^{\gamma-1}-\big\|e(\bar u_\tau)\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d})}^{\gamma-1}\Big)\Big(\big\|e(\bar u_{\tau h})\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d})}-\big\|e(\bar u_\tau)\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d})}\Big)\\
&\le\int_Q\mathbb{C}e(\bar u_{\tau h}{-}\bar u_\tau){:}e(\bar u_{\tau h}{-}\bar u_\tau)+\tau\Big(\big|e(\bar u_{\tau h})\big|^{\gamma-2}e(\bar u_{\tau h})-\big|e(\bar u_\tau)\big|^{\gamma-2}e(\bar u_\tau)\Big){:}e(\bar u_{\tau h}{-}\bar u_\tau)\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_\Omega\frac12\mathbb{D}e(u_{\tau h}(T){-}u_\tau(T)){:}e(u_{\tau h}(T){-}u_\tau(T))\,\mathrm{d}x\\
&\le\int_Q\mathbb{C}\big(\bar\pi_{\tau h}+\Theta(\bar w_{\tau h})\mathbb{E}\big){:}e(\bar u_{\tau h}{-}\widetilde u_{\tau h})-\varrho\mathrm{D}_t^2u_{\tau h}\cdot(\bar u_{\tau h}{-}\widetilde u_{\tau h})\\
&\quad-\Big(\mathbb{C}e(\bar u_\tau)+\tau\big|e(\bar u_\tau)\big|^{\gamma-2}e(\bar u_\tau)+\mathbb{D}e\big(\frac{\partial u_\tau}{\partial t}\big)\Big){:}e(\bar u_{\tau h}{-}\widetilde u_{\tau h})\,\mathrm{d}x\mathrm{d}t+\int_\Sigma\bar g_\tau{\cdot}(\bar u_{\tau h}{-}\widetilde u_{\tau h})\,\mathrm{d}S\mathrm{d}t\\
&\quad+\int_Q\mathbb{C}e(\bar u_{\tau h}{-}\bar u_\tau){:}e(\widetilde u_{\tau h}{-}\bar u_\tau)+\tau\Big(\big|e(\bar u_{\tau h})\big|^{\gamma-2}e(\bar u_{\tau h})-\big|e(\bar u_\tau)\big|^{\gamma-2}e(\bar u_\tau)\Big){:}e(\widetilde u_{\tau h}{-}\bar u_\tau)\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_\Omega\frac12\mathbb{D}e(u_{\tau h}(T){-}u_\tau(T)){:}e(\widetilde u_{\tau h}(T){-}u_\tau(T))\,\mathrm{d}x\to0.
\end{aligned}\tag{4.19}
$$

The second inequality in (4.19) is due to the inequality $\mathbb{D}e(\mathrm{D}_tu_\tau^k):e(u_\tau^k)\ge\frac12\mathrm{D}_t(\mathbb{D}e(u_\tau^k):e(u_\tau^k))$, which is just a generalization of the elementary algebraic inequality of the type $(a{-}b)a\ge\frac12a^2-\frac12b^2$. The convergence to zero in (4.19) for $h{\downarrow}0$ is because $\tau>0$ is fixed so that trivially $\mathrm{D}_t^2u_{\tau h}\to\mathrm{D}_t^2u_\tau$ in $L^2(Q;\mathbb{R}^d)$ due to the Rellich compact embedding $W^{1,2}(\Omega)\Subset L^2(\Omega)$, and furthermore also $\Theta(\bar w_{\tau h})\to\Theta(\bar w_\tau)$ certainly in $L^2(Q)$ (in fact even in a much smaller Lebesgue space $L^{2d\omega/(d-2)-\epsilon}(Q)$ with $\epsilon>0$) due to the compact embedding $W^{1,2}(\Omega)$ and $\bar\pi_{\tau h}\to\bar\pi_\tau$ in $L^2(Q;\mathbb{R}^{d\times d})$ due to the compact embedding $W^{a,2}(\Omega)$ so that $(\bar\pi_{\tau h}+\mathbb{C}\Theta(w_{\tau h})\mathbb{E}):e(\bar u_{\tau h}-\bar u_\tau)\to0$ weakly in $L^1(Q)$.

Then, we use the strong monotonicity of $e\mapsto\mathbb{D}e$ to estimate, for some $c>0$,

$$
\begin{aligned}
&c\Big\|\frac{\partial e(u_{\tau h}{-}u_\tau)}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^{d\times d})}^2\le\int_Q\mathbb{D}\frac{\partial e(u_{\tau h}{-}u_\tau)}{\partial t}{:}\frac{\partial e(u_{\tau h}{-}u_\tau)}{\partial t}\mathrm{d}x\mathrm{d}t\\
&\le\int_Q\mathbb{D}\frac{\partial e(u_{\tau h}{-}u_\tau)}{\partial t}{:}\frac{\partial e(u_{\tau h}{-}u_\tau)}{\partial t}\,\mathrm{d}x\mathrm{d}t+\int_\Omega\frac12\mathbb{C}e(u_{\tau h}(T){-}u_\tau(T)){:}e(u_{\tau h}(T){-}u_\tau(T))\,\mathrm{d}x\\
&=\int_Q-\varrho\mathrm{D}_t^2u_{\tau h}\frac{\partial(u_{\tau h}{-}\widetilde u_{\tau h})}{\partial t}\\
&\qquad-\Big(\mathbb{C}\pi_{\tau h}+\mathbb{C}\Theta(w_{\tau h})\mathbb{E}+\mathbb{D}\frac{\partial e(u_\tau)}{\partial t}+\tau\big|e(u_{\tau h})\big|^{\gamma-2}e(u_{\tau h})\Big){:}\frac{\partial e(u_{\tau h}{-}\widetilde u_{\tau h})}{\partial t}\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_\Sigma\bar g_\tau\cdot\frac{\partial(u_{\tau h}{-}\widetilde u_{\tau h})}{\partial t}\,\mathrm{d}S\mathrm{d}t-\int_\Omega\frac12\mathbb{C}e(u_\tau(T)){:}e(u_{\tau h}(T){-}u_\tau(T))\,\mathrm{d}x\\
&\qquad+\int_Q\mathbb{D}\frac{\partial e(u_{\tau h}{-}u_\tau)}{\partial t}{:}\frac{\partial e(\widetilde u_{\tau h}{-}u_\tau)}{\partial t}\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_\Omega\frac12\mathbb{C}e(u_{\tau h}(T){-}u_\tau(T)){:}(e(\widetilde u_{\tau h}(T){-}u_\tau(T)))\to0.
\end{aligned}\tag{4.20}
$$

The convergence to zero for $h{\downarrow}0$ again relies on $\tau>0$ fixed so that again $\mathrm{D}_t^2u_{\tau h}\to\mathrm{D}_t^2u_\tau$, and $\pi_{\tau h}\to\pi_\tau$ in $L^2(Q;\mathbb{R}^{d\times d})$ due to Rellich's compact embedding $W^{1,2}(\Omega)\Subset L^2(\Omega)$, and because that the strong convergence $e(u_{\tau h}^k)\to e(u_\tau^k)$ in $L^\gamma(Q;\mathbb{R}^{d\times d})$ has already been proved.

Furthermore, we use again the compact embedding $W^{a,2}(\Omega)\Subset L^2(\Omega)$ so that, thanks to the regularizing operator $\mathscr{S}$, we have convergence also in

$$\zeta_1(\mathrm{D}_t\pi_{\tau h}^k,\mathrm{D}_t\eta_{\tau h}^k)\to\zeta_1(\mathrm{D}_t\pi_\tau^k,\mathrm{D}_t\eta_\tau^k)\qquad\text{strongly in }\ L^\infty(I;L^2(\Omega))\tag{4.21}$$

because $\tau>0$ is considered fixed. Altogether, we proved strong convergence of the heat sources in $L^1(Q)$.

Then, using still the weak upper semicontinuity argument for $\mathscr{S}$-terms in (4.11a) summed over particular time levels, the claimed limit passage from (4.10) to the boundary-value problem (4.17)–(4.18) formulated weakly is easy to be seen. In particular, the limit passage from (4.2b) to (4.17b) uses also the approximation property (4.9b). $\square$

Note that (4.2c) has the right-hand side in $L^2(\Omega)$ since $\gamma\ge4$ and since $\pi_\tau^k-\pi_\tau^{k-1}$ and $\eta_\tau^k-\eta_\tau^{k-1}$ are certainly in $L^2(\Omega;\mathbb{R}^{d\times d})$ and $L^2(\Omega)$, respectively, hence the weak formulation of (4.2c) is understood standardly.

Let us abbreviate the regularized stored energy by Φ_τ, i.e.

$$\begin{aligned}\Phi_\tau(u,\pi,\eta)&:=\Phi(u,\pi,\eta)+\frac{\tau}{\gamma}\big\|e(u)\big\|_{L^\gamma(\Omega;\mathbb{R}^{d\times d})}^\gamma+\frac12\big|(\pi,\eta)\big|_{W^{a,2}(\Omega;\mathbb{R}^{d\times d+1})}^2\\&=\int_\Omega\frac12\mathbb{C}(e(u){-}\pi):(e(u){-}\pi)+\frac12\mathbb{H}\pi:\pi+\frac{b}{2}\eta^2\\&\qquad\qquad+\frac{\tau}{\gamma}|e(u)|^\gamma\,\mathrm{d}x+\frac{\tau}{2}\big|\pi\big|_{W^{a,2}(\Omega;\mathbb{R}^{d\times d})}^2+\frac{\tau}{2}\big|\eta\big|_{W^{a,2}(\Omega)}^2.\end{aligned}\tag{4.22}$$

Lemma 4.3 (Still further a-priori information) *For any $k=1,...,K_\tau$, the following "discrete mechanical energy" balance holds:*

$$\begin{aligned}&T_{\rm kin}\big(\mathrm{D}_tu_\tau^k\big)+\Phi_\tau\big(u_\tau^k,\pi_\tau^k,\eta_\tau^k\big)+\tau\sum_{l=1}^k\int_\Omega\zeta_1\big(\mathrm{D}_t\pi_\tau^k,\mathrm{D}_t\eta_\tau^k\big)+\mathbb{D}e\big(\mathrm{D}_tu_\tau^k\big):e\big(\mathrm{D}_tu_\tau^k\big)\,\mathrm{d}x\\&\quad\le T_{\rm kin}\big(\dot u_0\big)+\Phi_\tau\big(u_{0,\tau},\pi_{0,\tau},\eta_{0,\tau}\big)+\tau\sum_{l=1}^k\bigg(\int_\Omega\Theta(w_\tau^l)\mathbb{E}{:}\mathbb{C}e\big(\mathrm{D}_tu_\tau^k\big)\,\mathrm{d}x+\int_\Gamma g_\tau^l{\cdot}\mathrm{D}_tu_\tau^l\,\mathrm{d}S\bigg)\end{aligned}\tag{4.23}$$

as well as the following "discrete total energy" balance holds:

$$\begin{aligned}T_{\rm kin}\big(\mathrm{D}_tu_\tau^k\big)+\Phi_\tau\big(u_\tau^k,\pi_\tau^k,\eta_\tau^k\big)+\int_\Omega w_\tau^k\,\mathrm{d}x\le T_{\rm kin}\big(\dot u_0\big)+\Phi_\tau\big(u_{0,\tau},\pi_{0,\tau},\eta_{0,\tau}\big)+\int_\Omega w_0\,\mathrm{d}x\\+\ \tau\sum_{l=1}^k\bigg(\int_\Gamma g_\tau^l{\cdot}\mathrm{D}_tu_\tau^l+f_{{\rm ext},\tau}^l\mathrm{d}S-\varepsilon(\tau)\int_\Omega|w_\tau^l|^{\beta-2}w_\tau^l\,\mathrm{d}x\bigg),\end{aligned}\tag{4.24}$$

and also the "discrete semistability"

$$\Phi_\tau(u_\tau^k,\pi_\tau^k,\eta_\tau^k)\le\Phi_\tau(u_\tau^k,\tilde\pi,\tilde\eta)+\int_\Omega\xi_1(\tilde\pi-\pi_\tau^k,\tilde\eta-\eta_\tau^k)\,\mathrm{d}x\tag{4.25}$$

holds for any $(\tilde\pi,\tilde\eta)\in W^{a,2}(\Omega;\mathbb{R}_{\rm dev}^{d\times d}{\times}\mathbb{R})$, where Φ_τ and $T_{\rm kin}$ are from (4.22) and (2.13), respectively.

Proof. Let us use a short-hand notation $z:=(\pi,\eta)$ for this proof. Taking $(u_\tau^k,z_\tau^k,w_\tau^k)$ solving (4.2), we can test (4.2a,b) respectively by $\mathrm{D}_tu_\tau^k$ and $\mathrm{D}_tz_\tau^k$. By using the convexity of Φ_τ from (4.22) and by summation over time steps, we obtain (4.23).

Now, to get (4.24), we still add (4.2d) tested by 1 and summed over time steps to (4.23); here it is important that the dissipative/adiabatic terms mutually cancel in the mechanical and the thermal parts.

As for (4.25), we use that (4.2b) is the sufficient (and, of course, also necessary) optimality condition for z_τ^k to minimize the convex functional $\Phi_\tau(u_\tau^k,\cdot)+\int_\Omega \zeta_1(\cdot-z_\tau^{k-1})\mathrm{d}x$, which gives

$$\Phi_\tau\big(u_\tau^k,z_\tau^k\big)+\int_\Omega\tau\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)\,\mathrm{d}x\le\Phi_\tau\big(u_\tau^k,\tilde z\big)+\int_\Omega\tau\zeta_1\Big(\frac{\tilde z-z_\tau^{k-1}}{\tau}\Big)\,\mathrm{d}x$$

for any $\tilde z$ and then, by using that ζ_1 is homogeneous degree-1 and thus satisfies the triangle inequality $\zeta_1(z_\tau^k-z_\tau^{k-1})\le\zeta_1(\tilde z-z_\tau^k)+\zeta_1(z_\tau^k-z_\tau^{k-1})$, which altogether gives

$$\begin{aligned}\Phi_\tau\big(u_\tau^k,z_\tau^k\big)&\le\Phi_\tau\big(u_\tau^k,\tilde z\big)+\int_\Omega\zeta_1\big(\tilde z-z_\tau^{k-1}\big)-\zeta_1\big(z_\tau^k-z_\tau^{k-1}\big)\,\mathrm{d}x\\&\le\Phi_\tau\big(u_\tau^k,\tilde z\big)+\int_\Omega\zeta_1\big(\tilde z-z_\tau^k\big)\,\mathrm{d}x,\end{aligned}\tag{4.26}$$

arriving thus just to (4.25). $\square$

Proposition 4.4 (Uniform a-priori estimates) *Let, beside the assumptions from Lemma 4.1, also* (2.3) *hold and the exponent* ω *from (3.9b) satisfy*

$$\omega>\frac{2d}{d+2}.\tag{4.27}$$

Then, for some C *and* C_τ*, it holds*

$$\big\|u_\tau\big\|_{W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^d))}\le C,\tag{4.28a}$$
$$\big\|\bar\pi_\tau\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))\,\cap\,\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))}\le C,\tag{4.28b}$$
$$\big\|\bar\eta_\tau\big\|_{L^\infty(I;L^2(\Omega))\,\cap\,\mathrm{BV}(\bar I;L^1(\Omega))}\le C,\tag{4.28c}$$
$$\big\|\bar w_\tau\big\|_{L^\infty(I;L^1(\Omega))\,\cap\,L^r(I;W^{1,r}(\Omega))}\le C_r\quad\textit{with any }1\le r<\tfrac{d+2}{d+1},\tag{4.28d}$$
$$\Big\|\frac{\partial w_\tau}{\partial t}\Big\|_{L^1(I;W^{1+d,2}(\Omega)^*)}\le C,\tag{4.28e}$$
$$\Big\|\varrho\frac{\partial u_\tau}{\partial t}\Big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))\,\cap\,\mathrm{BV}(\bar I;W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^d)^*)}\le C\,,\tag{4.28f}$$
$$\big\|u_\tau\big\|_{L^\infty(I;W^{1,\gamma}(\Omega;\mathbb{R}^d))}\le C\tau^{-1/\gamma},\tag{4.28g}$$
$$\big\|\bar\pi_\tau\big\|_{L^\infty(I;W^{a,2}(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))}\le C\tau^{-1/2},\tag{4.28h}$$
$$\big\|\bar\eta_\tau\big\|_{L^\infty(I;W^{a,2}(\Omega))}\le C\tau^{-1/2},\tag{4.28i}$$
$$\big\|\bar w_\tau\big\|_{L^\beta(Q)}\le C_\tau\varepsilon(\tau)^{-1/\beta}.\tag{4.28j}$$

Note that $\frac{\partial}{\partial t}u_\tau$ is piece-wise constant in time with possible jumps at times $t=k\tau$, so that $\varrho\frac{\partial^2}{\partial t^2}u_\tau$ is a measure, which is why (4.28f) involves BV-space.

Ideas of the proof. For particular details as far as (4.28a-i) concerns, see [24, proof of Proposition 4.2]; in contrast to [24], here we have used g qualified as $L^2(I;L^q(\Gamma;\mathbb{R}^d))$ in (3.10a) instead of a bulk force in $L^1(I;L^2(\Omega;\mathbb{R}^d))$ and also we have admitted $\varrho=0$, which however is just a simplified case. Let us just outline the scenario: As we already got rid of the spatial discretization, from the maximum principle one can see that $w_\tau \geq 0$. ¿From (4.24) then one gets the L^∞-parts of (4.28b-d) and (4.28f-i). Then one uses the L^1-theory for the evolutionary heat equation [4, 5] based on the test by $1 - 1/(1{+}\bar{w}_\tau)^\delta$, $\delta > 0$, combined with the interpolation of the adiabatic term by using several-times Gagliardo-Nirenberg inequality as in [22, 24], which eventually gives (4.28a) and the rest of (4.28b-d). Then (4.28e) and the BV-part of (4.28f) follow by the already obtained estimates.

Eventually, for $\tau > 0$ fixed, we already mentioned that the right-hand side of the discrete heat equation (4.2d) is in $L^2(\Omega)$ so that the right-hand side of (4.18a), let us denote it by $\bar{r}_\tau$ for a moment, is in $L^\infty(I; L^2(\Omega))$ so that we can still test the discrete heat equation by w_τ to obtain (4.28j). $\square$

In general, C_τ in (4.28j) may depend not only on τ but implicitly also on $\varepsilon = \varepsilon(\tau)$ and then (4.28j) says nothing more than $w_\tau \in L^\gamma(Q)$ only, unless one has some more specific information about this implicit dependence. Here, it is, however, for γ sufficiently large, one can show that C_τ even does not depend on $\varepsilon(\tau)$ at all. Let us present the calculations only for the physically relevant case, i.e. $d = 3$ (while for $d \leq 2$ it is even less restrictive).

Lemma 4.5 *Let $d = 3$, let the assumptions from Lemma 4.1 and* (4.27) *hold, and let, in addition, $\gamma > 76/17$. Then* (4.28j) *holds with $C_\tau = C\tau^{-2(\gamma+1)/(2\beta\gamma)}$ for some C independent of τ.*

Note that the condition $\gamma > 76/17 \doteq 4.47$ may (but need not, depending on ω) slightly strengthen (4.14).

Proof of Lemma 4.5. In fact, by testing the heat equation (4.2d) by w_τ, we can see that the constant C_τ is proportional to $\|\bar{r}_\tau\|^{2/\beta}_{L^2(I;L^{6/5}(\Omega))}$. Thus it is desirable to estimate $\bar{r}_\tau$ in $L^2(I; L^{6/5}(\Omega))$ independently of $\varepsilon = \varepsilon(\tau)$.

By (4.28a), we have $\|e(\frac{\partial u_\tau}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d})} \leq C$. By (4.28g), we have $\|e(u_\tau)\|_{L^\infty(I;L^\gamma(\Omega;\mathbb{R}^{d\times d}))} \leq C/\tau^{1/\gamma}$, so that, realizing the equidistance of the partition with the time-step τ, we have also $\|e(\frac{\partial u_\tau}{\partial t})\|_{L^\infty(I;L^\gamma(\Omega;\mathbb{R}^{d\times d}))} \leq C/\tau^{1+1/\gamma}$. Interpolating these estimates with the weights $\frac{1}{2}$ and $\frac{1}{2}$ yields

$$\Big\|e\Big(\frac{\partial u_\tau}{\partial t}\Big)\Big\|_{L^4(I;L^{4\gamma/(\gamma+2)}(\Omega;\mathbb{R}^{d\times d}))} \leq \frac{C}{\tau^{(\gamma+1)/(2\gamma)}}. \tag{4.29}$$

Thus we have the estimate for the viscous part of the dissipative heat

$$\Big\|\mathbb{C}e\Big(\frac{\partial u_\tau}{\partial t}\Big) : e\Big(\frac{\partial u_\tau}{\partial t}\Big)\Big\|_{L^2(I;L^{2\gamma/(\gamma+2)}(\Omega))} \leq \frac{C}{\tau^{(\gamma+1)/\gamma}}. \tag{4.30}$$

Note that the desired embedding $L^2(I; L^{2\gamma/(\gamma+2)}(\Omega)) \subset L^2(I; L^{6/5}(\Omega))$ needs here $\gamma \geq 3$, which always holds due to (4.14). As to the adiabatic heat, we use the interpolation between two estimates in (4.28d), i.e. between $L^\infty(I; L^1(\Omega))$ and $L^r(I; W^{1,r}(\Omega))$ with $r < 5/4$, with the weight λ and $1{-}\lambda$ with sufficiently small $\lambda > 5/8$ to obtain $w_\tau \in L^{10/3}(I; L^q(\Omega))$ with $q < 5/4$. As Θ has the sublinear polynomial growth with an exponent less than 5/6, c.f. (4.16) with $\omega > 6/5$ due to (4.27), we then have the temperature $\Theta(\bar{w}_\tau) \in L^4(I; L^{6q/5}(\Omega))$. By (4.29), we have

$$\Big\|\Theta(\bar{w}_\tau)\mathbb{E} : \mathbb{C}e\Big(\frac{\partial u_\tau}{\partial t}\Big)\Big\|_{L^2(I;L^{12\gamma q/(2\gamma\omega+19)}(\Omega))} \leq \frac{C}{\tau^{(\gamma+1)/(2\gamma)}}. \tag{4.31}$$

Here the desired embedding $L^4(I; L^{12\gamma q/(2\gamma\omega+19)}(\Omega)) \subset L^2(I; L^{6/5}(\Omega))$ is possible provided γ is sufficiently large, namely $\gamma > 76/17$.

Eventually, by the L^∞-parts of (4.28b,c), we have also $\|\frac{\partial\pi_\tau}{\partial t}\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d+1}))}\le C/\tau$ which can be still interpolated (with the weight $\frac12$ and $\frac12$) with $\|\frac{\partial\pi_\tau}{\partial t}\|_{L^1(I;L^1(\Omega;\mathbb{R}^{d\times d+1}))}\le C$ due to the BV-part of (4.28b), and similarly for $\frac{\partial\pi_\tau}{\partial t}$ due to the BV-part of (4.28c), so that the remaining contribution to the dissipative heat can be estimated as

$$\Big\|\zeta_1\big(\frac{\partial\pi_\tau}{\partial t},\frac{\partial\eta_\tau}{\partial t}\big)\Big\|_{L^2(I;L^{4/3}(\Omega))}\le\frac{C}{\tau^{1/2}}. \tag{4.32}$$

Thus

$$C_\tau\sim\|\bar r_\tau\|^{2/\beta}_{L^2(I;L^{6/5}(\Omega))}=\mathscr{O}\Big(\frac{1}{\tau^{2(\gamma+1)/(2\beta\gamma)}}\Big). \tag{4.33}$$

□

Proposition 4.6 (Convergence for $\tau\downarrow0$) *Let $\varepsilon=\varepsilon(\tau)$ be chosen to converge to 0 for $\tau\downarrow0$ sufficiently fast so that*

$$\lim_{\tau\downarrow0}C_\tau^{\beta-1}\varepsilon(\tau)^{1/\beta}=0, \tag{4.34}$$

with C_τ refering to (4.28j). Then there is a subsequence of $\{(u_\tau,\pi_\tau,\eta_\tau,w_\tau)\}_{\tau>0}$ weakly convergent in the topologies indicated in (4.28a-f), cf. Remark 4.7, to some (u,π,η,w) and, if the initial conditions (π_0,η_0) are semistable with respect to u_0 in the sense*

$$\Phi(u_0,\pi_0,\eta_0)\le\Phi(u_0,\tilde\pi,\tilde\eta)+\int_\Omega\zeta_1(\tilde\pi-\pi_0,\tilde\eta-\eta_0)\,\mathrm{d}x \tag{4.35}$$

for all $(\tilde\pi,\tilde\eta)\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R})$, then (u,π,η,w) is an energetic solution according Definition 3.1.

Sketch of the proof. For some details see [24, proof of Proposition 4.3 and Remark 4.5], the essential differences are the regularizing $\mathscr{S}$- and β-terms. Let us again use the short-hand notation $z:=(\pi,\eta)$ for this proof.

First, by Banach's selection principle, we select a weakly* convergent subsequence. By the generalized Helly principle $\bar z_\tau(t)\to z(t)=(\pi(t),\eta(t))$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R})$ for all $t\in[0,T]$ as well as $\bar u_\tau(t)\to u(t)$ weakly in $W^{1,2}(\Omega;\mathbb{R}^d)$, and also $\bar w_\tau(t)\to w(t)$ weakly* in $\mathscr{M}(\bar\Omega)$.

To pass to the limit in (4.17a) by-part integrated over I to the weakly formulated momentum equation (3.7a) is simple because all terms are either linear, or enjoys compactness (which concerns $\Theta(w)$-term), or vanishes due to the estimate (4.28g) since

$$\Big|\int_Q|e(\bar u_\tau)|^{\gamma-2}e(\bar u_\tau){:}e(v)\,\mathrm{d}x\mathrm{d}t\Big|\le\big\|e(\bar u_\tau)\big\|^{\gamma-1}_{L^\gamma(Q;\mathbb{R}^{d\times d})}\big\|e(v)\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d})}=\mathscr{O}(\tau^{1/\gamma})\to0.$$

To pass to the limit in the semi-stability (4.25) towards (3.7e), we need to construct a so-called joint-recovery sequence, cf. [18]. Here it essentially means that, for any $\hat z=(\hat\pi,\hat\eta)\in L^2(\mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R})$ with $\hat\eta-\eta(t)\ge\delta^*_{P_0}(\hat\pi-\pi(t))$, we need to find a sequence $\hat z_\tau=(\hat\pi_\tau,\hat\eta_\tau)$ in $W^{a,2}(\Omega;\mathbb{R}^{d\times d+1})$ such that

$$\begin{aligned}\limsup_{\tau\downarrow0}\Phi_\tau(u_\tau(t),\hat z_\tau)-\Phi_\tau(u_\tau(t),z_\tau(t))+\int_\Omega\xi_1(\hat z_\tau-z_\tau(t))\mathrm{d}x&\\ \le\Phi(u(t),\hat z)-\Phi(u(t),z(t))+\int_\Omega\xi_1(\hat z-z(t))\mathrm{d}x.&\end{aligned} \tag{4.36}$$

In fact, the true Gibbs' stored energy would still yield the term $\int_\Gamma g_\tau(t)\cdot(\hat z_\tau - z_\tau(t))\mathrm{d}S$ which, however, could easily be shown to converge to zero if the construction (4.37) below is adopted.

Let us denote the standard mollifier $[\cdot]_\delta$ by convolution with a standard positive kernel whose support is of diameter proportional to δ. Thus we can rely on $\|\,[z]_\delta\|_{W^{1,2}(\Omega)} \le C\delta^{-1}\|z\|_{L^2(\Omega)}$ so that, by interpolation with $\|\,[z]_\delta\|_{L^2(\Omega)} \le C\|z\|_{L^2(\Omega)}$, one gets $\|\,[z]_\delta\|_{W^{a,2}(\Omega)} \le C\delta^{-a}\|z\|_{L^2(\Omega)}$. Then, for $z\in L^2(\Omega)$, we have also $[z]_\delta \to z$ in $L^2(\Omega)$ for $\delta\to 0$. We take the joint-recovery sequence as

$$\hat z_\tau := z_\tau(t) + \big[\hat z - z(t)\big]_{\delta(\tau)} \qquad \text{with } \ \delta(\tau) := \tau^{1/(4a)}. \tag{4.37}$$

We rely on the quadratic form of Φ_τ, which by the binomial formula results to

$$\begin{aligned}\Phi_\tau(u,\hat z) - \Phi_\tau(u,z) = \int_\Omega \frac12(\mathbb{C}{+}\mathbb{H})(\hat\pi{-}\pi):(\hat\pi{+}\pi) - \mathbb{C}e(u):(\hat\pi{-}\pi)&\\ + \frac{b}{2}(\hat\eta{-}\eta)(\hat\eta{+}\eta) + \frac{\tau}{2}\mathscr{S}^{1/2}(\hat z{-}z):\mathscr{S}^{1/2}(\hat z{+}z)\,\mathrm{d}x.&\end{aligned} \tag{4.38}$$

By (4.37), we have

$$\hat z_\tau - z_\tau(t) = [\hat z - z(t)]_{\delta(\tau)} \to \hat z - z(t) \qquad \text{strongly in } \ L^2(\Omega;\mathbb{R}^{d\times d+1}), \tag{4.39}$$

which causes the convergence

$$\begin{aligned}(\mathbb{C}{+}\mathbb{H})\hat\pi_\tau:\hat\pi_\tau - (\mathbb{C}{+}\mathbb{H})\pi_\tau(t):\pi_\tau(t) &= (\mathbb{C}{+}\mathbb{H})(\hat\pi_\tau{-}\pi_\tau(t)):(\hat\pi_\tau{+}\pi_\tau(t))\\ &= (\mathbb{C}{+}\mathbb{H})[\hat\pi{-}\pi(t)]_{\delta(\tau)}:(\hat\pi_\tau{+}\pi_\tau(t))\\ &\to (\mathbb{C}{+}\mathbb{H})(\hat\pi{-}\pi(t)):(\hat\pi{+}\pi(t)) = (\mathbb{C}{+}\mathbb{H})\hat\pi:\hat\pi - (\mathbb{C}{+}\mathbb{H})\pi(t):\pi(t)\end{aligned} \tag{4.40}$$

weakly in $L^1(\Omega)$. Similarly, we can converge the term $\frac{b}{2}|\hat\eta_\tau|^2 - \frac{b}{2}|\eta_\tau(t)|^2 = \frac{b}{2}(\hat\eta_\tau{-}\eta_\tau(t))(\hat\eta_\tau{+}\eta_\tau(t))$. The further term in the difference $\Phi_\tau(u_\tau(t),\hat z_\tau) - \Phi_\tau(u_\tau(t),z_\tau(t))$ in (4.36) admits the limit

$$\mathbb{C}e(u_\tau(t)):(\hat\pi_\tau{-}\pi_\tau(t)) = \mathbb{C}e(u_\tau(t)):[\hat\pi{-}\pi(t)]_{\delta(\tau)} \to \mathbb{C}e(u(t)):(\hat\pi{-}\pi(t)) \tag{4.41}$$

weakly in $L^1(\Omega)$, where we used (4.39). Moreover, to limit the $\mathscr{S}$-term in (4.36), by (4.6) we have $|z|_{W^{a,2}(\Omega)} = \|\mathscr{S}^{1/2}z\|_{L^2(\Omega)}$ and, by the choice of $\delta(\tau)$ in (4.37), we have also $\|\,[z]_{\delta(\tau)}\|_{W^{a,2}(\Omega)} = \mathscr{O}(\tau^{-1/4})$. This implies that

$$\begin{aligned}\big\|\mathscr{S}^{1/2}(\hat z_\tau{-}z_\tau(t))\big\|_{L^2(\Omega;\mathbb{R}^{d\times d+1})} &= \big\|\mathscr{S}^{1/2}([\hat z{-}z(t)]_{\delta(\tau)})\big\|_{L^2(\Omega;\mathbb{R}^{d\times d+1})}\\ &= \big\|\mathscr{S}^{1/2}\big\|_{\mathscr{L}(W^{a,2}(\Omega),L^2(\Omega))}\big\|[\hat z{-}z(t)]_{\delta(\tau)}\big\|_{W^{a,2}(\Omega);\mathbb{R}^{d\times d+1})} = \mathscr{O}(\tau^{-1/4})\end{aligned}$$

while

$$\big\|\mathscr{S}^{1/2}(\hat z_\tau{+}z_\tau(t))\big\|_{L^2(\Omega;\mathbb{R}^{d\times d+1})} = \big\|\mathscr{S}^{1/2}(2z_\tau(t){+}[\hat z{-}z(t)]_{\delta(\tau)})\big\|_{L^2(\Omega;\mathbb{R}^{d\times d+1})} = \mathscr{O}(\tau^{-1/2})$$

due to (4.28h,i). Thus the remaining term in the difference $\Phi_\tau(u_\tau(t),\hat z_\tau) - \Phi_\tau(u_\tau(t),z_\tau(t))$ in (4.36) can be estimated as

$$\int_\Omega \tau\mathscr{S}^{1/2}(\hat z_\tau{-}z_\tau(t)):\mathscr{S}^{1/2}(\hat z_\tau{+}z_\tau(t))\,\mathrm{d}x = \mathscr{O}(\tau^{1/4}) \to 0. \tag{4.42}$$

Still we need to pass in the ξ_1-term in (4.36) but, by (4.39), we have also $\xi_1(\hat z_\tau{-}z_\tau(t)) = \xi_1([\hat z{-}z(t)]_{\delta(\tau)}) \to \xi_1(z{-}z(t))$ certainly in $L^1(\Omega)$ (in fact even in $L^2(\Omega)$).

Here we also used that, as the kernel in the mollifier is positive, $[\cdot]_\delta$ remains in the convex set $\mathrm{dom}(\xi_1)$, hence $\xi_1(\hat{z}_\tau-z_\tau(t))<\infty$ a.e. on Ω provided $\xi_1(\hat{z}-z(t))<\infty$. Altogether, we can show (4.36) even as an equality with "lim", and thus we also proved the semistability (3.7e) instead of just mere inequality with "limsup".

The limit passage in the energy inequality (4.24) for $k=K_\tau$ to (3.7d) with "$\le$" is due to weak lower semicontinuity together with the convergence $\Phi_\tau(u_{0,\tau},z_{0,\tau})\to\Phi(u_0,z_0)$ which uses (4.8a-c).

Having already proved the semistability (3.7e), we can show the lower energy estimate (3.7d) with "$\ge$" by a Riemann-sum approximation of Lebesgue integral and thus energy equality as far as z-component concerns, i.e.

$$\Phi(u(T),z(T))+\mathrm{Var}_{\zeta_1}(z;0,T)\ge\Phi(u_0,z_0)+\int_Q\mathbb{C}(e(u)-\pi){:}e\big(\frac{\partial u}{\partial t}\big)\,\mathrm{d}x\mathrm{d}t;\tag{4.43}$$

for this rather technical argument we refer to [16], or in this "semi-stable" context rather to [24, Step 7 in the proof of Proposition 4.3]. Here it is also important that we have already proved (3.7a), from which we can also get the information $\frac{\partial^2}{\partial t^2}u\in L^2(I;W^{1,\infty}_{\Gamma_0}(\Omega;\mathbb{R}^d)^*)$ (which does not follow directly from (4.28f)), and then we can test it by $v:=\frac{\partial}{\partial t}u$ which is in duality with $\frac{\partial^2}{\partial t^2}u$ to get the energy balance (as an equality) as far as the u-component concerns. By summing it with (4.43), we thus obtain the mechanical energy balance (cf. (4.23)) with "$\ge$", i.e.

$$\begin{aligned}T_{\mathrm{kin}}\Big(\frac{\partial u}{\partial t}(T)\Big)+\Phi\big(u(T),z(T)\big)+\mathrm{Var}_{\zeta_1}(z;0,T)+\int_{\bar{Q}}\mathbb{D}e\big(\frac{\partial u}{\partial t}\big):e\big(\frac{\partial u}{\partial t}\big)\,\mathrm{d}x\\ \ge T_{\mathrm{kin}}\big(\dot{u}_0\big)+\Phi\big(u_0,z_0\big)+\int_Q\Theta(w)\mathbb{E}:\mathbb{C}e\big(\frac{\partial u}{\partial t}\big)\,\mathrm{d}x+\int_\Gamma g\cdot\frac{\partial u}{\partial t}\,\mathrm{d}S.\end{aligned}\tag{4.44}$$

Now, referring to the measure $\mathfrak{h}_z$ corresponding to $\zeta_1(\frac{\partial z}{\partial t})$ defined in (3.7c), then like in [24] we have

$$\begin{aligned}&\int_{\bar{Q}}\mathfrak{h}_z(\mathrm{d}x\mathrm{d}t)+2\int_Q\zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t=\mathrm{Var}_{\zeta_1}(z;0,T)+2\int_Q\zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le\liminf_{\tau\downarrow0}\int_Q\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+2\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t\le\limsup_{\tau\downarrow0}\int_Q\zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)+2\zeta_2\Big(e\big(\frac{\partial u_\tau}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le\limsup_{\tau\downarrow0}\bigg(\int_\Omega\frac{\varrho}{2}|\dot{u}_0|^2-\frac{\varrho}{2}\Big|\frac{\partial u_\tau}{\partial t}(T)\Big|^2\,\mathrm{d}x+\Phi_\tau(_{0,\tau},z_{0,\tau})\\
&\qquad-\Phi_\tau\big(u_\tau(T),z_\tau(T)\big)+\int_Q\Theta(\bar{w}_\tau)\mathbb{E}{:}\mathbb{C}e\big(\frac{\partial u_\tau}{\partial t}\big)\,\mathrm{d}x\mathrm{d}t-\int_\Sigma\bar{g}_\tau\cdot\frac{\partial u_\tau}{\partial t}\,\mathrm{d}S\mathrm{d}t\bigg)\\
&\quad\le\int_\Omega\frac{\varrho}{2}|\dot{u}_0|^2-\frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2\mathrm{d}x+\Phi(u_0,z_0)-\Phi\big(u(T),z(T)\big)+\int_Q\Theta(w)\mathbb{E}{:}\mathbb{C}e\big(\frac{\partial u}{\partial t}\big)\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_\Sigma g\cdot\frac{\partial u}{\partial t}\,\mathrm{d}S\mathrm{d}t\le\mathrm{Var}_{\zeta_1}(z;0,T)+2\int_Q\zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.45}$$

The inequalities in (4.45) are successively by the lower weak* semicontinuity, by general comparison "liminf$\le$limsup", by the discrete mechanical-energy inequality (4.23) for $k=K_\tau$, by the upper weak* semicontinuity and the obvious non-negativity $\Phi_\tau-\Phi\ge0$ and by the convergence

$$\Theta(\bar{w}_\tau)\mathbb{E}{:}\mathbb{C}e\big(\frac{\partial u_\tau}{\partial t}\big)\to\Theta(w)\mathbb{E}{:}\mathbb{C}e\big(\frac{\partial u}{\partial t}\big)\qquad\text{weakly in }L^1(Q)\tag{4.46}$$

and also by (4.8b) so that $\tau|z_{0,\tau}|^2_{W^{a,2}(\Omega;\mathbb{R}^{d\times d+1})}\to 0$, and finally by (4.44). Thus we have equality in the above chain of inequalities (4.45). This allows us to say that $\zeta_1(\frac{\partial z_\tau}{\partial t})\to\mathfrak{h}_z$ weakly* in measures on $\bar Q$ and $\zeta_2(\frac{\partial e(u_\tau)}{\partial t})\to\zeta_2(\frac{\partial e(u)}{\partial t})$ even strongly in $L^1(Q)$.

This allows for the limit passage in the enthalpy equation. In addition, by using (4.34), we also get rid of the regularizing β-term. More specifically, for any smooth z, we can estimate this term by using (4.28j) as

$$\Big|\int_Q\varepsilon(\tau)\big|\bar w_\tau\big|^{\beta-2}\bar w_\tau z\,\mathrm{d}x\mathrm{d}t\Big|\le\varepsilon(\tau)\big\|\bar w_\tau\big\|^{\beta-1}_{L^\beta(Q)}\big\|z\big\|_{L^\beta(Q)}\le\varepsilon(\tau)\big(C_\tau\varepsilon(\tau)^{-1/\beta}\big)^{\beta-1}\\ =C_\tau^{\beta-1}\varepsilon(\tau)^{1/\beta}\to 0. \tag{4.47}$$

Having (3.7b) already at disposal, we also obtain (3.6) and we can test (3.7b) by $v:=1$ which is obviously in duality with $\frac{\partial w}{\partial t}\in L^1(I;W^{1+d}(\Omega)^*)$, and summing it with (4.44), we obtain (3.7d) with "$\ge$". As the opposite inequality has already been discussed, altogether we proved the total energy equality (3.7d). □

Remark 4.7 The weak* topologies mentioned in Proposition 4.6 are meant, of course, in suitably extended spaces because (4.28b,c,e) involves L^1-spaces on which weak* topology is not defined at all. As to (4.28e), we consider $\mathscr{M}(\bar I;W^{1+d,2}(\Omega)^*)$ rather than $L^1(I;W^{1+d,2}(\Omega)^*)$, as used already in (3.6f). As to (4.28b,c), we enlarge $\mathscr{M}(\bar I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))$ and $\mathscr{M}(\bar I;L^1(\Omega))$ to the Borel measures $\mathscr{M}(\bar I\times\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ and $\mathscr{M}(\bar I\times\bar\Omega)$ so that the rate of plastic deformation $\frac{\partial\pi_\tau}{\partial t}$ and hardening $\frac{\partial\eta_\tau}{\partial t}$ are a-priori bounded in $C(\bar I\times\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})^*$ and $C(\bar I\times\bar\Omega)^*$, respectively. Then, after having the information that the limit is a solution, one can a-posteriori obtain the L^1-information as far as $\pi(t)$ and $\eta(t)$ concern.

Corollary 4.8 (Conditional convergence for $h\downarrow0$ and $\tau\downarrow0$) *Let $d\le3$, let the assumptions from Lemma 4.1 and (4.27) hold with $\gamma>76/17$, let (4.35) hold, and let*

$$\varepsilon(\tau)=o\big(\tau^{(\gamma+1)(\beta-1)/(\beta^2\gamma)}\big). \tag{4.48}$$

Then:

(i) *The convergence (in terms of subsequences) of the weak solutions to (4.17)–(4.18) with (4.4) towards energetic solutions according Definition 3.1 for $\tau\downarrow0$, claimed in Proposition 4.6, holds.*

(ii) *There is a function $H:\mathbb{R}^+\to\mathbb{R}^+$ such that every subsequence in the set $\big\{(u_{\tau h},\pi_{\tau h},\eta_{\tau h},w_{\tau h})\big\}_{h>0,\tau>0,h\le H(\tau)}$ of the Galerkin approximate solutions obtained by (4.10) which converges for $h\downarrow0$ and $\tau\downarrow0$ weakly* in the topologies indicated in (4.28a-f) yields, as its limit (u,π,η,w), an energetic solution according Definition 3.1.*

Proof. Note that, using Lemma 4.5, we have (4.33) which, together with (4.48), guarantees (4.34). Then Proposition 4.6 guarantees the claimed convergence.

To prove (ii), let us first note that all spaces involved in (4.28a-f) have separable preduals; here we again have in mind the extension the L^1-space occuring in (4.28b,c,e) as in Remark 4.7. In this way, we ensure all occuring weak* topologies compact and metrizable if restricted on any closed ball $B_\rho(0)$ centered at the origin 0 of the radius ρ referring to norms in (4.28a-f). We use ρ so large that all estimates (4.28a-f) yield a subset of $B_{\rho-1}(0)$; as to (4.28d), we can consider just one r which is sufficiently large (with respect to ω) that is used for interpolation which yields (4.28d), cf. again [22, 24] for details.

Then we consider the set $\mathfrak{S}_0$ of all energetic solution in accord to Definition 3.1 which lie $B_{\rho-1}(0)$. We have already proved that $\mathfrak{S}_0$ is non-empty. Similarly, for $\tau > 0$, we consider the set $\mathfrak{S}_\tau$ of the solutions $(u_\tau, \pi_\tau, \eta_\tau, w_\tau) \in B_{\rho-1}(0)$ to the problem (4.17)–(4.18) with (4.4). In Lemmas 4.1-4.2 and Proposition 4.4, we proved that the sets $\mathfrak{S}_\tau$ are nonempty for any $\tau > 0$. Then, considering again the metric generating the mentioned weak* topology on $B_\rho(0)$, we denote by $N_\epsilon(S) \subset B_\rho(0)$ a ϵ-neighbourhood of a set $S \subset B_\rho(0)$, i.e. $N_\epsilon(S) := \cup_{s\in S}\mathscr{N}_\epsilon(s) \cap B_\rho(0)$ where $\mathscr{N}_\epsilon(s)$ is an ϵ-neighbourhood of s with respect to the above mentioned metric. Note that $N_\epsilon(S)$, being a union of open sets, is always open in $B_\rho(0)$ and thus $B_\rho(0)\backslash N_\epsilon(S)$ is always compact, if nonempty.

For all $\epsilon > 0$, there is $\tau_\epsilon > 0$ such that $\mathfrak{S}_\tau \subset N_\epsilon(\mathfrak{S}_0)$ for all $0 < \tau \le \tau_\epsilon$; indeed, if ϵ is so large that $N_\epsilon(\mathfrak{S}_0) = B_\rho(0)$, there is nothing to prove since always $\mathfrak{S}_\tau \subset B_\rho(0)$, while in the opposite case, supposing the contrary, we would find a sequence in the nonempty compact set $B_\rho(0)\backslash N_\epsilon(\mathfrak{S}_0)$ and, again by arguments as in Proposition 4.6, we could show that (even all) its cluster point(s) for $\tau \to 0$ would again be the solution(s), i.e. belong to $\mathfrak{S}_0$, which is however a contradiction with being in $B_\rho(0)\backslash N_\epsilon(\mathfrak{S}_0) \subset B_\rho(0)\backslash\mathfrak{S}_0$. Beside, we can assume $\tau_\epsilon \to 0$ for $\epsilon \to 0$, e.g. $\tau_\epsilon \le \epsilon$.

Let us now denote by $\mathfrak{S}_{\tau h}$ the set of the solutions $(u_{\tau h}, \pi_{\tau h}, \eta_{\tau h}, w_{\tau h})$ whose existence has been proved in Lemma 4.1. It should be emphasized that we even cannot exclude that $\mathfrak{S}_{\tau h} \cap B_\rho(0) = \emptyset$. Anyhow, fixing $\tau > 0$, we can show that there is $H(\tau) > 0$ such that, for any $0 < h \le H(\tau)$, even $\mathfrak{S}_{\tau h} \subset N_\tau(\mathfrak{S}_\tau)$. Assume the contrary, i.e. for each $H > 0$ one can find some $0 < h_H \le H$ such that $(u_{\tau h_H}, \pi_{\tau h_H}, \eta_{\tau h_H}, w_{\tau h_H})$ lies outside $N_\tau(\mathfrak{S}_\tau)$. By Lemma 4.2, we could then take a subsequence converging for $H \to 0$ in the weak* topology indicated in (4.15) to some limit lying in $\mathfrak{S}_\tau$. As this topology is finer than the metrizable topology considered so far, this subsequence would converge in this coarser topology and eventually (i.e. for H small enough) would lie in $B_\rho(0)$ or, more precisely, in the compact set $B_\rho(0)\backslash N_\tau(\mathfrak{S}_\tau)$, which would show that this limit is simultaneously in $B_\rho(0)\backslash N_\tau(\mathfrak{S}_\tau)$ and in $\mathfrak{S}_\tau$, which is not possible.

Merging the obtained inclusions $\mathfrak{S}_{\tau h} \subset N_\tau(\mathfrak{S}_\tau)$ and $\mathfrak{S}_\tau \subset N_\epsilon(\mathfrak{S}_0)$, we can deduce $\mathfrak{S}_{\tau h} \subset N_\tau(\mathfrak{S}_\tau) \subset N_\tau(N_\epsilon(\mathfrak{S}_0)) = N_{\tau+\epsilon}(\mathfrak{S}_0) \subset N_{\tau_\epsilon+\epsilon}(\mathfrak{S}_0) \subset N_{2\epsilon}(\mathfrak{S}_0)$. Altogether, we thus have shown that for this $H(\tau)$ and for any $0 < h \le H(\tau)$, any discrete solution $(u_{\tau h}, \pi_{\tau h}, \eta_{\tau h}, w_{\tau h}) \in \mathfrak{S}_{\tau h} \subset N_{\tau_\epsilon+\epsilon}(\mathfrak{S}_0)$, and that this holds for any $\tau \le \tau_\epsilon$. As we can push $\epsilon \to 0$ (and also $\tau_\epsilon \to 0$), we verify the convergence claimed in (ii). $\square$

5 Computational implementation and 3D simulations

In our implementation we made the simplification $\mathscr{S} = 0$ and solved the variational inclusion exactly, making use of the fact that $\omega_{\tau h}^k \in \partial\delta_S^*(d_t z_{\tau h}^k)$ holds if and only if $d_t z_{\tau h}^k \in \partial\delta_S(\omega_{\tau h}^k)$, where $z_{\tau h}^k = (\pi_{\tau h}^k, \eta_{\tau h}^k)$ and $\omega_{\tau h}^k = (\widetilde{\sigma}_{\tau h}^k, \xi_{\tau h}^k) = (\mathbb{C}(e(u_{\tau h}^k) - \pi_{\tau h}^k) - \mathbb{H}\pi_{\tau h}^k, -b\eta_{\tau h}^k)$. For our numerical simulation, we neglected the kinematic hardening by putting $\mathbb{H} = 0$. We introduce $A_{\tau h}^k := d_t e(u_{\tau h}^k) - \tau^{-1}\mathbb{C}^{-1}\widetilde{\sigma}_{\tau h}^{k-1}$ and use the identity $\pi_{\tau h}^k = e(u_{\tau h}^k) - \mathbb{C}^{-1}\omega_{\tau h}^k$ to recast the flow rule as

$$\big(A_{\tau h}^k - \tau^{-1}\mathbb{C}^{-1}\widetilde{\sigma}_{\tau h}^k, d_t\eta_{\tau h}^k\big) \in \partial\delta_S(\widetilde{\sigma}_{\tau h}^k, \xi_{\tau h}^k). \tag{5.1}$$

For certain material laws and stress-strain relations it is possible to derive an explicit formula for the unique solution $\widetilde{\sigma}_{\tau h}^k$, $\eta_{\tau h}^k$ of (5.1) in terms of (given) $A_{\tau h}^k$, $\xi_{\tau h}^{k-1}$, and τ. As above, we employ the linear stress-strain relation $\widetilde{\sigma}_{\tau h}^k = \mathbb{C}\,\varepsilon_{\tau h}^k = \lambda_{\rm e}\operatorname{tr}\varepsilon_{\tau h}^k\mathbb{I} + 2\mu_{\rm e}\,\varepsilon_{\tau h}^k$ for the elastic strain tensor $\varepsilon_{\tau h}^k = e(u_{\tau h}^k) - \pi_{\tau h}^k$. We consider mere isotropic hardening defined through the von-Mises yield function $\Phi(\widetilde{\sigma}, \xi) := |\operatorname{dev} \widetilde{\sigma}| - \widetilde{\sigma}_y(1 + q_H \xi)$ and the corresponding set of admissible pairs of elastic stresses and driving forces for hardening

$$S := \big\{(\widetilde{\sigma}, \xi) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R};\ |\operatorname{dev} \widetilde{\sigma}| \le \widetilde{\sigma}_y(1 + q_H \xi)\big\}, \tag{5.2}$$

where $\widetilde{\sigma}_y$ is the yield stress, q_H the hardening parameter, and "dev" denotes the trace free part of a tensor. With these definitions we are in the setting of [10, Theorem 3.2] and may deduce that for given $A^k_{\tau h}$, $\xi^{k-1}_{\tau h}$, and $\tau > 0$ there exists a unique solution $(\widetilde{\sigma}^k_{\tau h}, \eta^k_{\tau h})$ of (5.1) given by

$$\widetilde{\sigma}^k_{\tau h} = \Sigma(A^k_{\tau h}, \xi^{k-1}_{\tau h}, \tau) := (\lambda_{\mathrm{e}} + 2\mu_{\mathrm{e}}/d)\mathrm{tr}\left(\tau A^k_{\tau h}\right)\mathbb{I} + F(A^k_{\tau h}, \xi^{k-1}_{\tau h}, \tau)\operatorname{dev}\left(\tau A^k_{\tau h}\right) \tag{5.3}$$

where

$$F(A^k_{\tau h}, \xi^{k-1}_{\tau h}, \tau) = \begin{cases} \frac{\widetilde{\sigma}_y}{(1+bq_H^2\widetilde{\sigma}_y^2)}\left(\frac{(1+q_H\xi^{k-1}_{\tau h})}{|\operatorname{dev}\left(\tau A^k_{\tau h}\right)|} + bq_H^2\widetilde{\sigma}_y\right) & \text{for } |\operatorname{dev}\left(\tau A^k_{\tau h}\right)| \ge \frac{\widetilde{\sigma}_y(1+q_H\xi^{k-1}_{\tau h})}{2\mu_{\mathrm{e}}}, \\ 2\mu & \text{for } |\operatorname{dev}\left(\tau A^k_{\tau h}\right)| \le \frac{\widetilde{\sigma}_y(1+q_H\xi^{k-1}_{\tau h})}{2\mu_{\mathrm{e}}}, \end{cases}$$

and

$$\xi^k_{\tau h} = \begin{cases} \frac{1}{q_H\widetilde{\sigma}_y}(|\operatorname{dev}\widetilde{\sigma}^k_{\tau h}| - \widetilde{\sigma}_y) & \text{for } |\operatorname{dev}\left(\tau A^k_{\tau h}\right)| \ge \frac{\widetilde{\sigma}_y(1+q_H\xi^{k-1}_{\tau h})}{2\mu_{\mathrm{e}}}, \\ \xi^{k-1}_{\tau h} & \text{for } |\operatorname{dev}\left(\tau A^k_{\tau h}\right)| < \frac{\widetilde{\sigma}_y(1+q_H\xi^{k-1}_{\tau h})}{2\mu_{\mathrm{e}}}, \end{cases}$$

and $\eta^k_{\tau h} = -b^{-1}\xi^k_{\tau h}$. In particular, the plastic phase occurs for $|\operatorname{dev}\left(\tau A^k_{\tau h}\right)| \ge \widetilde{\sigma}_y(1 + q_H\xi^{k-1}_{\tau h})/(2\mu_{\mathrm{e}})$. For explicit formulas in case of other plastic material behavior such as plasticity with linear kinematic hardening we refer the reader to [10].

In addition to the simplifications $\mathscr{S} = 0$ and $\mathbb{H} = 0$, we neglect inertial and viscous effects, kinematic hardening, and temperature dependence of the heat capacity. Thus, in the numerical experiments reported below, we consider $c_{\mathrm{v}} > 0$ constant and set $\varrho := 0$ and $\mathbb{D} := 0$. The discrete scheme (4.2a)-(4.2d) then reduces to the following coupled quasi-stationary, displacement and temperature formulation: Given $(u^{k-1}_{\tau h}, \xi^{k-1}_{\tau h}, \omega^{k-1}_{\tau h}, \theta^{k-1}_{\tau h}) \in V^d_{1,h} \times V_{0,h} \times V^{d\times d}_{0,h} \times V_{1,h}$ find $(u^k_{\tau h}, \theta^k_{\tau h}) \in V^d_{1,h} \times V_{1,h}$ such that $u^k_{\tau h}|_{\Gamma_0} = u_{\mathrm{D},\tau h_1}$ and

$$\int_\Omega \Sigma\big(A^k_{\tau h}\big[u^k_{\tau h}\big], \xi^{k-1}_{\tau h}, \tau\big) : e(v)\,\mathrm{d}x = \int_\Omega \mathbb{C}\mathbb{E}\theta^k_{\tau h} : e(v)\,\mathrm{d}x, \tag{5.4}$$

$$c_{\mathrm{v}}\big(d_t\theta^k_{\tau h}, w\big) + \int_\Omega \mathbb{K}\nabla\theta^k_{\tau h}\cdot\nabla w\,\mathrm{d}x = \int_\Omega \widetilde{\sigma}^k_{\tau h}{:}d_t\pi^k_{\tau h} w\,\mathrm{d}x - b^{-1}\int_\Omega \xi^k_{\tau h} d_t\xi^k_{\tau h} w\,\mathrm{d}x \tag{5.5}$$

for all $v \in V^d_{1,h}$ with $v|_{\Gamma_0} = 0$ and all $w \in V_{1,h}$.

The implementation of the approximation scheme was done in MATLAB in the spirit of [2, 10] and equations (5.4)-(5.5) were decoupled and solved with a fixed-point iteration. In this implementation, the nonlinear system of equations (5.4) is approximated with a Newton iteration and all occurring systems of linear equations are solved using MATLAB's backslash operator. In our experiments the Newton scheme always terminated within at most 4 iterations to achieve an ℓ^2 norm of the residual vector (defined through nodal basis functions) less than 10^{-7}J. Moreover, in all time steps, less than 6 fixed point iterations were sufficient to achieve an absolute change of the temperature in the H^1 norm less than $10^{-6}\mathrm{Km}^{1/2}$.

We used the scheme (5.4)–(5.5) to simulate the plasticization through thermal expansion of a steel cubic-shaped specimen subject to an external heating, starting from room temperature and without initial plastic strain. Focusing on this process, we neglect surface loading, i.e. $g = 0$. To demonstrate interesting rate-dependence of the whole system, we considered different speeds of the heating regime but with the same total energy pumped into the specimen. This is specified in the following example:

- *Material data*: heat capacity $c_\mathrm{v} = 3.2\mathrm{MJm^{-3}K^{-1}}$, heat transfer coefficient $\kappa = 80\mathrm{Wm^{-1}K^{-1}}$, thermal-expansion coefficient $\alpha = 2\cdot 10^{-5}\mathrm{K^{-1}}$, the Young's modulus $E = 137\mathrm{GPa}$, the Poisson ratio $\nu = 0.3$. The set of admissible stresses is defined through $\widetilde{\sigma}_y := 450\mathrm{MPa}$ and $q_H = 10^{-3}\mathrm{Pa^{-1}}$. The plastic part of the free energy is defined through the parameter $b = 10^{-3}\mathrm{Pa}$.

- *Geometry of the specimen*: $d := 3$, $\Omega := (-L/2, L/2)^3$ for $L = 2\cdot 10^{-2}\mathrm{m}$.

- *Initial conditions*: $u_0(x) = \alpha\theta_0 x$ for $x \in \Omega$, $\pi_0 := 0$, $\eta_0 := 0$, and $\theta_0 = 300\,\mathrm{K}$.

- *Heating regime*: considering $T = 1.5\mathrm{s}$ and given $t_* \in [0, T]$, we put the heat flux

$$f(t,x) := \begin{cases} t_*^{-1}\cdot 10^6\mathrm{Jm^{-2}} & \text{for } t < t_* \\ 0 & \text{for } t \geq t_* \end{cases} \tag{5.6}$$

  for $t \in [0, T]$ and $x \in \Gamma$.

Let us remark that the Lamé constants used in (2.19) are calculated, as standard, $\lambda_\mathrm{e} = \nu E/((1+\nu)(1-2\nu))$ and $\mu_\mathrm{e} = E/(2(1+\nu))$. The value of the heat capacity c_v corresponds to the capacity per mass $400\mathrm{J/kg\,K}$ if the mass density of the conventional steel $8\cdot 10^3\mathrm{kg/m^3}$ is considered. The overall energy pumped into the body $\int_0^T\int_\Gamma f\,\mathrm{d}S\mathrm{d}t = \int_0^{t_*}\int_\Gamma f\,\mathrm{d}S\mathrm{d}t = 10^6\mathrm{Jm^{-2}}\mathrm{meas}_2(\Gamma)$ is thus 2400J independently of t^*.

We simplify computationally this model problem by exploiting the symmetry of data, i.e. both of geometry and of the initial conditions as well as of the heating sources, and restricting to the subdomain $\Omega' := (0, L/2)^3$. This enforces us to implement gliding boundary conditions along the three sides $S_i := \{(x_1, x_2, x_3) \in \bar{\Omega}';\ x_i = 0\}$ with $i = 1, 2, 3$, i.e., to impose (homogeneous) Dirichlet conditions on u_i on S_i and a (homogeneous) Neumann condition on the remaining components of u as well as on w. Thus we pre-select only some symmetrical solutions of the original problem on Ω. One should realize that, due to lack of rigorous uniqueness proof, only the whole set of solutions must be symmetric and non-symmetric solutions may exist. Anyhow, this set contains also some symmetric solutions, which can be proved just by applying the previous arguments to the problem reduced on Ω'.

For a triangulation of Ω' into 2560 tetrahedra obtained from three uniform refinements of a coarse triangulation of Ω' into 5 tetrahedra (i.e. $h = 2^{-3}\sqrt{3}L/2 \approx 0.2\cdot 10^{-2}\mathrm{m}$) and used for both equations (5.4) and (5.5), we employed the time-step size $\tau = vh$ with $v = 0.05\mathrm{m\,s^{-1}}$. Figure 1 illustrates the evolution defined through $t_* = 0.075\mathrm{s}$. The heat energy is pumped through the sides of the body, which leads to higher temperatures along the sides and especially the edges and the corners. This non-uniform temperature (and thus thermal expansion) distribution enforces elastic stresses which are large along the edges of Ω and cause an expansion of the body. At $t \approx 0.035\mathrm{s}$ the stresses attain the yield stress in vicinity of the edges and trigger plastic strain evolution. In contrast with it, there is no plastic strain around the free corners (only one of which is depicted on Figures 1 and 2 due to the smaller computational domain $\Omega' \subsetneqq \Omega$, which is is due to the fact that the deformation is there locally a compression and no shear forces occur. When the external heat flux f is switched off at $t_* = 0.075\mathrm{s}$, the average temperature in Ω' no longer increases and the temperature equidistributes after some time. In contrast, the stresses cannot equidistribute and the specimen cannot entirely return towards its initial stress-free state if plasticised at some regions (here along edges) during the fast heating process. Figure 1(middle bottom snapshot) indeed shows remaining elastic stress

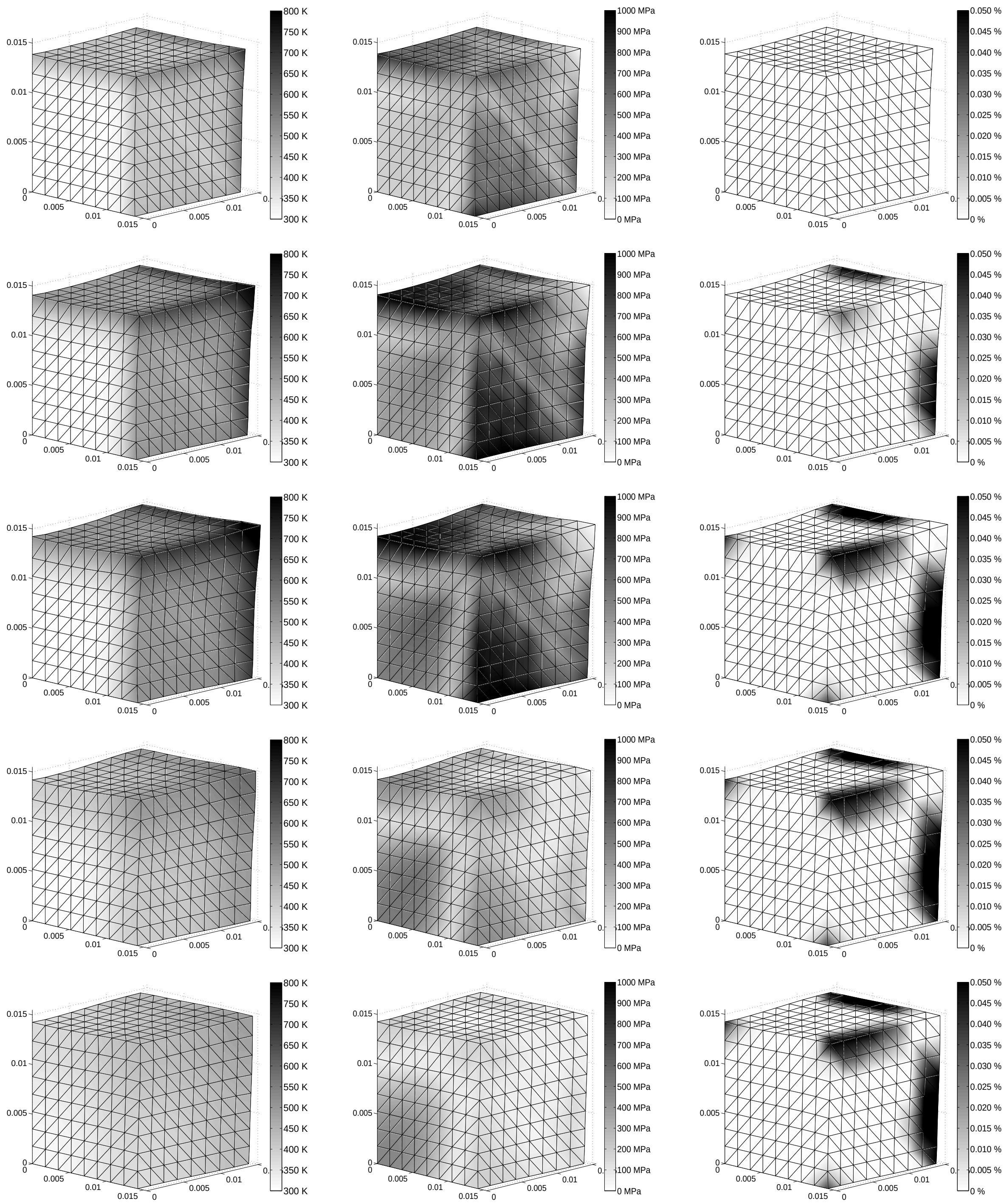

Figure 1: Displacement (magnified by factor 60) together with temperature $\theta_{\tau h}(t,\cdot)$, postprocessed modulus of stresses $|\sigma_{\tau h}(t,\cdot)|$, and postprocessed modulus of plastic strain $|\pi_{\tau h}(t,\cdot)|$ (from left to right) for $t = 0.025$, 0.05, 0.075, 0.15, 0.3s (from top to bottom) with f defined through (5.6) with $t_* = 0.075$s.

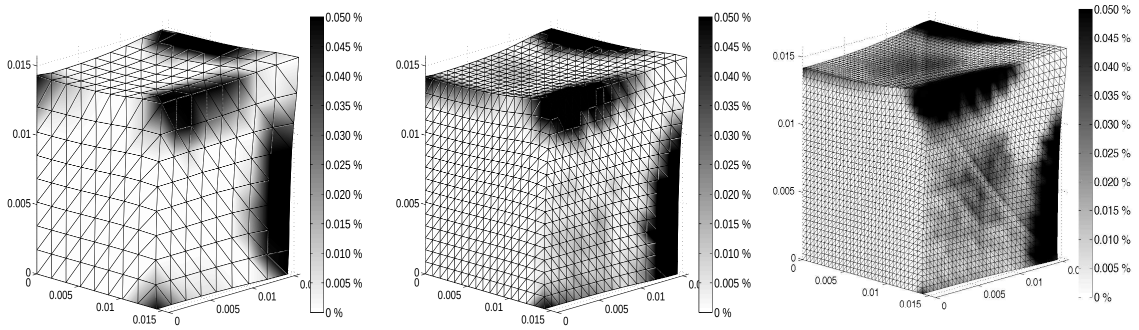

Figure 2: Postprocessed modulus of plastic strain $|\pi_{\tau h}(t,\cdot)|$ for $t = t_* = 0.075$s and the mesh sizes $h = 2^{-\ell}\sqrt{3}10^{-2}$m, $\ell = 3,4,5$ (from left to right). The finest mesh has 163 840 tetrahedral elements.

especially in the central region of the specimen.

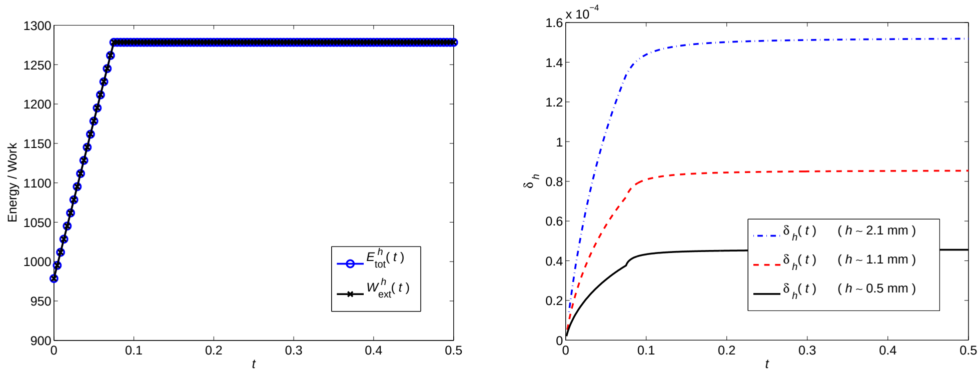

Figure 3: Total discrete energy and work of external heat (left). The relative difference between the total discrete energy and the external forces is small and converges linearly to zero as $h \to 0$ (right).

Figure 2 illustrates, in particular, that for decreasing mesh-sizes the plastic strain becomes more and more symmetric. The asymmetry on coarse meshes is expectedly due to the anisotropy of the underlying triangulation, although, due to lack of rigorous uniqueness proof, only the whole set of solutions to the limit problem (i.e. $h = 0$) must be symmetric and non-symmetric solutions may exist, and we thus even cannot claim that the concrete approximate solutions approximate any symmetric solution and exhibit some tendency for symmetry.

In Figure 3 we graphically studied the validity of a discrete energy balance analogous to the continuous one in (3.7d). The left plot of Figure 3 shows the total discrete energy $E^h_{\rm tot}$ and the work of external heat $W^h_{\rm ext}$ plus initial energy $E^h_{\rm tot}(0)$, defined by

$$E^h_{\rm tot}(t_k) := \int_\Omega c_{\rm v}\theta^k_{\tau h} + \frac{1}{2}\mathbb{C}^{-1}\widetilde{\sigma}^k_{\tau h} : \sigma^k_{\tau h} + b|\xi^k_{\tau h}|^2\,{\rm d}x, \quad W^h_{\rm ext}(t_k) := \sum_{\ell=0}^{k}\tau\int_\Gamma g\,{\rm d}s.$$

The two quantities almost coincide for all $t \in [0,T]$ and the right plot of Figure 3 shows their relative distance δ_h defined through

$$\delta_h(t_k) := \frac{\big|E^h_{\rm tot}(t_k) - E^h_{\rm tot}(t_0) - W^h_{\rm ext}(t_k)\big|}{\big|E^h_{\rm tot}(0) + W^h_{\rm ext}(t_k)\big|}$$

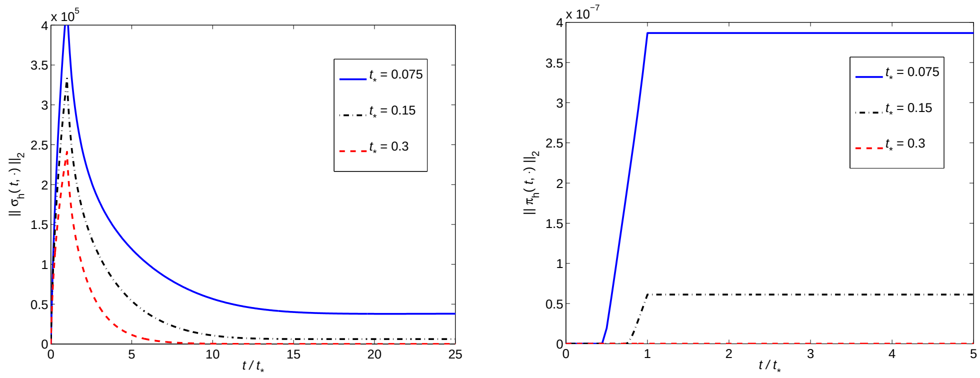

Figure 4: L^2 norms of discrete stresses (left) and plastic strains (right) as functions of the relative time t/t_* for heating times $t_* = 0.075, 0.15, 0.3$s. The plastic material behaviour becomes less pronounced as the external heating happens slower.

for $h \doteq 2^{-\ell}\sqrt{3}10^{-2}$m, $\ell = 3, 4, 5$. We observe that the relative difference is small and decays linearly to zero as the mesh-size becomes small. The increase of the quantities by approximately 300J corresponds to an eighth of the total energy pumped into the entire specimen Ω.

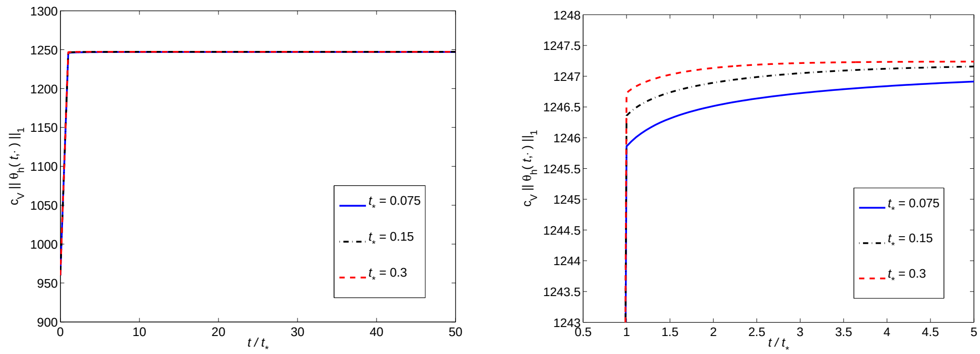

Figure 5: Thermal part of the energy for different speed of the heating regimes given by (5.6) for $t_* = 0.075, 0.15, 0.3$s. The detailed picture (right) shows that final temperature is slightly lower if the material was more plasticized during the heating.

In Figure 4 we displayed for different values of t_* the L^2 norms of the stresses $\widetilde{\sigma}^k_{\tau h} - \mathbb{C}\mathbb{E}P_{h,0}\theta^k_{\tau h}$ and plastic strains $\pi^k_{\tau h}$ as functions of t/t_*. The L^2 norm of the stresses increases within the relative-time interval $[0, 1]$. For small values of t_*, i.e., for a faster heating of the specimen, the material is plasticised in large domains. Since for slow heating of the specimen, the temperature rather equidistributes and does not lead to large elastic stresses so that no plasticity occurs at all, if t_* is 0.3 s or bigger. Finally, in Figure 5 we plotted the thermal part of the energy, i.e., the quantity $c_\mathrm{v} \int_\Omega \theta^k_{\tau h}\,\mathrm{d}x$ (i.e., up to a factor $c_\mathrm{v}|\Omega| = 25.6$J/K, the average temperature) as a function of t/t_*. We see that the achieved average temperature is slightly lower for more pronounced plastic process, i.e. for faster heating (=a shorter time t_*) because bigger part of the heat energy pumped into the body is converted into remaining plastic changes of the material and to the elastic stored energy due to the mentioned remaining elastic stress. This effect is, however, relatively very small (cf. Fig. 5–left) because the energetics of mechanical processes is “cheaper” than the thermal energetics, and can only be made visible on some detailed zoom (cf. Fig. 5–right).

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