NUMERICAL APPROACHES TO THERMALLY COUPLED PERFECT PLASTICITY

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Abstract. The partial differential equations describing viscoelastic solids in Kelvin-Voigt rheology at small strains exhibiting also stress-driven Prandtl-Reuss perfect plasticity are considered and are coupled with a heat-transfer equation through the dissipative heat produced by viscoplastic effects and through thermal expansion and corresponding adiabatic effects. Numerical discretization of the resulting thermodynamically consistent model is proposed by implicit time discretization, suitable regularization, and finite elements in space. Numerical stability is shown and computational simulations are reported to illustrate the practical performance of the method. In a quasistatic case, convergence is proved by careful successive limit passage.

Keywords: Thermodynamics, Prandtl-Reuss plasticity, Kelvin-Voigt rheology, thermal expansion, adiabatic effects, bounded deformations, finite elements, convergence.

AMS classification: 35K85, 49S05, 65M60, 74C05, 80A17.

1. Introduction

Plasticity is an important *inelastic dissipative* phenomenon in *solid mechanics*. The dissipation of the mechanical energy by plastification may lead to an increase of stored energy by permanent structural changes or, and typically, to heat production. In some applications, heat produced by this way is not transferred away sufficiently fast and leads to temperature variations, which influence in reverse the plastic processes and makes the problem thermally coupled.

Often, plasticity is accompanied by hardening effects, which makes the analysis simpler. Yet, in some materials as (some) metals or rocks, these hardening effects are negligible. This typically leads to the occurrence of shear bands (called also slip bands or, in rock mechanics, faults). We then speak about *perfect plasticity*. It introduces serious mathematical difficulties related to the localization of mechanical and thermal processes to these shear bands. This phenomenon does not seem rigorously treatable by existing mathematical methods at large strains, however. Therefore, we confine ourselves to *small strains* where the so-called *bounded-deformation spaces*, invented by P.-M. Suquet [32], can advantageously be exploited. Perfect plasticity has mathematically been studied in the isothermal case e.g. in [2, 11, 12, 33].

The key feature in the modelling of thermodynamics of perfect plasticity, while not destroying the characteristic phenomenon of possible localization of thermomechanical process to shear bands, is to involve stress viscosity and also stress-driven perfect plasticity rather than strain viscosity and strain-driven plasticity, as recently proposed in [30].

It is needless to emphasize that the problem of thermodynamics of perfect plasticity combines a lot of difficult phenomena and this is why various simplifications

Date: November 23, 2011.

must be made. Here, besides small strains, we confine ourselves to *linearized additive plasticity* and a *linear viscoelastic* response. On the other hand, we allow for a fully *rate-independent plastic* flow rule although, of course, the whole system is necessarily rate dependent due to the heat transfer, viscous effects, and possibly also inertial effects (if considered).

The goal of this article is to propose an implementable numerical scheme for the above problem that is unconditionally stable in an appropriate sense (and, in particular, a discrete solution exists) and, under certain specific conditions, is even convergent.

The paper is organized as follows: The equations (or rather also inclusions) and the particular initial/boundary-value problem is formulated in Section 2 together with a discussion of its energetics and thermodynamics. The problem is then slightly transformed by re-scaling temperature (=a so-called enthalpy transformation) and by shifting the Dirichlet boundary conditions to make them temporarily constant. The discretization by using a fully-implicit scheme in time, finite elements in space, regularization, and projection of the transformed discrete temperature to a non-negative cone is proposed in Section 3 where also the existence of approximate solutions and certain unconditional a-priori estimates are shown. Section 4 demonstrates an implementation of this scheme and shows illustrative examples. A convergence analysis is then outlined in Section 5 for special situations. An important issue is to prove positivity of the rescaled temperature to avoid the aforementioned projection in the limit.

2. The model within thermodynamics

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$. The state variables will be the *displacement* $u : \Omega \to \mathbb{R}^d$, the *plastic strain* $\pi : \Omega \to \mathbb{R}^{d\times d}_{\rm dev}$, and the *temperature* $\theta : \Omega \to \mathbb{R}$, where

$$\mathbb{R}^{d\times d}_{\rm dev} := \big\{A\in\mathbb{R}^{d\times d}_{\rm sym};\ {\rm tr}(A) = 0\big\} \quad \text{and} \quad \mathbb{R}^{d\times d}_{\rm sym} := \big\{A\in\mathbb{R}^{d\times d};\ A^\top = A\big\}. \tag{2.1}$$

The variable π plays the role of a matrix-valued *internal parameter*. Considering a Kelvin-Voigt-type viscous material, our model will consist of the *equilibrium equation* balancing viscous and elastic stresses,

$$\varrho\ddot{u} = {\rm div}\,\sigma, \qquad \sigma = \sigma_{\rm vi} + \sigma_{\rm el}, \qquad \sigma_{\rm vi} = \mathbb{D}\dot{\varepsilon}, \qquad \sigma_{\rm el} = \mathbb{C}\varepsilon_0, \tag{2.2a}$$

$$\varepsilon+\pi = e(u) := \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u, \qquad \varepsilon = \varepsilon_0+\mathfrak{e}, \qquad \mathfrak{e} = \mathbb{E}\theta, \tag{2.2b}$$

where the dot denotes the time derivative, ϱ is the mass density, $\mathbb{D}$ the tensor determining the viscous-type response, $\mathbb{C}$ the tensor determining the elastic response, and $\mathbb{E}$ the thermal expansion tensor, i.e. the stress σ is

$$\sigma = \mathbb{D}(e(\dot{u}) - \dot{\pi}) + \mathbb{C}(e(u) - \pi) - \mathbb{B}\theta, \quad \text{where } \mathbb{B} = \mathbb{C}\mathbb{E}. \tag{2.3}$$

We consider plastic response determined by a convex closed neighbourhood of the origin, say $S \subset \mathbb{R}^{d\times d}_{\rm dev}$, defining an *elasticity domain*, while its boundary is called the *yield surface* and has the meaning of the stress that triggers the evolution of plastic strains. Let δ_S denote its indicator function and δ_S^* the Fenchel-Legendre conjugate functional to δ_S with respect to the inner product $\sigma{:}e = \sum_{i,j=1}^d \sigma_{ij}e_{ij}$. Note that the physical dimension of $\sigma{:}e$ is Pa=J/m^3 so that S determining the degree-1 positively homogeneous “plastic” dissipation potential δ_S^*, acting on the dimensionless tensor π, has indeed the dimension J/m^3. Usually, S is considered to be bounded, which implies that δ_S^* is finite. We remark that the condition $0 \in {\rm int}(S)$ implies that

$\delta_S^* : \mathbb{R}^{d\times d}_{\rm dev} \to \mathbb{R}$ is coercive. The *evolution of the internal parameters*, i.e. here the plastic strain π, is then governed by the inclusion

$$\partial\delta_S^*\big(\dot\pi\big) \ni \mathrm{dev}\,\sigma, \tag{2.4}$$

where $\mathrm{dev}\,\sigma := \sigma - \sigma^{\rm s}$ is the deviatoric part of σ with $\sigma^{\rm s} = \frac{1}{d}(\mathrm{tr}\,\sigma)\mathbb{I}$ the spherical part of σ. The *heat transfer/production* is governed by the equation

$$c_{\rm v}(\theta)\dot\theta - \mathrm{div}\big(\mathbb{K}(\theta)\nabla\theta\big) = \delta_S^*(\dot\pi) + \mathbb{D}\dot\varepsilon{:}\dot\varepsilon + \theta\mathbb{B}\dot\varepsilon \quad \text{ with } \dot\varepsilon = \mathbb{C}\big(e(\dot u) - \dot\pi\big), \tag{2.5}$$

where $c_{\rm v} = c_{\rm v}(\theta)$ is the *heat capacity*, and $\mathbb{K} = \mathbb{K}(\theta)$ is the *thermal conductivity* tensor. The above equations/inclusion (2.2)–(2.5) are to hold on the space/time domain $Q := (0,T)\times\Omega$ with $T > 0$ a fixed time horizon. Using the identity $[\partial\delta_S^*]^{-1} = \partial([\delta_S^*]^*) = \partial\delta_S^{**} = \partial\delta_S$, the inclusion (2.4) can equivalently be written in a form which is more standard in the engineering literature, namely

$$\dot\pi \in N_S(\mathrm{dev}\,\sigma) \tag{2.6}$$

where $N_S = \partial\delta_S$ is the normal cone to S.

As we focus on processes in the bulk, we consider only the simplest boundary conditions, namely a prescribed normal stress on $\Gamma_{\rm Neu}$ and a heat flux on $\Gamma := \partial\Omega$:

$$u = u_{\rm Dir} \qquad \text{on } \Gamma_{\rm Dir}, \tag{2.7a}$$

$$\sigma\nu = 0 \qquad \text{on } \Gamma_{\rm Neu}, \tag{2.7b}$$

$$\big(\mathbb{K}(\theta)\nabla\theta\big)\cdot\nu = f \qquad \text{on } \Gamma, \tag{2.7c}$$

where "$\cdot$" denotes the scalar product of two vectors and ν is the outward normal to Γ.

Throughout this paper, we assume purely isotropical thermal expansion, i.e. $\mathbb{B}$ is purely spherical tensor, or equivalently

$$\mathrm{dev}\,\mathbb{B} = 0. \tag{2.8}$$

Together with $\pi^{\rm s} = 0$, this ensures the orthogonality $\pi{:}\mathbb{B}\theta = 0$, which is essential for our analysis because of the lack of sufficient a-priori estimates on the temperature to deduce compactness in an L^∞-space. Due to this orthogonality, the driving force σ in the flow rule (2.4) can effectively be replaced by $\sigma_{\rm vi}{+}\sigma_{\rm el}$ with exactly the same effect.

The energetics of the model can be obtained by testing (2.2) and (2.4) respectively by the "shifted" velocity $\dot u{-}\dot u_{\rm Dir}$ (which has zero traces on $\Gamma_{\rm Dir}$) and by the plastic strain rate $\dot\pi$, which gives after an application of Green's formula in (2.2) together with the boundary conditions (2.7) and eventually by summation the *mechanical energy balance*

$$\int_\Omega \delta_S^*(\dot\pi) + \mathbb{D}\dot\varepsilon{:}\dot\varepsilon + \frac{1}{2}\frac{\partial}{\partial t}\Big(\varrho|\dot u|^2 + \mathbb{C}\varepsilon{:}\varepsilon\Big)\,\mathrm{d}x$$
$$= \int_\Omega \mathbb{B}\theta{:}\big(\dot\varepsilon{-}e(\dot u_{\rm Dir})\big) + \big(\mathbb{D}\dot\varepsilon{+}\mathbb{C}\varepsilon\big){:}e(\dot u_{\rm Dir}) - \varrho\,\ddot u\cdot\dot u_{\rm Dir}\,\mathrm{d}x. \tag{2.9}$$

Testing further (2.5) by 1 and using again Green's formula gives, when summing with (2.9), the *total energy balance*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\underbrace{\int_\Omega \frac{\varrho}{2}|\dot u|^2\,\mathrm{d}x}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{\int_\Omega \frac{1}{2}\mathbb{C}\varepsilon{:}\varepsilon + C_{\rm v}(\theta)\,\mathrm{d}x}_{\substack{\text{stored and heat parts}\\ \text{of the internal energy}}}\bigg) = \underbrace{\int_\Omega \sigma{:}e(\dot u_{\rm Dir}) - \varrho\,\ddot u\cdot\dot u_{\rm Dir}\,\mathrm{d}x}_{\substack{\text{power of the external}\\ \text{mechanical loading}}} + \underbrace{\int_\Gamma f\,\mathrm{d}S}_{\substack{\text{power of the}\\ \text{external heating}}} \tag{2.10}$$

cf. also (5.21c) below written for the shifted displacement u. In (2.10), we denoted

$$C_{\rm v}(\theta) := \int_0^\theta c_{\rm v}(r)\,{\rm d}r\,; \tag{2.11}$$

thus $C_{\rm v}$ is a primitive function to $c_{\rm v}$ normalized such that $C_{\rm v}(0) = 0$. In fact, in (2.10) we also assumed ${\rm div}\, u_{\rm Dir} = 0$ which we will later use to facilitate certain a-priori estimates, cf. (3.11f), and which leads to $\mathbb{B}\theta{:}e(\dot u_{\rm Dir}) = 0$ due to (2.8).

To simplify the treatment of the nonlinear term $c_{\rm v}(\theta)\dot\theta$, we introduce a new variable w substituting the temperature, namely

$$w = C_{\rm v}(\theta). \tag{2.12}$$

The use of this re-scaled temperature is called the *enthalpy transformation*. Further, we define

$$\mathscr{K}(w) := \frac{\mathbb{K}(\Theta(w))}{c_{\rm v}(\Theta(w))} \quad \text{and} \quad \mathscr{B}(w) := \Theta(w)\mathbb{B} \quad \text{with} \quad \Theta(w) := \begin{cases} C_{\rm v}^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \tag{2.13}$$

where $C_{\rm v}^{-1}$ here denotes the inverse function to $C_{\rm v}$. We incorporate the Dirichlet condition by an additive shift, i.e. instead of the original u, we consider $u{+}u_{\rm Dir}$ with a suitable prolongation of $u_{\rm Dir}$ inside the domain; in particular, without any loss of generality, we will assume that

$$\big(\mathbb{D}e(\dot u_{\rm Dir}) + \mathbb{C}e(u_{\rm Dir})\big)\nu = 0 \qquad \text{on } \Gamma_{\rm Neu}, \tag{2.14}$$

which will simplify some formulae below. In terms of this shifted displacement, denoted again by u, and the re-scaled temperature w, the system (2.2)–(2.4)–(2.5) writes as

$$\begin{aligned}
&\varrho\ddot u = {\rm div}\,\sigma + f_{\rm Dir}, && \sigma = \mathbb{D}\dot\varepsilon + \mathbb{C}\varepsilon - \mathscr{B}(w), && (2.15a)\\
& && \varepsilon = e(u){-}\pi, && (2.15b)\\
& && f_{\rm Dir} = {\rm div}\,\sigma_{\rm Dir} - \varrho\ddot u_{\rm Dir}, && (2.15c)\\
&\partial\delta_S^*(\dot\pi) \ni {\rm dev}(\sigma + \sigma_{\rm Dir}), && \sigma_{\rm Dir} = \mathbb{D}e(\dot u_{\rm Dir}) + \mathbb{C}e(u_{\rm Dir}), && (2.15d)\\
&\dot w - {\rm div}\big(\mathscr{K}(w)\nabla w\big) = \delta_S^*(\dot\pi) + \big(\mathbb{D}\dot\varepsilon{+}\mathbb{D}e(\dot u_{\rm Dir}){+}\mathscr{B}(w)\big){:}(\dot\varepsilon{+}\mathbb{D}e(\dot u_{\rm Dir})). && && (2.15e)
\end{aligned}$$

We will call (2.15e) shortly the *enthalpy equation* rather than the heat-transfer equation in the enthalpy formulation. The boundary conditions (2.7) transform to

$$\begin{aligned}
&u = 0 && \text{on } \Gamma_{\rm Dir}, && (2.16a)\\
&\sigma\nu = \sigma_{\rm Dir}\nu && \text{on } \Gamma_{\rm Neu}, && (2.16b)\\
&\big(\mathscr{K}(w)\nabla w\big)\cdot\nu = f && \text{on } \Gamma. && (2.16c)
\end{aligned}$$

We complete this transformed system with the initial conditions

$$u(0,\cdot) = u_0, \qquad \dot u(0,\cdot) = \dot u_0, \qquad \pi(0,\cdot) = \pi_0, \qquad w(0,\cdot) = w_0 \qquad \text{on } \Omega. \tag{2.17}$$

Of course, the last condition means in fact that the initial temperature is prescribed as θ_0 and then $w_0 = C_{\rm v}(\theta_0)$.

Remark 2.1. Later, we will assume a non-cooling regime $f \geq 0$ in (2.7c). However, the considerations can be routinely generalized for a more general Robin-type condition $\big(\mathbb{K}(\theta)\nabla\theta\big)\cdot\nu = f - a\theta$ with $a \geq 0$, which effectively allows also for cooling while preserving positivity of temperature.

Remark 2.2. The *thermodynamics* of the original model (2.2)–(2.5) can be derived by postulating the Helmholtz *free energy* ψ_0 as $\psi_0(\varepsilon_{\rm el},\theta) = \frac{1}{2}\mathbb{C}\varepsilon_{\rm el}{:}\varepsilon_{\rm el} - \phi(\theta)$ with $\varepsilon_{\rm el} = \varepsilon_{\rm el}(\varepsilon,\theta) = \varepsilon - \theta\mathbb{E}$. By substituting for $\varepsilon_{\rm el}$, we also denote

$$\psi(\varepsilon,\theta) = \psi_0(\varepsilon_{\rm el}(\varepsilon,\theta),\theta) = \frac{1}{2}\mathbb{C}(\varepsilon{-}\theta\mathbb{E}){:}(\varepsilon{-}\theta\mathbb{E}) - \phi(\theta). \tag{2.18}$$

Then, *entropy* is given by

$$s = s(\varepsilon,\theta) := -\psi_\theta'(\varepsilon,\theta) = \phi'(\theta) - \mathbb{E}\mathbb{B}\theta + \mathbb{B}\varepsilon. \tag{2.19}$$

The so-called *entropy equation* reads

$$\theta\dot{s} = \xi(\dot{\pi},\dot{\varepsilon}) + {\rm div}\,\jmath \tag{2.20}$$

with the heat flux governed by the Fourier law $\jmath = \mathbb{K}(\theta)\nabla\theta$ and with the mechanical dissipation rate $\xi(\dot{\pi},\dot{\varepsilon}) = \delta_S^*(\dot{\pi}) + \mathbb{D}\dot{\varepsilon}{:}\dot{\varepsilon}$. This then takes the form of the *heat-transfer equation*

$$\theta\psi_{\theta\theta}''(\theta)\dot{\theta} - {\rm div}\big(\mathbb{K}(\theta)\nabla\theta\big) = \xi - \theta\psi_{\theta\varepsilon}''(\varepsilon,\theta){:}\dot{\varepsilon}. \tag{2.21}$$

Combining (2.18) with $c_{\rm v} = c_{\rm v}(\varepsilon,\theta) := \theta\psi_\theta''(\varepsilon,\theta) = \theta\phi''(\theta) - \mathbb{E}\mathbb{B}\theta$, we arrive at (2.5). Assuming positivity of temperature (as indeed proved later in Section 5), $f \geq 0$, positive-definiteness of $\mathbb{K}$, and realizing that always $\xi(\dot{\pi},\dot{\varepsilon}) \geq 0$, from (2.20) we can formally deduce the *Clausius-Duhem inequality*

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\int_\Omega s\,{\rm d}x &= \int_\Omega \frac{\xi(\dot{\pi},\dot{\varepsilon}) + {\rm div}\,\jmath}{\theta}\,{\rm d}x = \int_\Omega \frac{\xi(\dot{\pi},\dot{\varepsilon})}{\theta} + {\rm div}\Big(\mathbb{K}\frac{\nabla\theta}{\theta}\Big) + \frac{\mathbb{K}\nabla\theta{\cdot}\nabla\theta}{\theta^2}\,{\rm d}x\\ &= \int_\Omega \frac{\xi(\dot{\pi},\dot{\varepsilon})}{\theta} + \frac{\mathbb{K}\nabla\theta{\cdot}\nabla\theta}{\theta^2}\,{\rm d}x + \int_\Gamma \frac{f}{\theta}\,{\rm d}S \,\geq\, 0,\end{aligned} \tag{2.22}$$

i.e. the system complies with the 2nd law of thermodynamics. Using (2.8), (2.18), and (2.19), the *internal energy* standardly given by $\psi{+}\theta s$ results to

$$\begin{aligned}\psi + \theta s &= \frac{1}{2}\mathbb{C}(\varepsilon{-}\theta\mathbb{E}){:}(\varepsilon{-}\theta\mathbb{E}) - \phi(\theta) + \theta\big(\phi'(\theta) - \mathbb{E}\mathbb{B}\theta + \mathbb{B}\varepsilon\big)\\ &= \frac{1}{2}\mathbb{C}\varepsilon{:}\varepsilon - \phi(\theta) + \theta\phi'(\theta) - \frac{1}{2}\mathbb{E}\mathbb{B}\theta^2 = \frac{1}{2}\mathbb{C}\varepsilon{:}\varepsilon + C_{\rm v}(\theta) - \phi(0),\end{aligned} \tag{2.23}$$

which is, up to the constant $\phi(0)$, the quantity occurring in (2.10).

3. Discretization of the system (2.15)–(2.17)

It is not entirely easy to design numerically stable and convergent discrete schemes in coupled systems with super-linear growth of nonmonotone terms, as it is typically the case of thermodynamically consistent continuum-mechanical problems. Even mere existence of discrete solutions is a rather fine issue, requiring careful regularizations of the discrete scheme. Also, it seems difficult to devise a spatial discretization of the term $-{\rm div}(\mathscr{K}(w)\nabla w)$ that is compatible with the maximum principle even on acute triangulations if $\mathscr{K}$ is nonconstant (as it is quite typical if $c_{\rm v}$ is nonconstant); here we overcome this issue by treating the discrete enthalpy equation rather as an inequality and later prove positivity of the limit enthalpy.

Convergence needs further various fine ingredients, in particular an estimate of the enthalpy gradient in situations that heat sources have only L^1-structure. This unfortunately requires special nonlinear tests and does not seem transferable to the spatially discrete case. Thus the convergence can only be expected to be conditional, as indeed presented in Section 5.

To cope with the above outlined delicate issues, we follow and modify ideas from [4, 29, 30]. We will use a *fully implicit time-discretization* with a constant time-step $\tau > 0$, assuming $K_\tau = T/\tau \in \mathbb{N}$ and defining the *backward difference operator* by

$$\mathrm{D}_t\phi^k := \frac{\phi^k - \phi^{k-1}}{\tau} \tag{3.1}$$

for any sequence $\{\phi^k\}_{k\geq 0}$, combined with a *regularization* of the momentum equation and of the flow rule (using as a regularization parameter a function of the time-step size $\eta(\tau) > 0$). More specifically, we consider the following recursive incremental formula

$$\varrho\mathrm{D}_t^2 u_\tau^k - \mathrm{div}\big(\sigma_\tau^k + \tau\big|\mathrm{D}_t\varepsilon_\tau^k\big|^{\gamma-2}\mathrm{D}_t\varepsilon_\tau^k\big) = \mathrm{div}\,\sigma_{\mathrm{Dir},\tau}^k - \varrho\mathrm{D}_t^2 u_{\mathrm{Dir},\tau}^k, \tag{3.2a}$$

$$\partial\delta_S^*\big(\mathrm{D}_t\pi_\tau^k\big) + \eta(\tau)\pi_\tau^k \ni \mathrm{dev}\big(\sigma_\tau^k + \sigma_{\mathrm{Dir},\tau}^k + \tau\big|\mathrm{D}_t\varepsilon_\tau^k\big|^{\gamma-2}\mathrm{D}_t\varepsilon_\tau^k\big), \tag{3.2b}$$

$$\begin{aligned}\mathrm{D}_t w_\tau^k - \mathrm{div}\big(\mathscr{K}(w_\tau^k)\nabla w_\tau^k\big) &= \delta_S^*\big(\mathrm{D}_t\pi_\tau^k\big)\\ &\quad + \big(\mathbb{D}\mathrm{D}_t\varepsilon_\tau^k + \mathbb{D}e(\mathrm{D}_t u_{\mathrm{Dir},\tau}^k) + \mathscr{B}(w_\tau^k)\big){:}\mathrm{D}_t\big(\varepsilon_\tau^k + e(u_{\mathrm{Dir},\tau}^k)\big),\end{aligned} \tag{3.2c}$$

$$\text{with}\ \ \varepsilon_\tau^k = e(u_\tau^k) - \pi_\tau^k, \qquad \sigma_\tau^k = \mathbb{D}\mathrm{D}_t\varepsilon_\tau^k + \mathbb{C}\varepsilon_\tau^k - \mathscr{B}(w_\tau^k), \tag{3.2d}$$

for $k = 1, ..., K_\tau = T/\tau$, where

$$u_{\mathrm{Dir},\tau}^k(t,x) := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} u_{\mathrm{Dir}}(t,x)\,\mathrm{d}t \quad \text{and} \quad \sigma_{\mathrm{Dir},\tau}^k = \mathbb{D}e(\mathrm{D}_t u_{\mathrm{Dir},\tau}^k) + \mathbb{C}e(u_{\mathrm{Dir},\tau}^k), \tag{3.3}$$

and where $\gamma > 1$ and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $\eta(0) = 0$, with the corresponding boundary conditions

$$u_\tau^k = 0 \qquad \text{on } \Gamma_{\mathrm{Dir}}, \tag{3.4a}$$

$$\big(\sigma_\tau^k + \tau\big|e(u_\tau^k)\big|^{\gamma-2}e(u_\tau^k)\big)\nu = 0 \qquad \text{on } \Gamma_{\mathrm{Neu}}, \tag{3.4b}$$

$$\big(\mathscr{K}(w_\tau^k)\nabla w_\tau^k\big)\cdot\nu = f_\tau^k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}\tilde{f}_\tau(t,x)\,\mathrm{d}t \qquad \text{on } \Gamma, \tag{3.4c}$$

starting for $k = 1$ by using

$$u_\tau^0 = u_{0,\tau}, \qquad u_\tau^{-1} = u_{0,\tau} - \tau\dot{u}_0, \qquad \pi_\tau^0 = \pi_{0,\tau}, \qquad w_\tau^0 = w_{0,\tau}. \tag{3.5}$$

Note that, in (3.5) and (3.4c), we regularized the initial values u_0, π_0, and w_0 and the boundary flux f by $u_{0,\tau}$, $\pi_{0,\tau}$, $w_{0,\tau}$, and $\tilde{f}_\tau$, respectively, cf. (3.11) below.

Let us comment on the purpose of the regularizing terms. The "γ-terms" in (3.2a) and (3.2b) are needed to compensate the superlinear growth of the right-hand-side terms in the heat equation; it has already been used in [29] for a mere time discretization. The term $\eta(\tau)\pi_\tau^k$ in (3.2b) helps to carry out a limit passage in space discretization, cf. (5.11a) below, and it seems also to have some positive impact on the calculations, cf. Section 4.3 below. This term is in a position of a *kinematic hardening* but with a *vanishing* coefficient $\eta(\tau)$ like in [2]. Here, this vanishing hardening is controlled directly by the time step τ.

We will further make a *spatial discretization.* For this, we assume that we are given a sequence of triangulations $\{\mathscr{T}_h\}_{h>0}$ of the polyhedral domain Ω without hanging nodes but otherwise entirely general. We suppose that the maximal diameters $h > 0$ range over a countable set of positive real numbers with accumulation point at 0, and that $\max_{E\in\mathscr{T}_h}\mathrm{diam}(E) \leq h$.

We consider C^0-conforming P1-elements for the approximation of u and w and P0-elements for the approximation of π. The finite-dimensional subspaces of $L^2(\Omega)$

and $W^{1,2}(\Omega)$ related to P0- and P1-elements and subordinate to the triangulation $\mathscr{T}_h$ respectively are denoted by $V_{0,h}$ and $V_{1,h}$.

Then we devise the Galerkin scheme as follows. We seek $(u_{\tau h}^k, \pi_{\tau h}^k, w_{\tau h}^k) \in V_{1,h}^d \times V_{0,h}^{d\times d} \times V_{1,h}$, with $\pi_{\tau h}^k(\cdot) \in \mathbb{R}_{\rm dev}^{d\times d}$ and $w_{\tau h}^k(\cdot) \geq 0$ a.e. on Ω, satisfying

$$
\begin{aligned}
&\int_\Omega \varrho \mathrm{D}_t^2\big(u_{\tau h}^k + u_{\mathrm{Dir},\tau}^k\big)\cdot v + \big(\sigma_{\tau h}^k + \sigma_{\mathrm{Dir},\tau}^k + \tau\big|\mathrm{D}_t\varepsilon_{\tau h}^k\big|^{\gamma-2}\mathrm{D}_t\varepsilon_{\tau h}^k\big){:}e(v)\,\mathrm{d}x = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for all } v \in V_{1,h}^d, \qquad\qquad\qquad\qquad (3.6a)\\
&\int_\Omega \delta_S^*(\tilde\pi) + \big(\sigma_{\tau h}^k + \sigma_{\mathrm{Dir},\tau}^k + \tau\big|\mathrm{D}_t\varepsilon_{\tau h}^k\big|^{\gamma-2}\mathrm{D}_t\varepsilon_{\tau h}^k\big){:}(\tilde\pi - \mathrm{D}_t\pi_{\tau h}^k) + \eta(\tau)\pi_{\tau h}^k{:}(\tilde\pi - \mathrm{D}_t\pi_{\tau h}^k)\,\mathrm{d}x \\
&\qquad \geq \int_\Omega \delta_S^*(\mathrm{D}_t\pi_{\tau h}^k)\,\mathrm{d}x \qquad\qquad \text{for all } \tilde\pi \in V_{1,h}^{d\times d},\ \tilde\pi(\cdot) \in \mathbb{R}_{\rm dev}^{d\times d} \text{ a.e. on } \Omega, \quad (3.6b)\\
&\int_\Omega \mathrm{D}_t w_{\tau h}^k(v - w_{\tau h}^k) + \mathscr{K}(w_{\tau h}^k)\nabla w_{\tau h}^k\cdot\nabla(v - w_{\tau h}^k) \\
&\qquad \geq \int_\Omega \Big(\big(\mathbb{D}\mathrm{D}_t\varepsilon_{\tau h}^k + \mathbb{D}e(\mathrm{D}_t u_{\mathrm{Dir},\tau}^k) + \mathscr{B}(w_{\tau h}^k)\big){:}\mathrm{D}_t\big(\varepsilon_{\tau h}^k + e(u_{\mathrm{Dir},\tau}^k)\big) \\
&\qquad + \delta_S^*\big(\mathrm{D}_t\pi_{\tau h}^k\big)\Big)(v - w_{\tau h}^k)\,\mathrm{d}x + \int_\Gamma f_\tau^k(v - w_{\tau h}^k)\,\mathrm{d}S \qquad \text{for all } v \in V_{1,h},\ v \geq 0, \quad (3.6c)\\
&\text{where}\quad \sigma_{\tau h}^k = \mathbb{D}\mathrm{D}_t\varepsilon_{\tau h}^k + \mathbb{C}\varepsilon_{\tau h}^k + \mathscr{B}(w_{\tau h}^k) \quad\text{and}\quad \varepsilon_{\tau h}^k = e(u_{\tau h}^k) - \pi_{\tau h}^k. \qquad (3.6d)
\end{aligned}
$$

Note that, in fact, (3.6c) uses a projection of the standard Galerkin discretization of (3.2c) to the cone of non-negative functions.

Let us define the piecewise affine interpolants $u_{\tau h}$, $\pi_{\tau h}$, $w_{\tau h}$, $\sigma_{\tau h}$, and $\varepsilon_{\tau h}$ by

$$
\begin{aligned}
\big[u_{\tau h}, \pi_{\tau h}, w_{\tau h}, \sigma_{\tau h}, \varepsilon_{\tau h}\big](t) &:= \frac{t-(k-1)\tau}{\tau}\big(u_{\tau h}^k, \pi_{\tau h}^k, w_{\tau h}^k, \sigma_{\tau h}, \varepsilon_{\tau h}\big) \\
&\quad + \frac{k\tau - t}{\tau}\big(u_{\tau h}^{k-1}, \pi_{\tau h}^{k-1}, w_{\tau h}^{k-1}, \sigma_{\tau h}^{k-1}, \varepsilon_{\tau h}^{k-1}\big) \quad \text{for} \quad t \in [(k-1)\tau, k\tau] \qquad (3.7)
\end{aligned}
$$

with $k = 0, ..., K_\tau := T/\tau$. Besides, we define also the backward piecewise constant interpolants $\bar u_{\tau h}$, $\bar\pi_{\tau h}$, $\bar w_{\tau h}$, $\bar\sigma_{\tau h}$, and $\bar\varepsilon_{\tau h}$ by

$$
\big[\bar u_{\tau h}, \bar\pi_{\tau h}, \bar w_{\tau h}, \bar\sigma_{\tau h}, \bar\varepsilon_{\tau h}\big](t) := \big(u_{\tau h}^k, \pi_{\tau h}^k, w_{\tau h}^k, \sigma_{\tau h}^k, \varepsilon_{\tau h}^k\big) \quad \text{for} \quad (k-1)\tau < t \leq k\tau \quad (3.8)
$$

with $k = 1, ..., K_\tau$. We will also use the notation $\bar g_\tau$ and $\bar f_\tau$ defined by $\bar g_\tau|_{((k-1)\tau, k\tau]} = g_\tau^k$ and $\bar f_\tau|_{((k-1)\tau, k\tau]} = f_\tau^k$ for $k = 1, ..., K_\tau$.

In terms of the interpolants, the scheme (3.6) can be written as

$$\int_Q \varrho\big[\dot{u}_{\tau h}{+}\dot{u}_{\mathrm{Dir},\tau}\big]^{\boldsymbol{\cdot}}_\tau{\cdot}v + \big(\bar\sigma_{\tau h}{+}\tau\big|\dot\varepsilon_{\tau h}\big|^{\gamma-2}\dot\varepsilon_{\tau h}{+}\bar\sigma_{\mathrm{Dir},\tau}\big){:}e(v)\,\mathrm{d}x\mathrm{d}t = 0 \quad \forall v{\in}L^1(I;V^d_{1,h}), \tag{3.9a}$$

$$\int_Q \delta^*_S(v) + \big(\bar\sigma_{\tau h}+\bar\sigma_{\mathrm{Dir},\tau}+\tau\big|\dot\varepsilon_{\tau h}\big|^{\gamma-2}\dot\varepsilon_{\tau h}\big){:}\big(\tilde\pi{-}\dot\pi_{\tau h}\big) + \eta(\tau)\bar\pi_{\tau h}{:}\big(\tilde\pi{-}\dot\pi_{\tau h}\big)\,\mathrm{d}x\mathrm{d}t$$
$$\geq \int_Q \delta^*_S\big(\dot\pi_{\tau h}\big)\,\mathrm{d}x\mathrm{d}t \qquad \forall\tilde\pi{\in}L^1(I;V^{d\times d}_{0,h}),\ \ \tilde\pi(\cdot){\in}\mathbb{R}^{d\times d}_{\mathrm{dev}}\ \text{ a.e. on } Q, \tag{3.9b}$$

$$\int_Q \dot w_{\tau h}\big(v{-}\bar w_{\tau h}\big) + \big(\mathscr{K}(\bar w_{\tau h})\nabla\bar w_{\tau h}\big){\cdot}\nabla\big(v{-}\bar w_{\tau h}\big)\,\mathrm{d}x\mathrm{d}t$$
$$\geq \int_Q \Big(\big(\mathbb{D}\dot\varepsilon_{\tau h}{+}\mathbb{D}e(\dot u_{\mathrm{Dir},\tau}){+}\mathscr{B}(\bar w_{\tau h})\big){:}\big(\dot\varepsilon_{\tau h}{+}e(\dot u_{\mathrm{Dir},\tau})\big) + \delta^*_S\big(\dot\pi_{\tau h}\big)\Big)\big(v{-}\bar w_{\tau h}\big)\,\mathrm{d}x\mathrm{d}t$$
$$+\int_\Sigma \bar f_\tau\big(v{-}\bar w_{\tau h}\big)\,\mathrm{d}S\mathrm{d}t \qquad \forall v{\in}L^1(I;V_{1,h}),\quad v(\cdot)\geq 0\ \text{ a.e. on } Q, \tag{3.9c}$$

with $\varepsilon_{\tau h} = e(u_{\tau h}){-}\pi_{\tau h},\quad \bar\sigma_{\tau h} = \mathbb{D}\dot\varepsilon_{\tau h}{+}\mathbb{C}\bar\varepsilon_{\tau h}{-}\mathscr{B}(\bar w_{\tau h}),\quad \bar\varepsilon_{\tau h} = e(\bar u_{\tau h}){-}\bar\pi_{\tau h}.$ (3.9d)

In (3.9a), $[\dot u_{\tau h}]_\tau$ denotes the linearly interpolated time-derivative so that its time derivative $[\dot u_{\tau h}]^{\boldsymbol{\cdot}}_\tau$ is thus piecewise constant and takes the values $\mathrm{D}^2_t u^k_{\tau h}$ on the subintervals $((k{-}1)\tau, k\tau)$, $k = 1, ..., T/\tau$. A similar meaning has also the term $[\dot u_{\tau h}{+}\dot u_{\mathrm{Dir},\tau}]_\tau$. Throughout this article we make the following assumptions:

$$c_{\mathrm{v}} : [0,+\infty) \to \mathbb{R}^+ \ \ \text{continuous}, \tag{3.10a}$$
$$\exists\,\omega > 1,\ c_0 > 0\ \ \forall\theta{\in}\mathbb{R}^+ : \quad c_{\mathrm{v}}(\theta) \geq c_0(1{+}\theta)^{\omega-1}, \tag{3.10b}$$
$$\mathbb{C}, \mathbb{D} \ \ \text{symmetric, positive definite}, \tag{3.10c}$$
$$\mathscr{K}{:}\mathbb{R} \to \mathbb{R}^{d\times d} \ \ \text{bounded, continuous, and} \inf_{(w,\xi)\in\mathbb{R}\times\mathbb{R}^d,\ |\xi|=1} \mathscr{K}(w)\xi{\cdot}\xi > 0; \tag{3.10d}$$

with $\mathscr{K}$ from (2.13); later in (5.18) we impose further restrictions on ω. As far as the initial conditions and loading qualification (and its regularization) are concerned, we assume

$$u_0 \in W^{1,1}(\Omega;\mathbb{R}^d), \tag{3.11a}$$
$$\pi_0 \in L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}), \qquad \big\{\sqrt{\eta(\tau)}\pi_{0,\tau}\big\}_{\tau>0} \text{ bounded in } L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}), \tag{3.11b}$$
$$e(u_0){-}\pi_0 \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}), \qquad \big\{e(u_{0,\tau}){-}\pi_{0,\tau}\big\}_{\tau>0} \text{ bounded in } L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{3.11c}$$
$$w_0 \in L^1(\Omega),\ w_{0,\tau} \in L^2(\Omega), \quad \big\{w_{0,\tau}\big\}_{\tau>0} \text{ bounded in } L^1(\Omega), \tag{3.11d}$$
$$f \in L^1(\Sigma), \qquad \big\{\tilde f_\tau\big\}_{\tau>0} \text{ bounded in } L^1(\Sigma), \tag{3.11e}$$
$$u_{\mathrm{Dir}} \in W^{2,1}(I;W^{1,2}(\Omega;\mathbb{R}^d)),\quad \mathrm{div}\,\dot u_{\mathrm{Dir}} = 0, \tag{3.11f}$$

where we denoted $\Sigma := I \times \Gamma$ in (3.11e).

Lemma 3.1 (Existence and estimates of discrete solutions)**.** *Let (3.10) and (3.11) hold. Moreover, let $\gamma > \max(4, \frac{2\omega}{\omega-1})$. Then there exists a solution $(u^k_{\tau h}, \pi^k_{\tau h}, w^k_{\tau h}) \in$*

$V_{1,h}^d \times V_{0,h}^{d\times d} \times V_{1,h}$, *with* $\pi_{\tau h}^k(\cdot) \in \mathbb{R}_{\rm dev}^{d\times d}$ *a.e. on* Ω, *for the system (3.6). Moreover,*

$$\big\|u_{\tau h}\big\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^d))} \le C \qquad \text{if } \varrho > 0, \tag{3.12a}$$

$$\big\|e(u_{\tau h}) - \pi_{\tau h}\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}_{\rm sym}^{d\times d}))} \le C, \tag{3.12b}$$

$$\big\|w_{\tau h}^k\big\|_{L^\infty(I;L^1(\Omega))} \le C \tag{3.12c}$$

$$\big\|e(\overset{\bullet}{u}_{\tau h}) - \overset{\bullet}{\pi}_{\tau h}\big\|_{L^\gamma(Q;\mathbb{R}_{\rm sym}^{d\times d})} \le C\tau^{-1/\gamma}, \tag{3.12d}$$

$$\big\|\bar\pi_{\tau h}\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}_{\rm dev}^{d\times d}))} \le C\eta(\tau)^{-1/2}, \tag{3.12e}$$

where C *does not depend on* τ *and* h*. Furthermore, we have*

$$\big\|u_{\tau h}\big\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^d))} \le C_\tau, \tag{3.13a}$$

$$\big\|e(u_{\tau h}) - \pi_{\tau h}\big\|_{W^{1,\infty}(I;L^\gamma(\Omega;\mathbb{R}_{\rm sym}^{d\times d}))} \le C_\tau, \tag{3.13b}$$

$$\big\|\pi_{\tau h}\big\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}_{\rm dev}^{d\times d}))} \le C_\tau, \tag{3.13c}$$

$$\big\|w_{\tau h}^k\big\|_{W^{1,\infty}(I;W^{1,2}(\Omega))} \le C_\tau \tag{3.13d}$$

with some C_τ *dependent on* τ *(but not on* h*). If* $\omega \ge 2$*, then also*

$$\big\|e(u_{\tau h})\big\|_{W^{1,1}(I;L^1(\Omega;\mathbb{R}_{\rm sym}^{d\times d}))} \le C, \tag{3.14a}$$

$$\big\|e(u_{\tau h}) - \pi_{\tau h}\big\|_{W^{1,2}(I;L^2(\Omega;\mathbb{R}_{\rm sym}^{d\times d}))} \le C, \tag{3.14b}$$

$$\big\|\pi_{\tau h}\big\|_{W^{1,1}(I;L^1(\Omega;\mathbb{R}_{\rm dev}^{d\times d}))} \le C, \tag{3.14c}$$

where, similarly as in (3.12)*,* C *does not depend on* τ *and* h*.*

Sketch of the proof. We can see existence of a solution to (3.2) by a standard argument for coercive pseudomonotone set-valued operators; cf. e.g. [17] for a general infinite-dimensional concept or, here, [26, Section5.3] for inclusions with pseudomonotone operators whose set-valued part has a convex potential. (Here, in fact, we need only the finite-dimensional variant which is even simpler.) The coercivity of the underlying operator can be shown by considering a scaling factor $\epsilon_\tau > 0$ and testing (3.6a), (3.6b), and (3.6c) by $u_{\tau h}^k$, 0, and $(1{+}\epsilon_\tau)w_{\tau h}^k$, respectively. Note that these test-functions live in the corresponding finite-dimensional spaces and $(1{+}\epsilon_\tau)w_{\tau h}^k \ge 0$, and are thus legal for this test. It is important that the right-hand sides of (3.2a,c) have the growth that can be dominated by the growth of the coercive terms in the left-hand sides if $\epsilon_\tau > 0$ is taken sufficiently small; this is ensured by having taken γ large enough and by the assumption (3.10b) which ensures a sublinear growth of Θ and thus also of $\mathscr{B}$, namely

$$\big|\mathscr{B}(w)\big| \le |\mathbb{B}|\Big(\frac{\omega w}{c_0}{+}1\Big)^{1/\omega} - |\mathbb{B}| \le |\mathbb{B}|\Big(\frac{\omega w}{c_0}\Big)^{1/\omega} \tag{3.15}$$

because obviously $C_{\rm v}(\theta) \ge c_0((1{+}\theta)^\omega{-}1)/\omega$, cf. the definition (2.13). Realize that the sum of the left-hand sides of (3.2) can be estimated (up to an additive constant) from below by

$$\tau|\varepsilon|^\gamma + \eta(\tau)|\pi|^2 + \epsilon_\tau|w|^2. \tag{3.16}$$

This indeed dominates the growth of the "right-hand-side terms" is of the type $|w|^{1/\omega}|\varepsilon| + |\pi|\,|w| + |\varepsilon|^2|w| + |w|^{1+1/\omega}|\varepsilon|$. More in detail, the heat-production δ_S^*-term can be estimated as

$$\delta_S^*\Big(\frac{\pi{-}\pi_{\tau h}^{k-1}}{\tau}\Big)\epsilon_\tau w \le \frac{1}{\tau}K\epsilon_\tau\big|\pi{-}\pi_{\tau h}^{k-1}\big|\,|w| \le \frac{K}{2\tau}\epsilon_\tau^{1/2}\big|\pi{-}\pi_{\tau h}^{k-1}\big|^2 + \frac{K}{2\tau}\epsilon_\tau^{3/2}|w|^2$$

with $K = \sup_{|\dot\pi|\le 1}\delta_S^*(\dot\pi)$, and then absorbed in the left-hand side (3.16) if $\epsilon_\tau < 4\tau^2\min(1,\eta(\tau)^2)/K^2$; note that $K<\infty$ because $\mathrm{int}(S)\ni 0$. Similarly $|\varepsilon|^2|w| \le \delta|\varepsilon|^\gamma+\delta|w|^2+C_\delta$ with any $\delta>0$ and some C_δ; here $\gamma>4$ has been used. The last term can be estimated as $|w|^{1+1/\omega}|\varepsilon| \le \delta|\varepsilon|^\gamma+\delta^{-1/(\gamma-1)}|w|^{(1+1/\omega)\gamma/(\gamma-1)} \le \delta|\varepsilon|^\gamma+\delta|w|^2+C_\delta$ for arbitrary $\delta>0$ and some $C_\delta\in\mathbb{R}$; here the condition $\gamma>2\omega/(\omega-1)$ has originated.

Knowing existence of the solution $u_{\tau h}^k$, $\pi_{\tau h}^k$, and $w_{\tau h}^k\ge 0$, we can also perform the test of (3.6a), (3.6b), and (3.6c) by $\mathrm{D}_t u_{\tau h}^k$, 0, and $1{+}w_{\tau h}^k\ge 0$, respectively. This leads to a cancellation of the dissipative and adiabatic terms. By using the convexity of the underlying regularized stored energy

$$\Phi_\tau(u,\pi) := \frac12\int_\Omega \mathbb{C}(e(u){-}\pi){:}(e(u){-}\pi)+\eta(\tau)|\pi|^2\,\mathrm{d}x, \tag{3.17}$$

and by summation over time steps, we obtain the following "discrete total energy" balance:

$$\begin{aligned}&T_{\rm kin}\big(\mathrm{D}_t u_{\tau h}^k\big)+\Phi_\tau\big(u_{\tau h}^k,\pi_{\tau h}^k\big)+\int_\Omega w_{\tau h}^k\,\mathrm{d}x+\tau^2\sum_{l=1}^k\int_\Omega\big|\mathrm{D}_t\varepsilon_{\tau h}^l\big|^\gamma\mathrm{d}x\\
&\quad\le T_{\rm kin}(\dot u_0)+\Phi_\tau\big(u_0,\pi_{0,\tau}\big)+\int_\Omega w_0\,\mathrm{d}x\\
&\quad+\tau\sum_{l=1}^k\bigg(\int_\Omega\big(\mathbb{D}\mathrm{D}_t e_{\mathrm{Dir},\tau}^l{+}\mathbb{C}e_{\mathrm{Dir},\tau}^l\big){:}\mathrm{D}_t\varepsilon_{\tau h}^l+\mathbb{D}\mathrm{D}_t e_{\mathrm{Dir},\tau}^l{:}\mathrm{D}_t e_{\mathrm{Dir},\tau}^l\mathrm{d}x+\int_\Gamma f_\tau^l\,\mathrm{d}S\bigg),\end{aligned}\tag{3.18}$$

where we abbreviated $e_{\mathrm{Dir},\tau}^l = e(u_{\mathrm{Dir},\tau}^l)$. Cf. [30, proof of Prop.1] for details. Importantly, we used also (2.8) and (3.11f) for the orthogonality $\mathscr{B}(w_{\tau h}^l)){:}e_{\mathrm{Dir},\tau}^l=0$. Note that (3.18) is indeed the discrete variant of (2.10) after the shift of the Dirichlet conditions. Then we execute the by-part summations $\sum_{l=1}^k\mathbb{D}\mathrm{D}_t e_{\mathrm{Dir},\tau}^l{:}\mathrm{D}_t\varepsilon_{\tau h}^l = \mathbb{D}\mathrm{D}_t e_{\mathrm{Dir},\tau}^k{:}\varepsilon_{\tau h}^k-\sum_{l=2}^k\mathbb{D}\mathrm{D}_t^2 e_{\mathrm{Dir},\tau}^l{:}\varepsilon_{\tau h}^{l-1}-\mathbb{D}\mathrm{D}_t e_{\mathrm{Dir},\tau}^1{:}\varepsilon_{\tau h}^0$ and $\sum_{l=1}^k\mathbb{C}e_{\mathrm{Dir},\tau}^l{:}\mathrm{D}_t\varepsilon_{\tau h}^l = \mathbb{C}e_{\mathrm{Dir},\tau}^k{:}\varepsilon_{\tau h}^k-\sum_{l=2}^k\mathbb{C}\mathrm{D}_t e_{\mathrm{Dir},\tau}^l{:}\varepsilon_{\tau h}^{l-1}-\mathbb{C}e_{\mathrm{Dir},\tau}^1{:}\varepsilon_{\tau h}^0$. The a-priori estimates (3.12) then follows from the above test by standard procedure, i.e. by using Hölder's, Young's, and the discrete Gronwall inequality. It is essential that we have $w_{\tau h}^k\ge 0$ guaranteed just by the hard constraints. Also note that, to make $\Phi_\tau(u_0,\pi_{0,\tau})$ bounded (uniformly in $\tau>0$), we needed to approximate π_0 by $\pi_{0,\tau}$ and assume (3.11b).

If $\omega\ge 2$, we can also execute the test of (3.6a), (3.6b), and (3.6c) by $\mathrm{D}_t u_{\tau h}^k$, 0, and $1/2{+}w_{\tau h}^k\ge 0$, respectively. We can then see parts of the dissipative terms on the left-hand side, namely

$$\begin{aligned}&T_{\rm kin}\big(\mathrm{D}_t u_{\tau h}^k\big)+\Phi_\tau\big(u_{\tau h}^k,\pi_{\tau h}^k\big)+\frac12\int_\Omega w_{\tau h}^k\,\mathrm{d}x\\
&\quad+\tau\sum_{l=1}^k\int_\Omega\frac12\mathbb{D}\mathrm{D}_t\big(\varepsilon_{\tau h}^k{+}e_{\mathrm{Dir},\tau}^k\big){:}\mathrm{D}_t\big(\varepsilon_{\tau h}^k{+}e_{\mathrm{Dir},\tau}^k\big)+\frac12\delta_S^*(\mathrm{D}_t\pi_{\tau h}^k)+\tau\big|\mathrm{D}_t\varepsilon_{\tau h}^l\big|^\gamma\mathrm{d}x\\
&\quad\le T_{\rm kin}(\dot u_0)+\Phi_\tau\big(u_0,\pi_{0,\tau}\big)+\frac12\int_\Omega w_0\,\mathrm{d}x+\tau\sum_{l=1}^k\bigg(\int_\Omega\frac12\mathscr{B}(w_{\tau h}^k){:}\mathrm{D}_t\big(\varepsilon_{\tau h}^k{+}e_{\mathrm{Dir},\tau}^k\big)\\
&\quad+\big(\mathbb{D}\mathrm{D}_t e_{\mathrm{Dir},\tau}^l{+}\mathbb{C}e_{\mathrm{Dir},\tau}^l\big){:}\mathrm{D}_t\varepsilon_{\tau h}^l+\mathbb{D}\mathrm{D}_t e_{\mathrm{Dir},\tau}^l{:}\mathrm{D}_t e_{\mathrm{Dir},\tau}^l\mathrm{d}x+\frac12\int_\Gamma f_\tau^l\,\mathrm{d}S\bigg).\end{aligned}\tag{3.19}$$

Now the adiabatic terms did not cancel, but can be estimated simply (without any interpolation) by the estimate

$$\begin{aligned}\mathscr{B}(w_{\tau h}^k){:}\mathrm{D}_t\big(\varepsilon_{\tau h}^k{+}e_{\mathrm{Dir},\tau}^k\big) &\le \frac{1}{4\epsilon}|\mathscr{B}(w_{\tau h}^k)|^2+\epsilon\big|\mathrm{D}_t(\varepsilon_{\tau h}^k{+}e_{\mathrm{Dir},\tau}^k)\big|^2\\ &\le \frac{|\mathbb{B}|^2}{4\epsilon}\Big(\frac{\omega w_{\tau h}^k}{c_0}\Big)^{2/\omega}+\epsilon\big|\mathrm{D}_t(\varepsilon_{\tau h}^k{+}e_{\mathrm{Dir},\tau}^k)\big|^2\\ &\le C_{\epsilon,\omega}+C_{\epsilon,\omega}w_{\tau h}^k+\epsilon\big|\mathrm{D}_t(\varepsilon_{\tau h}^k{+}e_{\mathrm{Dir},\tau}^k)\big|^2 \end{aligned}\tag{3.20}$$

with some $C_{\epsilon,\omega}$ depending on $\epsilon>0$ and $\omega\ge 2$, and then handled by the discrete Gronwall inequality. The dissipative terms then yield (3.12b,c) and, realizing $\|e(\dot u_{\tau h})\|_{L^1(Q;\mathbb{R}^{d\times d})}\le \mathrm{meas}(\Omega)^{1/2}\|e(\dot u_{\tau h}){-}\dot\pi_{\tau h}\|_{L^2(Q;\mathbb{R}^{d\times d})}+\|\dot\pi_{\tau h}\|_{L^1(Q;\mathbb{R}^{d\times d})}$, we eventually obtain also (3.14a). $\square$

Let us emphasize that the a-priori estimates (3.12) and, if $\omega\ge 2$, also (3.14) are uniform in both τ and h and guarantee thus a certain *unconditional numerical stability* of the approximation scheme as far as the elastic strain ε, velocity $\dot u$, enthalpy w, and (if $\omega\ge 2$) also plastic strain π and strain rate $\dot\varepsilon$ concerned. As already mentioned, the convergence will however be guaranteed only conditionally with h passing to 0 sufficiently fast with respect to τ, cf. Section 5.

4. Computational implementation and simulations

We want to illustrate the applicability of the above model and its approximation by implementing it and performing specific experiments documenting both thermomechanical phenomena of converting mechanical-to-thermal energy (Section 4.1) and thermal-to-mechanical energy (Section 4.2) together with a concentration tendency due to arising shear bands. Besides, we discuss and document some interesting numerical issues in Section 4.3.

In our implementation, we neglect inertial and viscous effects, and temperature dependence of the heat capacity. Thus, in the numerical experiments reported below, we consider $c_{\rm v}>0$ constant and set $\varrho:=0$ and $\mathbb{D}:=0$. We also consider an *isotropic material*, i.e. with the symmetric positive definite fourth order tensor $\mathbb{C}=[\mathbb{C}_{ijkl}]$, i.e.

$$\mathbb{C}_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu\big(\delta_{ik}\delta_{jl}{+}\delta_{il}\delta_{jk}\big)\qquad \text{with } \mu>0,\quad \lambda>-\frac{2}{d}\mu,\tag{4.1a}$$

$$\mathbb{E}_{ij}=\alpha\delta_{ij},\qquad \text{hence}\ \ \mathbb{B}_{ij}=\alpha(d\lambda+2\mu)\delta_{ij},\tag{4.1b}$$

$$\mathbb{K}_{ij}=\kappa\delta_{ij},\tag{4.1c}$$

$$S:=\big\{\tilde s\in\mathbb{R}^{d\times d}_{\rm dev};\ |\tilde s|\le\sigma_{\rm yield}\big\}\tag{4.1d}$$

with δ denoting here the Kronecker symbol, λ and μ are the Lamé constants, α the thermal-expansion coefficient, $|\cdot|$ the Frobenius norm, and $\sigma_{\rm yield}>0$ a so-called yield stress. Thus the elastic stress is $\mathbb{C}\varepsilon=\lambda{\rm tr}(\varepsilon)\mathbb{I}+2\mu\varepsilon$ with $\mathbb{I}=[\delta_{ij}]$ denoting the unit matrix, and the corresponding energy density is $\frac12\mathbb{C}\varepsilon{:}\varepsilon=\frac12\lambda|{\rm tr}(\varepsilon)|^2+\mu|\varepsilon|^2$ which defines a positive definite quadratic form of ε. Note that both (3.10c) for $\mathbb{C}$ and (2.8) are satisfied. We further made the simplification that we did not implement the nonlinear γ-terms in (3.6a,b); these terms were introduced in (3.6) for the only purpose to guarantee (anyhow nonconstructively) existence of a solution to this nonlinear system in the proof of Lemma 3.1 which we facilitate in numerical implementation quite explicitly by checking convergence of the iterative solver we use. Moreover, in our particular simulations, the discrete w was always well away from zero, hence we did not need to implement the enthalpy equation as a variational inequality (3.6c) and only a-posteriori check positivity of the obtained $w_{\tau h}$.

For our computational experiments, we considered a 2-dimensional specimen depicted on Figure 1 made of conventional steel. We summarize the employed data:

- *Material data*:
  heat capacity $c_{\rm v} = 3.2\mathrm{MJm^{-3}K^{-1}}$,
  heat transfer coefficient $\kappa = 80\mathrm{Wm^{-1}K^{-1}}$,
  thermal-expansion coefficient $\alpha = 2 \cdot 10^{-5}\mathrm{K^{-1}}$,
  Young's modulus $E = 137\mathrm{GPa}$, Poisson ratio $\nu = 0.3$,
  plastic yield stress $\sigma_{\rm yield} := 450\mathrm{MPa}$.
- *Geometry of the specimen*: $d = 2$, $\Omega = (-5\frac{L}{2}, 5\frac{L}{2}) \times (-\frac{L}{2}, \frac{L}{2})$ for $L = 10^{-2}$m, $\Gamma_{\rm Dir} = \Gamma_{\rm Dir}^{\rm top} \cup \Gamma_{\rm Dir}^{\rm bottom}$, $\Gamma_{\rm Dir}^{\rm top} = (-5\frac{L}{2}, \frac{L}{2}) \times \{\frac{L}{2}\}$, $\Gamma_{\rm Dir}^{\rm bottom} = (-\frac{L}{2}, 5\frac{L}{2}) \times \{-\frac{L}{2}\}$, cf. Figure 1.
- *Initial conditions*: $u_0(x) = \alpha\theta_0 x$ for $x \in \Omega$, $\pi_0 := 0$, and $\theta_0 = 300\,\mathrm{K}$.
- *Experiment 1 – mechanically induced plastification*: $u_{\rm Dir}(t, x) = 0$ for $x \in \Gamma_{\rm Dir}^{\rm bottom}$ and

$$(u_{\rm Dir})_2(t,x) = \begin{cases} -t\,2{\cdot}10^{-3}\mathrm{m/s} & \text{(fast loading regime)} \\ -t\,2{\cdot}10^{-4}\mathrm{m/s} & \text{(slow loading regime)} \end{cases} \quad \text{for } x \in \Gamma_{\rm Dir}^{\rm top}$$

  thermal isolation (i.e. $f(t,\cdot) = 0$ on Γ).
- *Experiment 2 – thermally induced plastification*: mechanical loading deactivated, i.e. $u_{\rm Dir}(t, x) = 0$ for $x \in \Gamma_{\rm Dir}^{\rm bottom}$ and $(u_{\rm Dir})_2(t, x) = 0$ for $x \in \Gamma_{\rm Dir}^{\rm top}$, and

$$f(t,x) = \begin{cases} 80\,\mathrm{MJ/m^2s} & \text{(fast heating regime)} \\ 8\ \mathrm{MJ/m^2s} & \text{(slow heating regime)} \end{cases} \quad \text{for } x \in \Gamma_{\rm Dir}^{\rm top} \cup \Gamma_{\rm Dir}^{\rm bottom}$$

  otherwise $f(t,\cdot) = 0$ on the rest of the boundary $\Gamma \backslash (\Gamma_{\rm Dir}^{\rm top} \cup \Gamma_{\rm Dir}^{\rm bottom})$.

The Lamé coefficients in (4.1a) are defined through $\lambda = \nu E/((1+\nu)(1-2\nu))$ and $\mu = E/(2(1+\nu))$. Notice that we consider slightly more general boundary conditions than (2.7), i.e., only the normal displacement on $\Gamma_{\rm Dir}^{\rm top}$ is prescribed (and gradually increasing in time). Yet it still allows for an extension satisfying (3.11f).

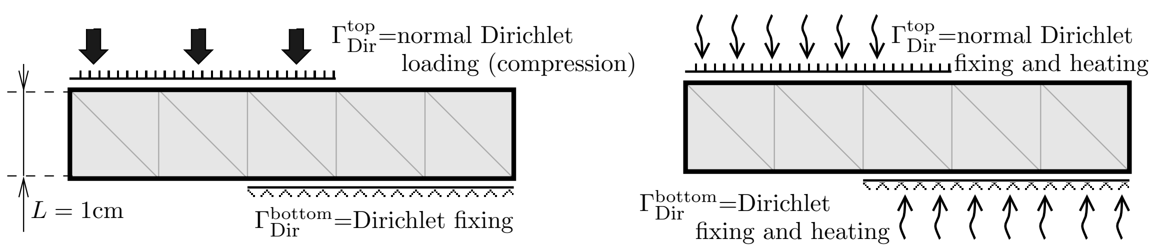

FIGURE 1. Geometry for the 2D example:
Left: Experiment 1 (mechanical-loading),
Right: Experiment 2 (heating).

For the experiments, we use $\tau = 2^{-m}T$ and $h = 2^{-m}L$ for $m = 6$, i.e. our triangulation consists of 40.960 triangles with edges of the length $2^{-m}L$ resulting from 6 so-called red-refinements of the coarse mesh with 10 triangles depicted on Figure 1.

The displacements in all figures below are depicted magnified by a factor 20. Always, four snapshots are depicted, namely for $t = 32\tau$, 64τ, 96τ, and $128\tau = T$. For slow experiments we used $T = 0.5$s (i.e. $\tau = T/128 \sim 4{\cdot}10^{-3}$s) while for fast experiments we used $T = 0.05$s (i.e. $\tau = T/128 \sim 0.4{\cdot}10^{-3}$s). Particular numerical issues about the choice of a triangulation with the interplay of the numerical hardening parameter η will be discussed in Section 4.3 below.

The nonlinear systems of mechanical and heat equations was decoupled and then alternatingly iterated at each time step. The equations arising at each iteration from the mechanical system were solved by a semismooth Newton method, cf. [9, 31]. The implementation of the Newton method was done in MATLAB in the spirit of [1, 9, 16]. In our experiments the Newton scheme always terminated within at most 8 iterations to achieve an ℓ^2-norm of the residual vector (defined through nodal basis functions) less than 10^{-7}J. Moreover, in all time steps, less than 7 fixed point iterations were sufficient to achieve an absolute change of the temperature in the H^1 norm less than $10^{-6}\mathrm{Km}^{1/2}$.

4.1. **Experiment 1: plastification via mechanical loading.** To illustrate the rate-dependence of the coupled model we displayed in Figure 2 snapshots of the evolution of the plastic strain for different speeds of mechanical loading. The fast process is shown in the left column and the 10 times slower one in the plots of the right column.

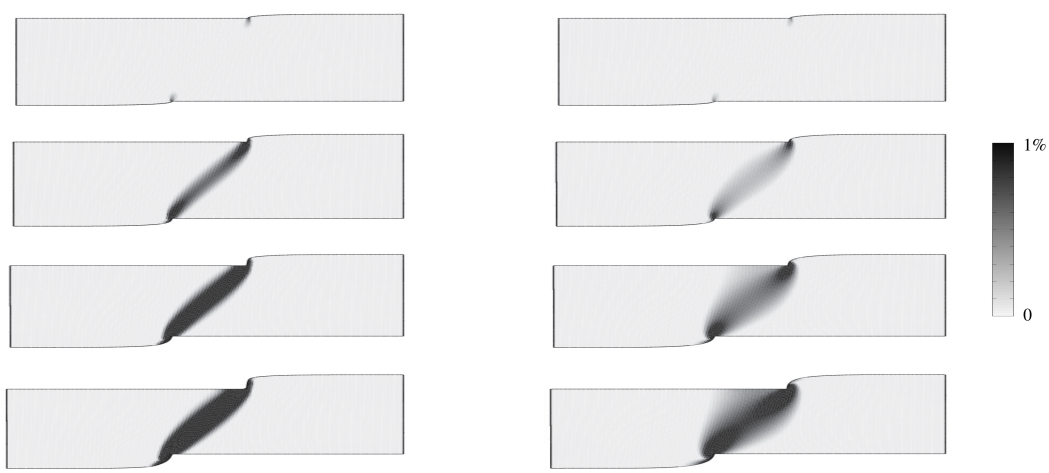

FIGURE 2. Experiment 1: Modulus of plastic strain $|\pi_h|$ evolving in time; fast loading (left column) and slow loading (right column).

In both cases a shear band develops, indicated by the localized plastic strain along the line that connects the points on the boundary where the type of boundary condition changes. The corresponding temperatures are shown in Figure 3 and we can see that the heating due to the heat generated by plastification is more localized around the shear band in the case of fast loading (left column) than for slow loading where it rather diffuses out of the shear band (right column). It is interesting to note that the displacements differ significantly at the critical points of the boundary. Owing to the rather localized temperature variations in the case of fast loading (left column) the body expands rather locally around the shear band and this leads to a different profile of the deformed boundary than in the slow loading process (right column). In other words, the difference between the left and the right columns on

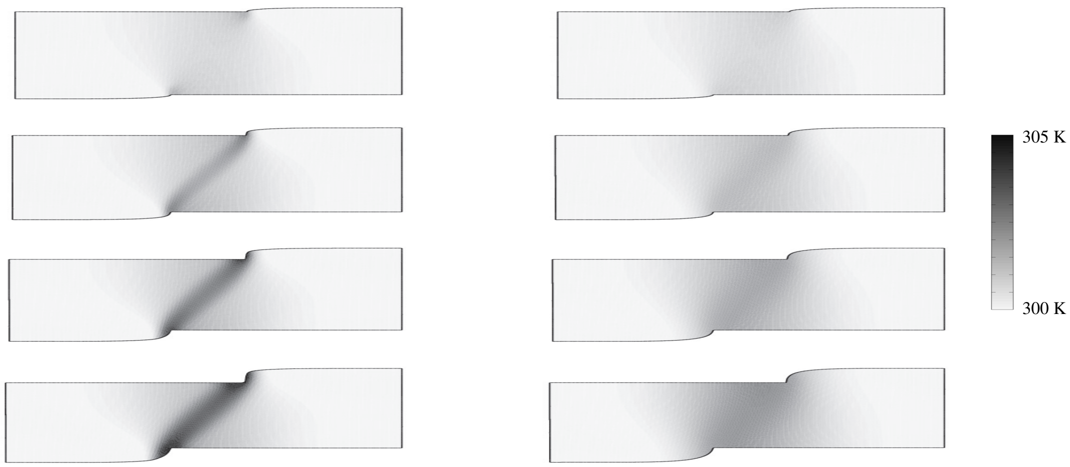

FIGURE 3. Experiment 1: Temperature θ_h evolving in time; fast loading (left column) and slow loading (right column).

Figure 2 documents the influence of the thermal coupling, and there would be no difference in an isothermal case.

4.2. **Experiment 2: plastification via external heating.** To illustrate the effect of thermal expansion, we "pumped" the heat through the Dirichlet part of the (mechanical) boundary instead of the mechanical load considered in Section 4.1. ¿From the snapshots shown in Figures 4 and 5 for fast (left column) and slow (right column) heating, we can see that the body expands where heat is pumped or generated and this is localized to the neighbourhoods of the fixed part of the body and later, to a smaller extent, also around the shear band, if it arises. In contrast to the first experiment, the formation of the shear band is now very rate dependent and we can see it pronounced especially under fast heating. A localization of plastic stresses is also observed for the slow heating (right column) but the magnitude of the plastic strain is considerably smaller. Another difference is that the shape of the plastified region is not exactly straight, i.e. the shear band is rather diffused, cf. also Figure 9 below.

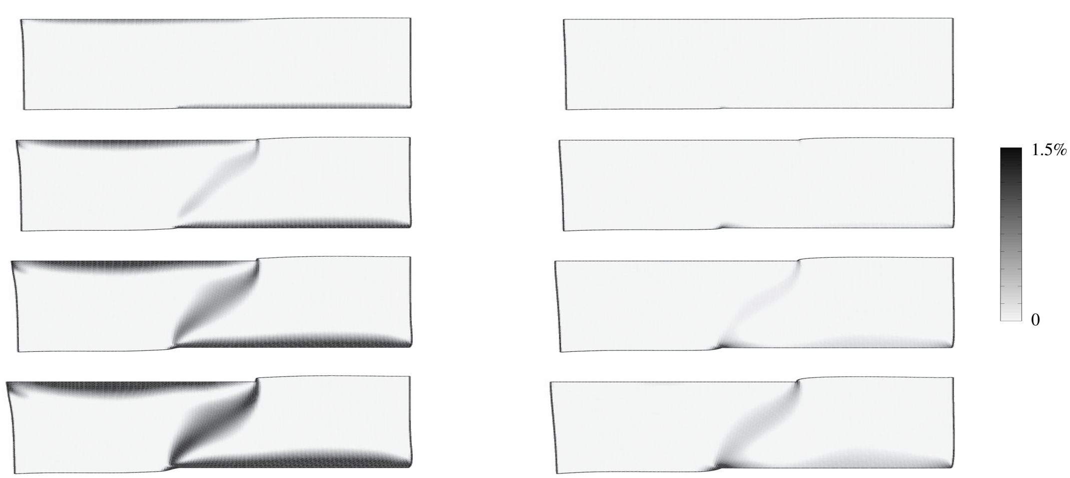

FIGURE 4. Experiment 2: Modulus of plastic strain $|\pi_h|$ evolving in time; fast external heating (left column) and slow external heating (right column).

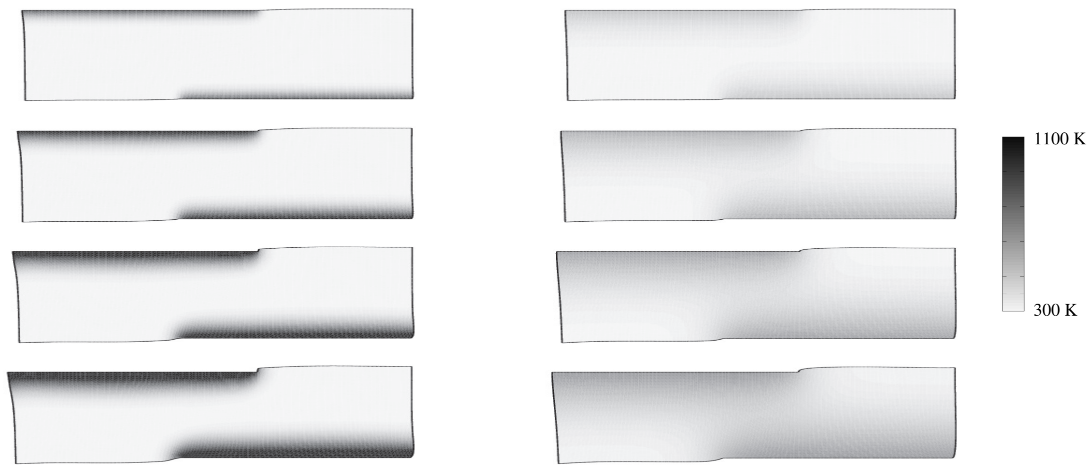

FIGURE 5. Experiment 2: Temperature θ_h evolving in time; fast external heating (left column) and slow external heating (right column).

4.3. **Suppressing mesh dependence.** In Experiment 1 from Section 4.1, we study the dependence of our numerical solution on the choice of the numerical hardening parameter $\eta = \eta(\tau)$ in (3.2b) and on geometric properties of the underlying triangulations. For the results shown in Figure 7 we used the hardening parameters $\eta(\tau) = \eta_0(\tau/\mathrm{s})^\beta$ for $\eta_0 = 10^8$MPa and for three choices of $\beta = 1,\ 3/2,\ 2$ (top to bottom), and we tried two choices of the triangulations whose directions either match the expected direction of the expected shear band (right) or form an angle of $\pi/4$ with it (left). These were obtained from six uniform refinements of the triangulations shown in Figure 6. In a sense, these are two extreme cases that may be non-generic.

FIGURE 6. Two different triangulations tested, depicted for $m = 0$ before their refinements for $m = 6$. Left: matching the expected shear band, right: not matching the expected shear band.

We observe that the displacement and the plastic strain do not differ significantly on the two triangulations, when the hardening parameter is given by $\eta(\tau) = \eta_0\tau/\mathrm{s}$ (upper row). For the choice $\eta(\tau) = \eta_0(\tau/\mathrm{s})^{3/2}$ we see moderate differences in the solution (middle row) while for $\eta(\tau) = \eta_0(\tau/\mathrm{s})^2$ critical mesh-dependence of the numerical solution occurs. The corresponding temperatures shown in Figure 8 show similar effects that are less pronounced. These results motivated us to employ $\eta(\tau) = \eta_0\tau^{3/2}$ in the above presented experiments, where we always intentionally used the triangulation that does not match the direction of the shear band to avoid an artificial improvement of the approximation.

In Experiment 2, the shear band does not have the shape of a straight line, which gave us another opportunity to study the influence of different triangulations. We compared the plastic strains at the final time for triangulations consisting of halved squares along the directions $(1,1)$ and $(-1,1)$ in Figure 9. Moreover, we used the small hardening parameter $\eta(\tau) = \eta_0(\tau/\mathrm{s})^2$. Surprisingly, we do not observe significant differences in the numerical solutions. The plastic strains are slightly more localized in the case of the triangulation with diagonals parallel to the direction

FIGURE 7. Modulus of plastic strain $|\pi_h|$ for Experiment 1 (fast) for $t = T$ for hardening parameters $\eta(\tau) = \eta_0(\tau/\mathrm{s})^\beta$ for $\beta = 1, 3/2, 2$ (from top to bottom) on non-matching (left) and matching triangulations (right). The mesh dependence is pronounced for smaller η.

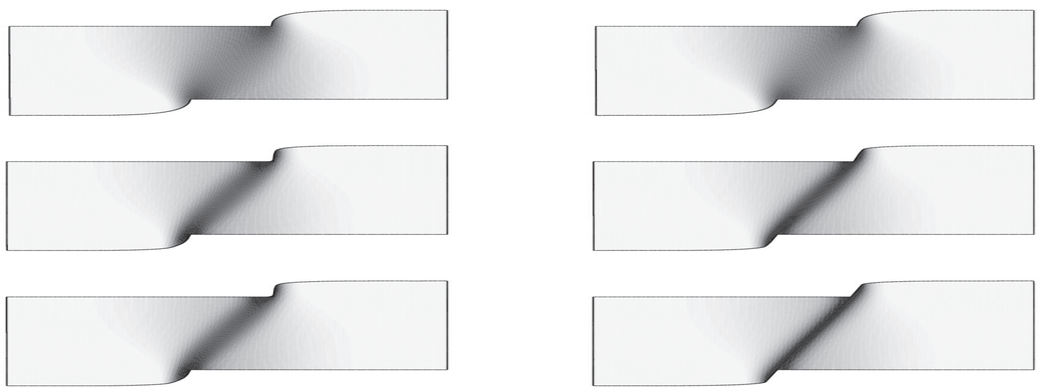

FIGURE 8. Temperature profiles corresponding to Figure 7.

$(1, 1)$ but the choice of triangulation does not seem to influence the geometry of the shear band. This is likely because the "S" shaped shear bend is not compatible with any of these two triangulations, and it also documents that this "S" shape is the actual mechanical phenomenon.

FIGURE 9. Modulus of plastic strain $|\pi_h|$ for $t = T$ in the Experiment 2 (fast) on triangulations resulting from six refinements of the ones shown in Figure 6 and for the small hardening parameter $\eta(\tau) = \eta_0(\tau/\mathrm{s})^2$.

5. Convergence analysis of the scheme (3.6) in special cases

Convergence ultimately needs estimates in particular on the gradient of enthalpy w to the limit especially in the non-linear Nemytskiĭ operators arising by temperature dependence of $c_{\rm v}$ which, in turn, is required to comply with (3.10b). Under L^1-heat sources, this needs special 'nonlinear" test 'functions $1-1/(1{+}\bar w)^\delta$ that do not seem to be available for spatially discrete problems. This is why the rigorous convergence proof seems possible only in two successive steps, first $h\to 0$ (holding still in rather general situations) and only afterwards $\tau\to 0$, which altogether yield only conditional convergence, cf. Remark 5.5 below. A further peculiarity is that, due to the degree-1 homogeneity of δ_S^*, the heat equation has its right-hand side not only in $L^1(Q)$ (as it would be in case of higher-degree homogeneity of dissipative-force potential) but even in measures. For this, the key trick is to recover the exact energy balance in the limit. Here another peculiarity occurs, namely that the required by-part integration in time does not seem to work if the strain rate concentrates, which allows only for rather very conditional results either considering only quasistatic problems ($\varrho=0$) or qualifying a-priori the limit in the dynamical case, cf. also Remark 5.6 below.

We consider an evolution in the time interval $I:=(0,T)$ with a fixed time horizon $T>0$ and denote $Q:=(0,T)\times\Omega$, $\Sigma:=(0,T)\times\partial\Omega$, and $\bar I:=[0,T]$. We will use a standard notation for function spaces, namely the space of the continuous $\mathbb{R}^k$-valued functions $C(\bar\Omega;\mathbb{R}^k)$, its dual $\mathscr{M}(\bar\Omega;\mathbb{R}^k)$ (i.e., up to an isometric isomorphism, the space of Borel measures), the continuously differentiable functions $C^1(\bar\Omega;\mathbb{R}^k)$, the Lebesgue space $L^p(\Omega;\mathbb{R}^k)$, the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^k)$, and the Bochner space of X-valued Bochner measurable p-integrable functions $L^p(I;X)$. If $X=(X')^*$, the notation $L^\infty_{\rm w*}(I;X)$ stands for space of weakly* measurable functions $I\to X$; this space is dual to the space $L^1(I;X')$ and, in general, is not equal to $L^\infty(I;X)$. If X is separable reflexive, then $L^\infty(I;X)=L^\infty_{\rm w*}(I;X)$ by Pettis' theorem, however. Moreover, we denote by $B(\bar I;X)$, $B_{\rm w*}(\bar I;X)$, ${\rm BV}(\bar I;X)$ or $C_{\rm w}(\bar I;X)$ the Banach space of functions $\bar I\to X$ that are bounded Bochner measurable, bounded weakly* measurable, have a bounded variation or are weakly continuous, respectively; note that all these functions are defined everywhere on $\bar I$. We will use the notation $q'=q/(q{-}1)$ for the conjugate exponent to q.

We will define the space of functions with bounded deformations and satisfying the Dirichlet boundary conditions (2.16) by

$$
{\rm BD}(\Omega;\mathbb{R}^d){:=}\big\{u\,{\in}\,L^1(\Omega;\mathbb{R}^d);\ \ e(u)\,{\in}\,\mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm sym})\big\}
\tag{5.1}
$$

and moreover we define the space of admissible pairs (u,π) satisfying also the Dirichlet boundary conditions (2.16) by

$$
\begin{aligned}
\mathfrak{Q}:=\big\{&(u,\pi)\in{\rm BD}(\Omega;\mathbb{R}^d){\times}\mathscr{M}(\Omega{\cup}\Gamma_{\rm Dir};\mathbb{R}^{d\times d}_{\rm dev});\\
&e(u){-}\pi|_\Omega\,{\in}\,L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}),\ \ u\,{\circledS}\,\nu{\rm d}S{+}\,\pi|_{\Gamma_{\rm Dir}}=0\ \text{ on }\Gamma_{\rm Dir}\big\},
\end{aligned}
\tag{5.2}
$$

where $a\,{\circledS}\,b$ means the symmetrized tensorial product $\frac12(a\otimes b+b\otimes a)$. We will also use the spaces where velocities will typically live:

$$
\mathscr{V}:=\big\{v\,{\in}\,L^\infty(I;L^2(\Omega;\mathbb{R}^d));\ e(v)\,{\in}\,L^1(Q;\mathbb{R}^{d\times d}_{\rm sym}),\ {\rm div}\,v\,{\in}\,L^2(Q),\ v|_{\Gamma_{\rm Dir}}=0\big\},
\tag{5.3}
$$

For $j=0,1$, the L^2-orthogonal projection onto $V_{j,h}$ is denoted by $P_{j,h}$. We have the following approximation property at our disposal for any $1\le\gamma<\infty$, cf. e.g. [8]:

$$
\forall v\in W^{1,\gamma}(\Omega):\qquad P_{1,h}v\to v\qquad\text{in }W^{1,\gamma}(\Omega),
\tag{5.4a}
$$

$$
\forall v\in L^2(\Omega):\qquad P_{0,h}v\to v\qquad\text{in }L^2(\Omega).
\tag{5.4b}
$$

Lemma 5.1 (Convergence for $h{\downarrow}0$). *Let again (3.10) and (3.11) hold, $\gamma > \max(4, \frac{2\omega}{\omega-1})$, and, in addition, let*

$$f(\cdot) \geq 0 \qquad \text{a.e. on } \Sigma, \tag{5.5a}$$

$$w_0(\cdot) \geq w_{0,\min} > 0 \qquad \text{a.e. on } \Omega. \tag{5.5b}$$

Then there is a subsequence of $\{(u_{\tau h}, \pi_{\tau h}, w_{\tau h})\}_{h>0}$ converging for $h{\downarrow}0$ weakly in the topologies indicated in (3.13) to some $(u_\tau, \pi_\tau, w_\tau)$ and each triple obtained by such way is a weak solution to (3.2)–(3.4), i.e. in term of the interpolants*

$$\varrho\big[\dot{u}_\tau\big]^{\bullet}_\tau - \mathrm{div}\big(\bar{\sigma}_\tau + \tau\big|\dot{\varepsilon}_\tau\big|^{\gamma-2}\dot{\varepsilon}_\tau\big) = \mathrm{div}\,\bar{\sigma}_{\mathrm{Dir},\tau} - \varrho\big[\dot{u}_{\mathrm{Dir},\tau}\big]^{\bullet}_\tau, \tag{5.6a}$$

$$\partial\delta^*_S\big(\dot{\pi}_\tau\big) + \tau\kappa_0\bar{\pi}_\tau \ni \mathrm{dev}\big(\bar{\sigma}_\tau + \bar{\sigma}_{\mathrm{Dir},\tau} + \tau\big|\dot{\varepsilon}_\tau\big|^{\gamma-2}\dot{\varepsilon}_\tau\big), \tag{5.6b}$$

$$\begin{aligned}\dot{w}_\tau - \mathrm{div}\big(\mathscr{K}(\bar{w}_\tau)\nabla\bar{w}_\tau\big) &= \delta^*_S\big(\dot{\pi}_\tau\big)\\ &\quad + \big(\mathbb{D}\dot{\varepsilon}_\tau + e(\dot{u}_{\mathrm{Dir},\tau}) + \mathscr{B}(\bar{w}_\tau)\big){:}\big(\dot{\varepsilon}_\tau + e(\dot{u}_{\mathrm{Dir},\tau})\big),\end{aligned} \tag{5.6c}$$

$$\varepsilon_\tau = e(u_\tau){-}\pi_\tau, \qquad \bar{\sigma}_\tau = \mathbb{D}\dot{\varepsilon}_\tau{+}\mathbb{C}\bar{\varepsilon}_\tau{-}\mathscr{B}(\bar{w}_\tau), \quad \bar{\varepsilon}_\tau = e(\bar{u}_\tau){-}\bar{\pi}_\tau, \tag{5.6d}$$

with the boundary conditions

$$u_\tau = 0 \qquad \text{on } \Gamma_{\mathrm{Dir}}, \tag{5.7a}$$

$$\big(\bar{\sigma}_\tau + \tau\big|\dot{\varepsilon}_\tau\big|^{\gamma-2}\dot{\varepsilon}_\tau\big)\nu = 0 \qquad \text{on } \Gamma_{\mathrm{Neu}}, \tag{5.7b}$$

$$\big(\mathscr{K}(\bar{w}_\tau)\nabla\bar{w}_\tau\big)\cdot\nu = \bar{f}_\tau \qquad \text{on } \Gamma, \tag{5.7c}$$

and with the initial conditions (3.5); here, $(\bar{u}_\tau, \bar{\pi}_\tau, \bar{w}_\tau)$ is the piece-wise constant interpolant in time corresponding to $(u_\tau, \pi_\tau, w_\tau)$ and simultaneously also the limit of the same subsequence of $\{(\bar{u}_{\tau h}, \bar{\pi}_{\tau h}, \bar{w}_{\tau h})\}_{h>0}$.

Note that (5.6c) has the right-hand side in $L^\infty(I; L^2(\Omega))$ since $\gamma \geq 4$ and since $\pi^k_\tau{-}\pi^{k-1}_\tau$ is certainly in $L^2(\Omega; \mathbb{R}^{d\times d})$ for any $k = 1, ..., K_\tau$, hence the weak formulation of (5.6c) is understood standardly; here also the regularization $w_{0\tau}$ and $\tilde{f}_\tau$ is used.

Proof of the Lemma 5.1. For clarity, we split the proof into four steps.

Step 1 – selection of subsequences: By Banach's selection principle, we first select a weakly* convergent subsequence in the spaces indicated in (3.13). Due to the construction of $V_{1,h}$, we have the approximation property (5.4) at our disposal. Hence we can consider also a sequence $\{(\widetilde{u}_{\tau h}, \widetilde{\pi}_{\tau h})\}_{h>0}$ converging strongly to (u_τ, π_τ) even in $W^{1,\infty}(I; W^{1,2}(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{dev}}))$ and simultaneously $e(\widetilde{u}_{\tau h}) - \widetilde{\pi}_{\tau h} \to e(u_\tau) - \pi_\tau$ strongly in $W^{1,\infty}(I; L^\gamma(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$ and such that $(\widetilde{u}_{\tau h}, \widetilde{\pi}_{\tau h}) : I \to V^d_{1,h} \times V^{d\times d}_0$; here one must take into account that $\tau > 0$ is fixed hence only a finite number of values of (u_τ, π_τ) is to be approximated by using (5.4). We can choose

$$\widetilde{u}_{\tau h} := P_{1,h}u_\tau \qquad \text{and} \qquad \widetilde{\pi}_{\tau h} := P_{0,h}\pi_\tau\,. \tag{5.8}$$

Step 2 – strong convergence $\dot{\varepsilon}_{\tau h} \to \dot{\varepsilon}_\tau$ and $\pi_{\tau h} \to \pi_\tau$: Due to the dissipative-heat terms in (5.6c), we need to prove the mentioned strong convergence. To achieve this goal, we test the Galerkin identity (3.6a) by $\mathrm{D}_t(u^k_{\tau h}{-}\widetilde{u}^k_{\tau h})$ and the Galerkin inequality (3.6b) by $\mathrm{D}_t\widetilde{\pi}^k_{\tau h}$; note that we approximated (u_τ, π_τ) by $(\widetilde{u}_{\tau h}, \widetilde{\pi}_{\tau h})$ in order to be able to make such a test. Then we sum it up for $k = 1, ..., l \leq T/\tau$ and use the so-called d-monotonicity of $\mathscr{D}_{\gamma,\tau} : L^\gamma(\Omega{\times}[0,t]; \mathbb{R}^{d\times d}) \to L^{\gamma/(\gamma-1)}(\Omega{\times}[0,t]; \mathbb{R}^{d\times d})$:

$\varepsilon \mapsto \mathbb{D}\varepsilon + \tau|\varepsilon|^{\gamma-2}\varepsilon$. Considering $t = l\tau$, this leads to

$$\tau\Big(\big\|\dot\varepsilon_{\tau h}\big\|^{\gamma-1}_{L^\gamma(\Omega\times[0,t];\mathbb{R}^{d\times d})} - \big\|\dot\varepsilon_{\tau}\big\|^{\gamma-1}_{L^\gamma(\Omega\times[0,t];\mathbb{R}^{d\times d})}\Big)\Big(\big\|\dot\varepsilon_{\tau h}\big\|_{L^\gamma(\Omega\times[0,t];\mathbb{R}^{d\times d})} - \big\|\dot\varepsilon_{\tau}\big\|_{L^\gamma(\Omega\times[0,t];\mathbb{R}^{d\times d})}\Big)$$
$$+ \mathsf{d}\big\|\dot\varepsilon_{\tau h}-\dot\varepsilon_\tau\big\|^2_{L^2(\Omega\times[0,t];\mathbb{R}^{d\times d})} + \frac{\eta(\tau)}{2}\big\|\pi_{\tau h}(t)-\pi_\tau(t)\big\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}$$
$$\le \int_0^t\!\!\int_\Omega \big(\mathscr{D}_{\gamma,\tau}(\dot\varepsilon_{\tau h})-\mathscr{D}_{\gamma,\tau}(\dot\varepsilon_\tau)\big){:}(\dot\varepsilon_{\tau h}-\dot\varepsilon_\tau)\,\mathrm{d}x\mathrm{d}t + \frac{\eta(\tau)}{2}\int_\Omega \big|\pi_{\tau h}(t)-\pi_\tau(t)\big|^2\mathrm{d}x$$
$$\le \int_0^t\!\!\int_\Omega \big(\mathscr{D}_{\gamma,\tau}(\dot\varepsilon_{\tau h})+\mathbb{C}\bar\varepsilon_{\tau h}-\mathscr{D}_{\gamma,\tau}(\dot\varepsilon_\tau)-\mathbb{C}\bar\varepsilon_\tau\big){:}(\dot\varepsilon_{\tau h}-\dot\varepsilon_\tau)$$
$$+ \varrho\big[\dot u_{\tau h}-\dot u_\tau\big]^{\boldsymbol\cdot}_\tau{\cdot}(\dot u_{\tau h}-\dot u_\tau) + \eta(\tau)(\bar\pi_{\tau h}-\bar\pi_\tau){:}(\dot\pi_{\tau h}-\dot\pi_\tau)\,\mathrm{d}x\mathrm{d}t$$
$$\le \int_0^t\!\!\int_\Omega \big(\mathscr{B}(\bar w_{\tau h})+\bar\sigma_{\mathrm{Dir},\tau}\big){:}\big(\dot\varepsilon_{\tau h}-\dot{\tilde\varepsilon}_{\tau h}\big) + \delta_S^*(\dot{\tilde\pi}_{\tau h}) - \delta_S^*(\dot\pi_{\tau h}) \tag{5.9a}$$
$$+\varrho\big[\dot u_{\tau h}\big]^{\boldsymbol\cdot}_\tau{\cdot}(\dot{\tilde u}_{\tau h}-\dot u_\tau)+\big(\mathscr{D}_{\gamma,\tau}(\dot\varepsilon_{\tau h})+\mathbb{C}\bar\varepsilon_{\tau h}\big){:}(\dot{\tilde\varepsilon}_{\tau h}-\dot\varepsilon_\tau)+\eta(\tau)\bar\pi_{\tau h}{:}(\dot{\tilde\pi}_{\tau h}-\dot\pi_\tau) \tag{5.9b}$$
$$-\varrho\big[\dot u_\tau\big]^{\boldsymbol\cdot}_\tau{\cdot}(\dot u_{\tau h}-\dot u_\tau)-\big(\mathscr{D}_{\gamma,\tau}(\dot\varepsilon_\tau)+\mathbb{C}\bar\varepsilon_\tau\big){:}(\dot\varepsilon_{\tau h}-\dot\varepsilon_\tau)-\eta(\tau)\bar\pi_\tau{:}(\dot\pi_{\tau h}-\dot\pi_\tau)\,\mathrm{d}x\mathrm{d}t \to 0, \tag{5.9c}$$

where $\mathsf{d} > 0$ denotes the positive-definiteness constant of $\mathbb{D}$, cf. (3.10c). We used the inequalities $\mathrm{D}_t^2 u_\tau^k{:}\mathrm{D}_t u_\tau^k \ge \frac12\mathrm{D}_t|\mathrm{D}_t u_\tau^k|^2$ and $\mathbb{D}\varepsilon_\tau^k{:}\mathrm{D}_t\varepsilon_\tau^k \ge \frac12\mathrm{D}_t(\mathbb{D}\varepsilon_\tau^k{:}\varepsilon_\tau^k)$ and similar also for the π-terms, which is just a generalization of the elementary algebraic inequality of the type $(a_k{-}a_{k-1})a_k \ge \frac12 a_k^2 - \frac12 a_{k-1}^2$, together with the "telescopical" effect $\sum_{k=1}^l \frac12 a_k^2 - \frac12 a_{k-1}^2 = \frac12 a_l^2 \ge 0$ if $a_0 = 0$.

Let us show the convergence to 0 in (5.9). We have $\mathscr{B}(\bar w_{\tau h}) \to \mathscr{B}(\bar w_\tau)$ certainly in $L^2(Q)$ (in fact even in a much smaller Lebesgue space $L^{2d\omega/(d-2)-\epsilon}(Q)$ with $\epsilon > 0$) due to the compact embedding $W^{1,2}(\Omega) \Subset L^2(\Omega)$ so that $\mathscr{B}(\bar w_{\tau h}){:}(\dot\varepsilon_{\tau h}{-}\dot{\tilde\varepsilon}_{\tau h}) \to 0$ weakly in $L^1(Q)$. Also, we use

$$\limsup_{h\to0}\int_0^t\!\!\int_\Omega \delta_S^*(\dot{\tilde\pi}_{\tau h}) - \delta_S^*(\dot\pi_{\tau h})\mathrm{d}x\mathrm{d}t = \lim_{h\to0}\int_0^t\!\!\int_\Omega \delta_S^*(\dot{\tilde\pi}_{\tau h})\mathrm{d}x\mathrm{d}t$$
$$-\liminf_{h\to0}\int_0^t\!\!\int_\Omega \delta_S^*(\dot\pi_{\tau h})\mathrm{d}x\mathrm{d}t \le \int_0^t\!\!\int_\Omega \delta_S^*(\dot\pi_\tau)\mathrm{d}x\mathrm{d}t - \int_0^t\!\!\int_\Omega \delta_S^*(\dot\pi_\tau)\mathrm{d}x\mathrm{d}t = 0. \tag{5.10}$$

Hence the terms in (5.9a) converge to 0 for $h \to 0$.

Furthermore, as $\tau > 0$ is fixed, $\{[\dot u_{\tau h}]^{\boldsymbol\cdot}_\tau\}_{h>0}$ is bounded in $L^\infty(0,t;L^2(\Omega;\mathbb{R}^d))$ and $\dot{\tilde u}_{\tau h} \to \dot u_\tau$ strongly in $L^\infty(0,t;L^2(\Omega;\mathbb{R}^d))$. Similar arguments hold for the other terms in (5.9b), which shows that also (5.9b) converge to 0 for $h \to 0$.

Also, again relying on $\tau > 0$ fixed, we have the weak L^2-convergence $\dot u_{\tau h} \to \dot u_\tau$, $\dot\varepsilon_{\tau h} \to \dot\varepsilon_\tau$, and $\dot\pi_{\tau h} \to \dot\pi_\tau$, which shows that also (5.9c) converge to 0.

Altogether, from (5.9) considered for $t = T$, one can see $\|\dot\varepsilon_{\tau h}\|_{L^\gamma(Q;\mathbb{R}^{d\times d})} \to \|\dot\varepsilon_\tau\|_{L^\gamma(Q;\mathbb{R}^{d\times d})}$, which, together with $\dot\varepsilon_{\tau h} \rightharpoonup \dot\varepsilon_\tau$ and the uniform convexity of the Banach space $L^\gamma(Q;\mathbb{R}^{d\times d})$, gives the strong convergence $\dot\varepsilon_{\tau h} \to \dot\varepsilon_\tau$ in $L^\gamma(Q;\mathbb{R}^{d\times d})$.

Considering (5.9) for arbitrary $t = l\tau$, $l = 1, ..., T/\tau$, one can also see $\pi_{\tau h} \to \pi_\tau$ in $L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}))$.

Step 3 – limit passage in (3.6): Based on the convergences proved in Step 2, we have

$$\mathbb{D}\dot\varepsilon_{\tau h}{:}\dot\varepsilon_{\tau h} \to \mathbb{D}\dot\varepsilon_\tau{:}\dot\varepsilon_\tau \qquad \text{strongly in } L^\infty(I;L^2(\Omega)), \tag{5.11a}$$
$$\delta_S^*(\dot\pi_{\tau h}) \to \delta_S^*(\dot\pi_\tau) \qquad \text{strongly in } L^\infty(I;L^2(\Omega)) \tag{5.11b}$$

because $\tau > 0$ is considered fixed. For (5.11a), we used that $\gamma \ge 4$. Altogether, we proved strong convergence of the heat sources in $L^1(Q)$.

Then, having such a strong convergence, we can perform the desired limit passage to the boundary-value problem (5.6a,b)–(5.7) formulated weakly completed with the limit of (3.9c), i.e. the variational inequality

$$
\begin{aligned}
&\int_Q \dot{w}_\tau\big(v-\bar{w}_\tau\big) + \big(\mathscr{K}(\bar{w}_\tau)\nabla\bar{w}_\tau\big)\cdot\nabla\big(v-\bar{w}_\tau\big)\,\mathrm{d}x\mathrm{d}t\\
&\quad\ge \int_Q \Big(\big(\mathbb{D}\dot{\varepsilon}_\tau+\mathbb{D}e(\dot{u}_{\mathrm{Dir},\tau})+\mathscr{B}(\bar{w}_\tau)\big):\big(\dot{\varepsilon}_\tau+e(\dot{u}_{\mathrm{Dir},\tau})\big)+\delta_S^*\big(\dot{\pi}_\tau\big)\Big)\big(v-\bar{w}_\tau\big)\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_\Sigma \bar{f}_\tau\big(v-\bar{w}_\tau\big)\,\mathrm{d}S\mathrm{d}t \qquad \forall v\in L^1(I;W^{1,2}(\Omega)),\quad v(\cdot)\ge 0\ \text{ a.e. on } Q,
\end{aligned}
\tag{5.12}
$$

In particular, the limit passage from (3.9a) to (5.6a) uses also the approximation property (5.4a) while (5.4b) was used for (3.9b) $\to$ (5.6b).

Step 4 – positivity of enthalpy w_τ: Eventually, we prove positivity of the enthalpy. We adapt a comparison argument from [14, Section 4.2.1] to the time-discrete setting, improving thus also [25]. Written (5.12) in the classical form, we obtain the estimate

$$
\begin{aligned}
\dot{w}_\tau-\mathrm{div}\big(\mathscr{K}(\bar{w}_\tau)\nabla\bar{w}_\tau\big) &= \delta_S^*\big(\dot{\pi}_\tau\big)+\big(\mathbb{D}\dot{\varepsilon}_\tau+\mathbb{D}e(\dot{u}_{\mathrm{Dir},\tau})+\mathscr{B}(\bar{w}_\tau)\big):\big(\dot{\varepsilon}_\tau+e(\dot{u}_{\mathrm{Dir},\tau})\big)+\bar{r}_\tau\\
&\ge \mathsf{d}\big|\dot{\varepsilon}_\tau+e(\dot{u}_{\mathrm{Dir},\tau})\big|^2+\mathscr{B}(\bar{w}_\tau):\big(\dot{\varepsilon}_\tau+e(\dot{u}_{\mathrm{Dir},\tau})\big)\\
&\ge \frac{\mathsf{d}}{2}\big|\dot{\varepsilon}_\tau+e(\dot{u}_{\mathrm{Dir},\tau})\big|^2-\frac{1}{2\mathsf{d}}|\mathscr{B}(\bar{w}_\tau)|^2 \ge -\frac{\omega^2|\mathbb{B}|^2}{2\mathsf{d}c_0^2}\bar{w}_\tau^2,
\end{aligned}
\tag{5.13}
$$

where $\bar{r}_\tau\ge 0$ is the "reaction" multiplier to the constraint $w_\tau^k\ge 0$ involved in the limit variational inequality arising from (3.6c) and $\mathsf{d}>0$ is as in (5.9). The equality in (5.13) is just (5.12) written in the classical form in the sense of distributions. Note that we used also the first inequality in (3.15) from which $\mathscr{B}(w)\le|\mathbb{B}|\omega|w|/c_0$ follows. We compare (5.13) with the solution to the difference equation

$$
\mathrm{D}_t\chi_\tau^k = -\frac{\omega^2|\mathbb{B}|^2}{2\mathsf{d}c_0^2}(\chi_\tau^k)^2 \qquad \forall\, k=1,\dots,K_\tau,
\tag{5.14}
$$

to be solved recursively starting from the initial datum $\chi_0 = w_{0,\min} > 0$ with $w_{0,\min}$ from (5.5b). In fact, this is an implicit discretization of the Riccati ordinary-differential equation $\dot{\chi}+\omega^2|\mathbb{B}|^2\chi^2/(2\mathsf{d}c_0^2)=0$ which, for $\chi(0)=w_{0,\min}>0$, gives a sub-solution of the (continuous) heat equation. This initial-value problem has the solution $\chi(t)=2\mathsf{d}c_0^2/(\omega^2|\mathbb{B}|^2t+2\mathsf{d}c_0^2/w_{0,\min})$. The implicit discretization of this decaying convex solution to the mentioned Riccati equation is always above in the sense $\chi_\tau^k>\chi(k\tau)$. Thus we have

$$
\chi_\tau^k>\chi^*:=\min_{t\in[0,T]}\chi(t)=\frac{2\mathsf{d}c_0^2}{\omega^2|\mathbb{B}|^2T+2\mathsf{d}c_0^2/w_{0,\min}}>0 \qquad \forall\, k=1,\dots,K_\tau.
\tag{5.15}
$$

We subtract (5.13) from (5.14) written as $\dot{\chi}_\tau=-\omega^2|\mathbb{B}|^2\bar{\chi}_\tau^2/(2\mathsf{d}c_0^2)$ the piecewise affine χ_τ taking values χ_τ^k at $t=k\tau$ and the corresponding piecewise constant $\bar{\chi}_\tau$. Both χ_τ and $\bar{\chi}_\tau$ are considered spatially constant. We then make the test by $(\bar{w}_\tau-\bar{\chi}_\tau)^-\le 0$, integrate over Ω at each $t\in I$, use Green's formula, and exploit the fact that $\bar{f}_\tau\ge 0$

a.e. in Σ. Thus, using also convexity of $((\cdot)^-)^2$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big((w_\tau{-}\chi_\tau)^-\big)^2\mathrm{d}x \le \int_\Omega (\bar w_\tau{-}\bar\chi_\tau)^-(\dot w_\tau{-}\dot\chi_\tau)\mathrm{d}x$$
$$\le \int_\Omega \frac{\omega^2|\mathbb{B}|^2}{2\mathsf{d}c_0^2}\big(\bar w_\tau^2{-}\bar\chi_\tau^2\big)\big(\bar w_\tau{-}\bar\chi_\tau\big)^-$$
$$-\mathscr{K}(\bar w_\tau)\nabla\bar w_\tau{\cdot}\nabla\big(\bar w_\tau{-}\bar\chi_\tau\big)^-\,\mathrm{d}x - \int_\Gamma \bar f_\tau\big(\bar w_\tau{-}\bar\chi_\tau\big)^-\mathrm{d}S \le 0, \tag{5.16}$$

where the last inequality also due to

$$\mathscr{K}(\bar w_\tau)\nabla\bar w_\tau{\cdot}\nabla\big(\bar w_\tau{-}\bar\chi_\tau\big)^- = \mathscr{K}(\bar w_\tau)\nabla(\bar w_\tau{-}\bar\chi_\tau){\cdot}\nabla\big(\bar w_\tau{-}\bar\chi_\tau\big)^-$$
$$= \mathscr{K}(\bar w_\tau)\nabla(\bar w_\tau{-}\bar\chi_\tau)^-{\cdot}\nabla\big(\bar w_\tau{-}\bar\chi_\tau\big)^- \ge 0 \tag{5.17}$$

a.e. on Ω, due to $w_\tau \ge 0$, and also due to (5.5a). Realizing the initial condition $(w_\tau(0){-}\chi_\tau(0))^- = (w_0{-}w_{0,\min})^- = 0$ due to (5.5b), we easily conclude that $(w_\tau - \chi_\tau)^- = 0$ a.e. on Q, whence $w_\tau \ge \chi_\tau \ge \chi^* > 0$ a.e. in Q. This shows, in particular, that the constraint $\bar w_\tau \ge 0$ in the variational inequality (5.12) is never active and thus the limit problem is an equality, as indeed formulated in (5.6c). $\Box$

The following Propositions 5.2–5.4 have essentially been proved in [30, Prop.2-3 and Rem.5]. For completeness, let us very briefly summarize the results and sketch basic ideas.

Proposition 5.2 (Uniform a-priori estimates for (5.6))**.** *Let (3.10), (3.11), and (5.5) hold, $\gamma > \max(4, \frac{2\omega}{\omega-1})$, and, in addition, let the exponent ω from (3.10b) satisfy*

$$\omega > \frac{2d}{d{+}2}. \tag{5.18}$$

Then, for some C and C_r, it holds

$$\big\|u_\tau\big\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^d))} \le C \quad \text{if } \varrho > 0, \tag{5.19a}$$
$$\big\|e(u_\tau)\big\|_{W^{1,1}(I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))} \le C, \tag{5.19b}$$
$$\big\|\mathrm{div}\,u_\tau\big\|_{W^{1,2}(I;L^2(\Omega))} \le C, \tag{5.19c}$$
$$\big\|\pi_\tau\big\|_{W^{1,1}(I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))} \le C, \tag{5.19d}$$
$$\big\|e(u_\tau) - \pi_\tau\big\|_{W^{1,2}(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))} \le C, \tag{5.19e}$$
$$\big\|\bar w_\tau\big\|_{L^\infty(I;L^1(\Omega))\,\cap\,L^r(I;W^{1,r}(\Omega))} \le C_r \quad \text{with any } 1 \le r < \tfrac{d+2}{d+1}, \tag{5.19f}$$
$$\big\|\dot w_\tau\big\|_{L^1(I;W^{1+d,2}(\Omega)^*)} \le C, \tag{5.19g}$$
$$\big\|\sigma_\tau\big\|_{L^2(Q;\mathbb{R}^{d\times d}_{\rm sym})} \le C, \tag{5.19h}$$
$$\big\|e(\dot u_\tau) - \dot\pi_\tau\big\|_{L^\gamma(Q;\mathbb{R}^{d\times d}_{\rm sym})} \le C\tau^{-1/\gamma}, \tag{5.19i}$$
$$\big\|\bar\pi_\tau\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))} \le C\eta(\tau)^{-1/2}. \tag{5.19j}$$

In particular, if $\varrho > 0$, the velocity $\dot u_\tau$ is bounded in the Banach space $\mathscr{V}$ from (5.3).

Sketch of the proof. First a-priori estimates, namely (5.19a), (5.19j), and the L^∞-part of (5.19f), can directly be inherited from (3.12). If $\omega \ge 2$, from (3.14), we can also inherit the estimates (5.19b), (5.19d), and (5.19e). Here we will get them even for smaller ω as specified in (5.18).

Then one uses the L^1-theory for the evolutionary heat equation [5, 6] based on the test by $1-1/(1+\bar{w}_\tau)^\delta$, $\delta>0$, combined with the interpolation of the adiabatic term by using repeatedly the Gagliardo-Nirenberg inequality as in [27, 29], which eventually allows us to bound the dissipation, yielding (5.19d) and (5.19e), and to bound the enthalpy gradient, yielding the second part of (5.19f). ¿From (5.19d) and (5.19e) one gets also (5.19b). Moreover, $\mathscr{B}(\bar{w}_\tau)$ is shown bounded in $L^2(Q)$ and then from (5.19e) we obtain (5.19h). As $\mathrm{tr}\,\dot{\pi}=0$, (5.19e) implies that $\mathrm{div}\,\dot{u}_\tau=\mathrm{tr}\,e(\dot{u}_\tau)=\mathrm{tr}(e(\dot{u}_\tau){-}\dot{\pi})=\mathrm{tr}\,\dot{\varepsilon}_\tau$ bounded in $L^2(Q)$, i.e. (5.19c). The boundedness of $\dot{u}_\tau$ in $\mathscr{V}$ follows from (5.19a-c). $\square$

It remains to carry out a final limit passage with $\tau\to0$ towards the continuous model. This requires a suitable definition of a weak-type solution. The following definition of a certain sort of a weak solution has been devised in [29], based on the concept of so-called energetic solution invented by Mielke at al. [15, 21, 23, 24] for the theory of rate independent processes and adapted also for the coupling with viscous/inertial effects in [28]. Here still further adaptations to cope with concentrations of strains is needed and implemented in the following definition; for a more detailed discussion especially about the semistability (5.21d) we refer to [30].

Definition 5.3. ***(Energetic solution.)*** *Assuming (4.1a) and (3.11), we call a triple (u,π,w) with*

$$u\in B(I;\mathrm{BD}(\Omega;\mathbb{R}^d)), \tag{5.20a}$$
$$\varepsilon=e(u){-}\pi\in W^{1,2}(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{5.20b}$$
$$\dot{u}\in\mathscr{V}\quad\textit{from } (5.3), \tag{5.20c}$$
$$\pi\in B(\bar{I};W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))\cap\mathrm{BV}(\bar{I};L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})), \tag{5.20d}$$
$$w\in L^r(I;W^{1,r}(\Omega))\cap L^\infty(I;L^1(\Omega))\cap B_{\mathrm{w}*}(\bar{I};\mathscr{M}(\bar{\Omega})), \tag{5.20e}$$
$$\dot{w}\in\mathscr{M}(\bar{I};W^{1+d,2}(\Omega)^*) \tag{5.20f}$$

with any $1\le r<\frac{d+2}{d+1}$ an energetic solution to (2.15) with the initial/boundary conditions (2.17) and (3.4) if the following five conditions hold:

(i) *the weakly formulated momentum-equilibrium equation (2.15a-c) with (2.16a,b) holds, i.e. for all $v{\in}C^1(\bar{Q};\mathbb{R}^d)$ such that $v(t,\cdot)|_{\Gamma_{\mathrm{Dir}}}{=}0$ for all t and $v(T,\cdot){=}0$,*

$$\int_Q\Big(\mathbb{D}\dot{\varepsilon}{+}\mathbb{C}\varepsilon{-}\mathscr{B}(w)\Big){:}e(v)-\varrho\dot{u}{\cdot}\dot{v}\,\mathrm{d}x\mathrm{d}t$$
$$=\int_\Omega\varrho\dot{u}_0{\cdot}v(0)\,\mathrm{d}x-\int_Q\Big(\mathbb{D}\varepsilon(\dot{u}_{\mathrm{Dir}}){+}\mathbb{C}\varepsilon(u_{\mathrm{Dir}})\Big){:}e(v)+\varrho\,\ddot{u}_{\mathrm{Dir}}{\cdot}v\,\mathrm{d}x\mathrm{d}t, \tag{5.21a}$$

(ii) *the weakly formulated enthalpy equation (2.15e) with (3.4c) holds, i.e. for all $v{\in}C^1(\bar{Q})$ with $v(T)=0$,*

$$\int_Q\mathscr{K}(w)\nabla w{\cdot}\nabla v-w\dot{v}-\big(\mathscr{B}(w)+\mathbb{D}\dot{\varepsilon}+\mathbb{D}\varepsilon(\dot{u}_{\mathrm{Dir}})\big){:}\big(\dot{\varepsilon}{+}\varepsilon(\dot{u}_{\mathrm{Dir}})\big)v\,\mathrm{d}x\mathrm{d}t$$
$$=\int_{\bar{Q}}v\,\big|\dot{\pi}\big|_S(\mathrm{d}x\mathrm{d}t)+\int_\Omega w_0v(0)\,\mathrm{d}x+\int_\Sigma fv\,\mathrm{d}S\mathrm{d}t, \tag{5.21b}$$

(iii) *the total energy equality holds:*

$$\int_\Omega \frac{\varrho}{2}|\dot{u}(T)|^2 + \frac{1}{2}\mathbb{C}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x + \int_{\bar\Omega} w(T,\mathrm{d}x) = \int_\Omega \frac{\varrho}{2}|\dot{u}_0|^2 + \frac{1}{2}\mathbb{C}\varepsilon_0{:}\varepsilon_0\,\mathrm{d}x$$
$$+\int_\Omega w_0\,\mathrm{d}x + \int_\Sigma f\,\mathrm{d}S\mathrm{d}t + \int_Q \mathbb{D}e(\dot{u}_{\mathrm{Dir}}){:}e(\dot{u}_{\mathrm{Dir}}) + \mathbb{C}e(u_{\mathrm{Dir}}){:}\dot{\varepsilon} - \varrho\,\ddot{u}_{\mathrm{Dir}}{\cdot}\dot{u}\,\mathrm{d}x\mathrm{d}t, \quad (5.21\mathrm{c})$$

(iv) *the "semistability" holds for any $\tilde{u}\in \mathrm{BD}(\Omega;\mathbb{R}^d)$ and $\tilde\pi \in \mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that $\tilde\varepsilon := e(\tilde u){-}\tilde\pi \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and for a.a. $t\in[0,T]$, i.e.*

$$\int_\Omega \frac{1}{2}\mathbb{C}\varepsilon(t){:}\varepsilon(t) + \mathfrak{s}(t){:}\varepsilon(t)\mathrm{d}x \le \int_\Omega \frac{1}{2}\mathbb{C}\tilde\varepsilon{:}\tilde\varepsilon + \mathfrak{s}(t){:}\tilde\varepsilon\mathrm{d}x + \int_{\bar\Omega}\delta_S^*(\cdot)\big[\tilde\pi - \pi(t)\big](\mathrm{d}x), \quad (5.21\mathrm{d})$$

with the "partial stress" $\mathfrak{s}(t) := \mathbb{D}\dot{\varepsilon}(t) - \mathscr{B}(w(t)) + \sigma_{\mathrm{Dir}}(t),$ (5.21e)

(v) *the initial conditions $u(0)=u_0$ and $\pi(0)=\pi_0$ hold.*

In (5.21b), $|\dot\pi|_S \in \mathscr{M}(\bar Q)$ denotes the total variation of the measure $\dot\pi \in \mathscr{M}(\bar Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ with respect to δ_S^*; in case of (4.1d), we have just $|\dot\pi|_S = \sigma_{\mathrm{yield}}|\dot\pi|$ with $|\dot\pi|$ the standard total variation, while in general $|\dot\pi|_S$ is defined by prescribing its values for every closed set of the type $A:=[t_1,t_2]{\times}B$ with B a Borel subset of $\bar\Omega$ by

$$\big|\dot\pi\big|_S(A) := \sup\sum_{i=1}^k \int_\Omega \delta_S^*\big(\pi(s_i,x) - \pi(s_{i-1},x)\big)\,\mathrm{d}x \quad (5.22)$$

where the supremum is taken over all partitions $t_1{\le}s_0{<}...{<}s_k{\le}t_2$, $k{\in}\mathbb{N}$.

It is not surprising that we will be able to prove existence of an energetic solution only for suitably qualified initial conditions (u_0,π_0,w_0) that can be interpreted as some sort of a "gentle" start. Standardly, we assume that the triple (u_0,π_0,w_0) is semistable at $t=0$, which needs here, however, a special care because $\dot\varepsilon(t)$ occurring in $\mathfrak{s}(t)$ in (5.21d) is well defined only for a.a. t and not just for $t=0$. The simplest option [30], consistent also with Section 4, how to overcome this trouble is to assume $\pi_0=0$ and to guarantee even $\pi(t,\cdot)=0$ and also $u(\cdot,x)$ constant for all $t\in[0,t_0]$ with some $t_0>0$ because then simply $\dot\varepsilon(0)=0$. This happens if u minimizes the stored energy, and temperature is equilibrated. We can equally require that such an extension exist on $[-t_0,0]$ and prescribe the initial condition (without any ambitions of generality) and their regularization used in (3.5) as

$$u_0 = u_{0,\tau} \in W^{1,\infty}(\Omega;\mathbb{R}^d), \qquad \pi_0 = \pi_{0,\tau}, \qquad w_0 = w_{0,\tau} \ge 0 \text{ constant on } \Omega, \quad (5.23\mathrm{a})$$
$$u_0 \ \text{ minimizes } \ u\mapsto \int_\Omega \frac{1}{2}\mathbb{C}e(u){:}e(u) + (\mathscr{B}(w_0) - \sigma_{\mathrm{Dir}}(0)){:}e(u)\,\mathrm{d}x. \quad (5.23\mathrm{b})$$

The other, rather implicit and technical assumption formulated in [2] is that, for all $t\in[0,T]$ and all $(u,\pi)\in\mathfrak{Q}$ the following holds:

$$\left.\begin{array}{l}\text{If } E(t,u,\pi) \le E(t,u{+}\hat u,\pi{+}\hat\pi) + R(\hat\pi) \text{ for all } (\hat u,\hat\pi)\in\mathfrak{Q}_0,\\ \text{then } E(t,u,\pi) \le E(t,u{+}\tilde u,\pi{+}\tilde\pi) + R(\tilde\pi) \text{ for all } (\tilde u,\tilde\pi)\in\mathfrak{Q},\end{array}\right\} \quad (5.24)$$

where we abbreviated $E(t,u,\pi) := \int_\Omega(\frac{1}{2}\mathbb{C}\varepsilon{+}\mathfrak{s}(t)){:}\varepsilon\,\mathrm{d}x$ with $\varepsilon = e(u){-}\pi$ and $R(\dot\pi) := \int_{\Omega\cup\Gamma_{\mathrm{Dir}}}\delta_S^*(\cdot)\dot\pi(\mathrm{d}x)$, and where $\mathfrak{Q}$ is from (5.2) and

$$\mathfrak{Q}_0 := \big\{(u,\pi)\in W^{1,1}(\Omega;\mathbb{R}^d)\times L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}});\ u{=}0 \text{ on } \Gamma_{\mathrm{Dir}},\ e(u){-}\pi{\in}L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\big\}.$$

It was proved essentially in [11] (cf. [2, Prop. 3.3]) that the condition (5.24) is satisfied for a certain special $\mathbb{C}$ (in particular for $\mathbb{C}$ from (4.1a)) and is Γ and the boundary of $\Gamma_{\rm Dir}$ is smooth. It was conjectured in [2] that (5.24) holds more generally.

As far as the (regularized) loading concerns, we assume

$$\tilde{f}_\tau{\in}L^\infty(\Sigma),\quad \tilde{f}_\tau\ge0,\quad \lim_{\tau\downarrow0}\sqrt{\tau}\big\|\tilde{f}_\tau\big\|_{L^2(I;L^{4/3}(\Gamma))}=0,\quad \lim_{\tau\downarrow0}\tilde{f}_\tau=f\ \text{ in } L^1(\Sigma). \tag{5.25}$$

Proposition 5.4 (Convergence for $\tau\downarrow0$)**.** *Let $d\le3$, let (3.10), (3.11), (5.5), (5.18), and (5.23)–(5.25) hold, and $\gamma>\max(4,\frac{2\omega}{\omega-1})$,. Then:*

(i) there is a subsequence of $\{(u_\tau,\pi_\tau,w_\tau)\}_{\tau>0}$ and (u,π,w) such that

$$(\bar{u}_\tau,\bar{\pi}_\tau)\to(u,\pi)\qquad \textit{weakly* in } L^\infty(I;{\rm BD}(\Omega;\mathbb{R}^d){\times}\mathscr{M}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm sym})), \tag{5.26a}$$

$$\varepsilon_\tau=e(u_\tau){-}\pi_\tau\to\varepsilon=e(u){-}\pi\quad \textit{weakly in } W^{1,2}(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})), \tag{5.26b}$$

$$\bar{w}_\tau\to w\qquad \textit{strongly in } L^r(Q),\quad 1\le r<(d{+}2)/d, \tag{5.26c}$$

$$\bar{w}_\tau(t)\to w(t)\qquad \textit{weakly* in } \mathscr{M}(\bar\Omega)\qquad \forall t\in\bar{I}. \tag{5.26d}$$

$$\bar{\pi}_\tau(t)\to \pi(t)\qquad \textit{weakly* in } \mathscr{M}(\bar\Omega)\qquad \forall t\in\bar{I}. \tag{5.26e}$$

$${\rm div}\,\dot{u}_\tau\to{\rm div}\,\dot{u}\qquad \textit{weakly in } L^2(Q). \tag{5.26f}$$

(ii) Moreover, if $\varrho=0$ (=the quasi-static case), then any (u,π,w) obtained in this way is an energetic solution of the problem (2.15) with the initial/boundary conditions (2.17) and (3.4) according Definition 5.3 and the stresses converges strongly, i.e.

$$\sigma_\tau\to\sigma=\mathbb{D}(e(\dot{u}){-}\dot{\pi}){+}\mathbb{C}(e(u){-}\pi){-}\mathscr{B}(w)\quad \textit{strongly in } L^2(Q;\mathbb{R}^{d\times d}_{\rm sym}), \tag{5.27}$$

(iii) Also, if $\varrho>0$ and ${\rm dev}\,\sigma_{\rm Dir}\in L^\infty(Q;\mathbb{R}^{d\times d}_{\rm dev})$ is assumed, and, in addition, $e(\dot{u})$ happens to be absolutely continuous, i.e. $e(\dot{u})\in L^1(Q;\mathbb{R}^{d\times d}_{\rm sym})$, then this triple (u,π,w) is an energetic solution to (2.15)-(2.17)-(3.4) and (5.27) holds, too.

The proof of the points (i)–(ii) can be found in [30]. The point (iii) can be proved by modification in the spirit of [30, Remark 5], the only important ingredients being the by-part integration formula

$$\begin{aligned}\int_\Omega\frac{\varrho}{2}|\dot{u}(T)|^2{\rm d}x-\int_\Omega\frac{\varrho}{2}|\dot{u}(0)|^2{\rm d}x=\big\langle\varrho\ddot{u},\dot{u}\big\rangle=\int_Q f_{\rm Dir}{\cdot}\dot{u}-\sigma_\tau{:}e(\dot{u})\,{\rm d}x{\rm d}t\\ =\int_Q f_{\rm Dir}{\cdot}\dot{u}-{\rm tr}(\sigma^{\rm s}){\rm div}\,\dot{u}-{\rm dev}\,\sigma{:}{\rm dev}\,e(\dot{u})\,{\rm d}x{\rm d}t\end{aligned} \tag{5.28}$$

where we used the decomposition of the stress to the spherical and the deviatoric parts, i.e. $\sigma_\tau=\sigma_\tau^{\rm s}+{\rm dev}\,\sigma_\tau$ and where $\langle\cdot,\cdot\rangle$ refers to the duality pairing on $\mathscr{V}^*\times\mathscr{V}$ from (5.3). Here, it is important that ${\rm div}\,\dot{u}\in L^2(Q)$ has been inherited from (5.19c) and that ${\rm dev}(\sigma{+}\sigma_{\rm Dir})\in S$ a.e. has already been proved so that ${\rm dev}\,\sigma\in L^\infty(Q;\mathbb{R}^{d\times d}_{\rm dev})$ since ${\rm dev}\,\sigma_{\rm Dir}\in L^\infty(Q;\mathbb{R}^{d\times d}_{\rm dev})$ is assumed. Standardly, (5.28) is first proved for mollified u's by conventional calculus and then by letting these mollifiers converge to the limit. Usually, one can rely on the strong convergence but it does not work if the original distribution that is mollified were not any L^1-function but only a measure. This is why we had to assume $e(\dot{u})\in L^1(Q;\mathbb{R}^{d\times d}_{\rm sym})$. This assumption $e(\dot{u})\in L^1(Q;\mathbb{R}^{d\times d}_{\rm sym})$ together with we have $\dot{u}\in\mathscr{V}$ give the sense to the integral on the right-hand side of (5.28), which allows the rigorous proof of (5.28). Let us emphasize that (5.28) is needed to keep the energy balance which is further needed to show (5.27) which is eventually vitally needed to converge the dissipative heat in the enthalpy equation.

Remark 5.5 (Merging space and time convergence)**.** The joint convergence for $h{\downarrow}0$ and $\tau{\downarrow}0$ is not obvious, unfortunately. Following [4, Corollary 4.8], one can at least prove existence of a function $H : \mathbb{R}^+ \to \mathbb{R}^+$ such that every subsequence in the set $\{(u_{\tau h}, \pi_{\tau h}, w_{\tau h})\}_{h>0,\tau>0,h\le H(\tau)}$ which converges for $h{\downarrow}0$ and $\tau{\downarrow}0$ weakly* in the topologies indicated in (3.12a-c) and, if $\omega \ge 2$, also in (3.14) yields, as its limit (u, π, η, w), an energetic solution according Definition 5.3. Yet, any more specific form of H seems difficult to obtain.

Remark 5.6 (Dynamical case)**.** The assumption of non-concentration of strain rate in Proposition 5.4 (or even rather only of dev $e(\dot u) \in L^1(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$, which would suffice for (5.28)) is hardly to be ensured a-priori; note that, by (5.19b) and (5.26b), we have only $\dot\pi \in \mathscr{M}(\bar Q; \mathbb{R}^{d\times d}_{\rm sym})$ and $e(\dot u) - \dot\pi \in L^2(Q; \mathbb{R}^{d\times d}_{\rm sym})$ so that $e(\dot u) \in \mathscr{M}(\bar Q; \mathbb{R}^{d\times d}_{\rm sym})$ in general. This makes the convergence assertion in the dynamical case rather vague. It is not clear whether it is “only” a mathematical difficulty or whether it is related with some physical phenomenon of dissipation of energy during impacts of elastic waves on shear bands, similarly like it may possibly happen during impacts on a unilateral Signorini boundary contact (which remains for a long time an open difficult problem).

Acknowledgment: A partial support from the grants A 100750802 (GA AV ČR), 201/09/0917 and 201/10/0357 (GA ČR), LC 06052 (MŠMT ČR) and from the research plan AV0Z20760514 “Complex dynamical systems in thermodynamics, mechanics of fluids and solids” (ČR), and CENTEM project no. CZ.1.05/21.00/03.0088 (cofounded from ERDF within the OP RDI programme, MŠMT ČR) is acknowledged. Besides, T.R. also particularly acknowledges the hospitality of SFB 611 “Singular phenomena and scaling in mathematical models” of the University of Bonn.

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