LOCAL COARSENING OF TRIANGULATIONS CREATED BY BISECTIONS

SÖREN BARTELS AND PATRICK SCHREIER

Abstract. A simple criterion that allows the efficient local coarsening of triangulations created by bisections is devised and analyzed. Under a mild condition on the initial triangulation the proposed criterion allows to gradually reverse the entire refinement without employing its history explicitly. Numerical experiments underline the efficiency of the resulting algorithm.

1. Introduction

Adaptive mesh refinement is a popular tool for the efficient discretization of partial differential equations and is now well understood for linear elliptic problems [Stev07, CKNS08]. For time-dependent and nonlinear problems coarsening is an important ingredient to develop efficient adaptive strategies, e.g., when interfaces or singularities advance in time or during an iterative method the refined region of the triangulation should follow the interface or singularity. Examples of partial differential equations for which such phenomena occur are phase field models and geometric evolution problems.

Most available coarsening strategies store the complete refinement history of the grid, i.e., the binary tree of bisections, explicitly and then reverse entire refinement steps in order to guarantee conformity of the resulting coarsened triangulation. A local coarsening strategy that allows to remove single nodes of a two-dimensional triangulation has been devised, analyzed, and successfully tested in [CZ07]. The criterion states that configurations of four or two neighbouring triangles around one node can be coarsened if the node is the newest vertex for all of those triangles. This coarsening criterion characterizes patches of nodes that result from the compatible bisection of two neighbouring triangles that share their refinement edge or of one triangle whose refinement edge belongs to the boundary. Related ideas for local coarsening have also been outlined in [Kos94].

In this article we demonstrate that the same criterion can be applied in any dimension. The generalized criterion states that a node can be coarsened locally if and only if it is the newest vertex of all elements it belongs to and is not a node of the initial triangulation. Such nodes are exactly those nodes that are created by compatible bisections of edge patches. By this we mean the bisection of an edge in a triangulation which is the refinement edge of all elements it belongs to. For the criterion to be applicable we assume that the given triangulation is obtained from an initial one by a successive bisection of compatible edge patches. Employing a recursive refinement algorithm from [Kos94, Stev08] we show that triangulations obtained with bisection algorithms that are equivalent to completion strategies satisfy this requirement under mild conditions on the initial triangulation. We thereby give a more constructive and general proof of the assertions in [CZ07].

The outline of this paper is as follows. We povide equivalent characterizations of nodes that can be coarsened locally in Section 2, discuss a refinement algorithm that leads to triangulations which can be entirely coarsened in Section 3, specify an appropriate choice of refinement edges in Section 4, and illustrate the performance of the coarsening criterion in numerical experiments in Section 5.

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2. Compatible edge patch bisection and local coarsening

Let $\mathcal{T}$ be a conforming triangulation of a polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$ consisting of simplices called *elements*. Here, conforming means that the boundary of Ω is matched exactly by subsimplices of elements and the intersection of two disjoint elements is either empty or an entire subsimplex of both elements. We let $\mathcal{E}(\mathcal{T})$ denote the set of all one-dimensional subsimplices in $\mathcal{T}$ called *edges* and define the *edge patch* $\mathcal{T}_E$ of an edge $E \in \mathcal{E}(\mathcal{T})$ as

$$\mathcal{T}_E := \{T \in \mathcal{T} : E \subset T\}.$$

Similarly, $\mathcal{N}(\mathcal{T})$ is the set of *nodes* (vertices of simplices) in $\mathcal{T}$ and the *node patch* $\mathcal{T}_z$ of a node $z \in \mathcal{N}(\mathcal{T})$ is defined as

$$\mathcal{T}_z := \{T \in \mathcal{T} : z \in T\}.$$

We assume that for any triangulation $\mathcal{T}$ we are given a function $R : \mathcal{T} \to \mathcal{E}(\mathcal{T})$ which associates to each element $T \in \mathcal{T}$ its *refinement edge* $R(T) \in \mathcal{E}(\mathcal{T})$. The *sons* $s_1(T)$ and $s_2(T)$ are the simplices that result from bisecting T along its refinement edge $R(T) \in \mathcal{E}(\mathcal{T})$. The newly created vertex on the edge $R(T)$ is called the *newest vertex of* $s_1(T)$ *and* $s_2(T)$. The *father* of a simplex T' is the element $T = f(T')$ such that either $T' = s_1(T)$ or $T' = s_2(T)$.

Definition 2.1. *We say that* the triangulation $\mathcal{T}_f$ results from the triangulation $\mathcal{T}_c$ by compatible bisection of the edge patch $\mathcal{T}_{f,E}$ *if $E \in \mathcal{E}(\mathcal{T}_c)$ is the refinement edge for all $T \in \mathcal{T}_{c,E}$ and if*

$$\mathcal{T}_f = \big(\mathcal{T}_c \setminus \mathcal{T}_{c,E}\big) \cup \big\{s_j(T) : T \in \mathcal{T}_{c,E},\ j = 1, 2\big\}.$$

In the following we let $\mathcal{T}_0$ be a fixed conforming triangulation of Ω.

Definition 2.2. *The class* $\mathbb{T} = \mathbb{T}(\mathcal{T}_0)$ *of* conforming triangulations that result from $\mathcal{T}_0$ by successive compatible bisection of edge patches *is the set of triangulations for which there exists a sequence of triangulations $(\mathcal{T}_j)_{j=0}^{L}$ and edges $(E_j)_{j=0}^{L-1}$ such that $\mathcal{T}_L = \mathcal{T}_f$ and for $j = 0, 1, ..., L-1$ we have $E_j \in \mathcal{E}(\mathcal{T}_j)$ and $\mathcal{T}_{j+1}$ is obtained from $\mathcal{T}_j$ by compatible bisection of the edge patch $\mathcal{T}_{j,E_j}$.*

We will show that triangulations in $\mathbb{T}$ can be coarsened locally in the sense of the following definition.

Definition 2.3. *Given a triangulation $\mathcal{T}_f \in \mathbb{T}$ we say that* the node $z \in \mathcal{N}(\mathcal{T}_f) \setminus \mathcal{N}(\mathcal{T}_0)$ can be coarsened locally *if there exists a triangulation $\mathcal{T}_c \in \mathbb{T}$ and an edge $E \in \mathcal{E}(\mathcal{T}_c)$ such that $\mathcal{T}_f$ results from $\mathcal{T}_c$ by compatible bisection of the edge patch $\mathcal{T}_{f,E}$ and $\mathcal{N}(\mathcal{T}_f) = \mathcal{N}(\mathcal{T}_c) \cup \{z\}$.*

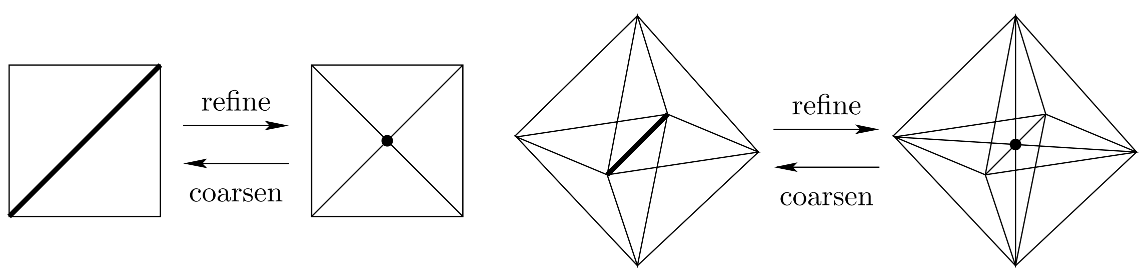

Figure 1. Compatible bisection of edge patches in two and three dimensions. All elements share the refinement edge (thick line) and the newly created vertex (filled circle) can be coarsened locally.

Proposition 2.4. *The node $z \in \mathcal{N}(\mathcal{T}_f) \setminus \mathcal{N}(\mathcal{T}_0)$ can be coarsened locally if and only if it is the newest vertex of all elements it belongs to.*

Proof. Suppose that $z \in \mathcal{N}(\mathcal{T}_f)$ can be coarsened locally, i.e., there exist $\mathcal{T}_c \in \mathbb{T}$ and $E \in \mathcal{E}(\mathcal{T}_c)$ such that $\mathcal{T}_f$ results from the triangulation $\mathcal{T}_c$ by compatible bisection of the edge patch $\mathcal{T}_{c,E}$ and z is newly created. Then z is the newest vertex of all elements it belongs to.

Assume now that $z \in \mathcal{N}(\mathcal{T}_f) \setminus \mathcal{N}(\mathcal{T}_0)$ is the newest vertex of all elements it belongs to. Since $z \notin \mathcal{N}(\mathcal{T}_0)$ and z is the newest vertex for each $T' \in \mathcal{T}_{f,z}$ we have that each such T' is the son of some simplex T, e.g., $T' = s_1(T)$. If $s_2(T) \notin \mathcal{T}_{f,z}$ then there exists a neighbouring element T'' to T' with $z \in T''$ which results from bisections of $s_2(T)$. But then z could not be the newest vertex of T''. We can thus define

$$\mathcal{T}_c := \big(\mathcal{T}_f \setminus \mathcal{T}_{f,z}\big) \cup \big\{f(T') : T' \in \mathcal{T}_{f,z}\big\}$$

and it follows that $\mathcal{T}_f$ results from $\mathcal{T}_c$ by a compatible bisection of the edge patch $\mathcal{T}_{c,E}$ where $E = R(f(T'))$ for an arbitrary element $T' \in \mathcal{T}_{f,z}$. It remains to be shown that $\mathcal{T}_c \in \mathbb{T}$. Since $\mathcal{T}_f \in \mathbb{T}$ there exist triangulations $(\mathcal{T}_j)_{j=0}^{L}$ and edges $(E_j)_{j=0}^{L-1}$ with $\mathcal{T}_L = \mathcal{T}_f$ and $\mathcal{T}_{j+1}$ is obtained from $\mathcal{T}_j$ by compatible bisection of the edge patch $\mathcal{T}_{j,E_j}$. We have $E = E_{j'}$ for some $0 \le j' \le L-1$ and since z is the newest vertex of all elements in $\mathcal{T}_{f,z}$ we have $\mathcal{T}_{f,z} \subset \mathcal{T}_k$ for $k = j'+1, ..., L$. We may therefore assume that $j' = L-1$, i.e., $E_{L-1} = E$ and $\mathcal{T}_{L-1} = \mathcal{T}_c$. Hence, $\mathcal{T}_c \in \mathbb{T}$. $\square$

Local coarsening of conforming refinements of arbitrary triangulations is not always possible. The triangulation shown in Figure 2 shows that this depends on the choice of refinement edges.

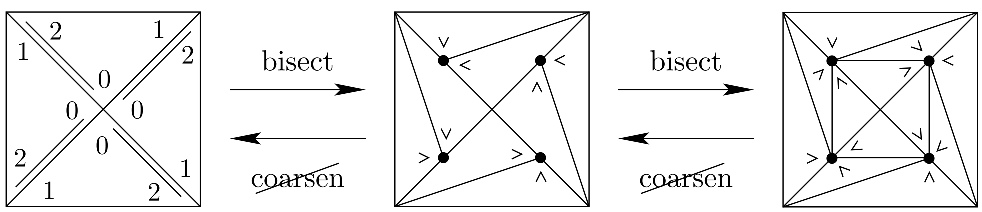

FIGURE 2. Triangulation $\mathcal{T}_0$ (left; refinement edges are indicated by bars) whose uniform refinement of level 1 is not conforming (middle). A completion strategy leads to a triangulation (right) which can not be coarsened locally, i.e., none of the newly created nodes (filled circles) is the newest vertex (indicated by arrows) of all elements it belongs to. The numbers in the left part define local orderings of the vertices of the elements.

3. Triangulations created by bisections

Under mild conditions on $\mathcal{T}_0$ and for an appropriate choice of refinement edges, cf. Lemma 3.1 below, the bisection algorithm of [Kos94, Stev08] specified below terminates and produces the smallest conforming refinement of the given triangulation $\mathcal{T}$ in which the element $T \in \mathcal{T}$ is refined. Given $T' \in \mathcal{T}$ the set $N(\mathcal{T}, T')$ consists of all elements $T'' \in \mathcal{T}$ that share a *side* (a $(d-1)$-dimensional subsimplex) with T' and contain the refinement edge $R(T')$ of T'; the element $T'' \in N(\mathcal{T}, T')$ is said to be *compatibly divisible with* T' if it has the same refinement edge as T', i.e., if $R(T'') = R(T')$. Note that the edge patch $\mathcal{T}_{R(T)}$ can be compatibly bisected if and only if for all $T' \in \mathcal{T}_{R(T)}$ and all $T'' \in N(\mathcal{T}, T')$ we have that $T'' \in \mathcal{T}_{R(T)}$ and T'' and T' are compatibly divisible.

function $\mathcal{T}' = \mathbf{refine}[\mathcal{T}, T]$
$K := \emptyset;\ F = \{T\}$
`do` $Fnew := \emptyset$
`forall` $T' \in F$ `do`
`forall` $T'' \in N(\mathcal{T}, T')$ *with* $T'' \notin F \cup K$ `do`
`if` T'' *compatibly divisible with* T'
$Fnew = Fnew \cup \{T''\}$
`else` $\mathcal{T} := \mathbf{refine}[\mathcal{T}, T'']$
add to $Fnew$ *the son of* T'' *that shares a side with* T'
`endif`
`endfor`
`endfor`
$K := K \cup F$
$F := Fnew$
`until` $F = \emptyset$
create $\mathcal{T}'$ *from* $\mathcal{T}$ *by simultaneously bisecting all* $T' \in K$

A call of the function **refine** bisects a compatible edge patch or recursively refines further elements until the edge patch $\widetilde{\mathcal{T}}_{R(T)}$ (with respect to the refined intermediate conforming triangulation) can be compatibly bisected. In two dimensions, the algorithm follows a chain of neighbouring elements connected by refinement edges until two neighbouring elements have the same refinement edge or the refinement edge belongs to the boundary. Then, the pair of compatibly divisible elements or the element at the boundary is bisected and recursively, the compatible pairs of previous elements in the chain and sons of bisected neighbours are refined, cf. Figure 3. In higher dimensions there are in general several elements that contain the refinement edge of a given element and the algorithm has to follow several chains which themselves may branch, cf. Figure 4.

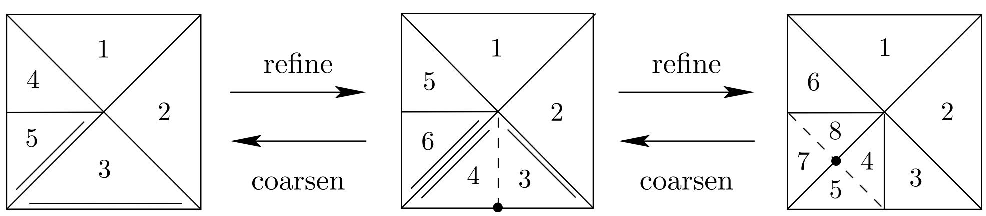

FIGURE 3. Elements 5 and 3 are not compatibly divisible (left) so that a recursive call of **refine** is required in order to refine element 5. After the compatible bisection of the refinement edge of element 3 (middle), a compatible bisection of the refinement edge of the initially marked element 5 is carried out. The refined triangulation (right) is a result of a sequence of compatible edge patch bisections.

In order to show that the triangulation $\mathcal{T}'$ and all intermediate triangulations, generated in the recursive execution of a call of **refine**, belong to $\mathbb{T}$ we need the following lemma. We say that the possibly nonconforming triangulation $\mathcal{T}$ is a *refinement* of $\mathcal{T}_0$ if it is obtained from $\mathcal{T}_0$ by repeated bisection of refinement edges. A refinement $\mathcal{T}$ of $\mathcal{T}_0$ is called a *uniform refinement* of $\mathcal{T}_0$ if it results from bisecting each element in $\mathcal{T}_0$ a fixed number of times, i.e., if all elements in $\mathcal{T}$ have the same refinement level.

Lemma 3.1 ([Stev08]). *Suppose that $\mathcal{T}_0$ and the function R are such that every uniform refinement of $\mathcal{T}_0$ is conforming. If $\mathcal{T}$ is a conforming refinement of $\mathcal{T}_0$ then for $T' \in \mathcal{T}$ and $T'' \in N(\mathcal{T}, T')$*

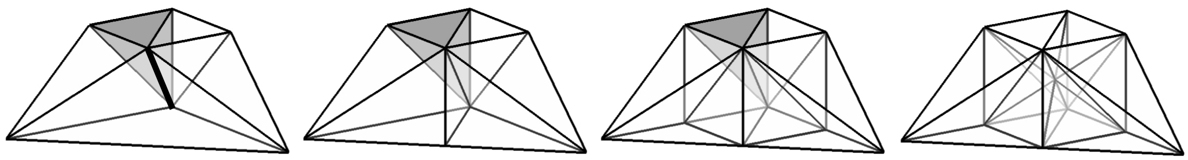

FIGURE 4. The refinement of T (shaded gray) requires recursive refinements of elements that share the refinement edge (thick line) with T. The newly created nodes can be coarsened locally in reverse order to their creation.

either T' and T'' are compatibly divisible or the son of T'' that shares a side with T' is compatibly divisible with T'.

Sufficient conditions on the triangulation $\mathcal{T}_0$ and an appropriate choice of the refinement edges that imply the conditions of the lemma are discussed in Section 4 below.

Proposition 3.2. *Suppose that the assumption of Lemma 3.1 is satisfied. Then, each call of the function* **refine** *with an element T and a triangulation $\mathcal{T}$ that is a conforming refinement of $\mathcal{T}_0$ leads to a compatible bisection of the edge patch $\widetilde{\mathcal{T}}_{R(T)}$ with respect to the current, possibly refined, conforming triangulation $\widetilde{\mathcal{T}}$. In particular, if $\mathcal{T}' =$ **refine**$[\mathcal{T}, T]$ then $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a successive compatible bisection of edge patches.*

Proof. Owing to the assumption the recursion of **refine** terminates, cf. [Stev08], and we argue by induction over the depth of the recursion. Suppose first that the call **refine**$[\mathcal{T}, T]$ does not lead to a recursion, i.e., no further calls of **refine**. Then, after termination of the `do` loop we have that K is the edge patch of the refinement edge of T, i.e., $K = \mathcal{T}_{R(T)}$, and $R(T)$ is the refinement edge of all elements in K. The last command in the function **refine** thus performs a compatible bisection of the edge patch $\mathcal{T}_{R(T)}$. Otherwise, if further, recursive calls of **refine** are required then the algorithm recursively bisects those neighbouring elements of T (elements that share a side with T) which are not compatibly divisible with T so that according to Lemma 3.1 the new neighbours are compatibly divisible with T. This procedure is repeated with the neighbours of the (new) neighbours of T until all neighbouring elements contained in the set K are compatibly divisible. Moreover, all elements in K have the same refinement edge and after termination K coincides with the edge patch $\widetilde{\mathcal{T}}_{R(T)}$ (with respect to the recursively refined triangulation $\widetilde{\mathcal{T}}$). ☐

The triangulation shown in the left part of Figure 2 shows that the recursion of the function **refine** may not terminate if the assumption of Lemma 3.1 is not satisfied, i.e., if a uniform refinement of $\mathcal{T}_0$ is not conforming.

4. CHOICE OF REFINEMENT EDGES

A *tagged element* $T = (z_0, ..., z_d)_\gamma$ is a pair of an ordered sequence $(z_0, .., z_d)$ of $d+1$ vectors in $\mathbb{R}^d$ such that $\operatorname{conv}\{z_0, ..., z_d\}$ is an element and a nonnegative integer $\gamma \in \{0, ..., d-1\}$ called *type* of the element. The refinement edge of a tagged element $T = (z_0, ..., z_d)_\gamma$ is the edge $\overline{z_0 z_d}$. The refinement edges of the two sons of a tagged element are then defined via the type γ of T by setting

$$s_1(T) := (z_0, y, z_1, ..., z_\gamma, z_{\gamma+1}, \dots, z_{d-1})_{\gamma+1 \bmod d}$$
$$s_2(T) := (z_d, y, z_1, ..., z_\gamma, z_{d-1}, \dots, z_{\gamma+1})_{\gamma+1 \bmod d}$$

If $\gamma = 0$ or $\gamma = d-1$ then the right-hand sides are to be understood as $s_1(T) = (z_0, y, z_1, ..., z_{d-1})_1$ and $s_2(T) = (z_d, y, z_{d-1}, ..., z_1)_1$ or $s_1(T) = (z_0, y, z_1, ..., z_{d-1})_0$ and $s_2(T) = (z_d, y, z_1, ..., z_{d-1})_0$, respectively. The *reflected* element $T_R := (z_d, ..., z_0)_\gamma$ has the same sons as $T = (z_0, ..., z_d)_\gamma$. Two neighbouring tagged elements T and T' are called *reflected neighbours* if the ordered sequences of T and T' or T_R and T' coincide on all but one position.

This bisection scheme was introduced and analyzed in [Mau95, Tra97] and it was shown that shape regularity is preserved. A motivation for this scheme is that for an initial partition of a cube into *Kuhn simplices* (cf. Definition 5.1 below) with assigned type 0, the scheme always bisects the longest edge of an element.

An appropriate local numbering and tagging of the elements in the initial triangulation $\mathcal{T}_0$ guarantees that the assumption of Lemma 3.1 is satisfied. In additon to the requirement that $\mathcal{T}_0$ is conforming, i.e., the intersection of two disjoint elements is either empty or a common subsimplex (i.e., a node, an edge, or a side), we assume that the elements in $\mathcal{T}_0$ are tagged elements with the following property:

(R) All tagged elements in $\mathcal{T}_0$ are of the same type γ. Two neighbouring tagged elements $T = (z_0, ..., z_d)_\gamma$ and $T' = (z'_0, ..., z'_d)_\gamma$ in $\mathcal{T}_0$ are reflected neighbours if $\overline{z_0 z_d}$ or $\overline{z'_0 z'_d}$ belongs to $T \cap T'$. Otherwise the pair of neighbouring children of T and T' are reflected neighbours.

For two-dimensional triangulations it is shown in [BDD04] that it is possible to choose an initial local numbering and tagging which implies condition (R). The triangulation shown in the left part of Figure 2 with the indicated orderings and with assigned types 0 shows that not every local numbering and tagging implies (R). For triangulations of three- or higher-dimensional domains a refinement of the initial triangulation may be necessary to define an appropriate local numbering and tagging, cf. [Kos94, Stev08] for details. If (R) is satisfied then the assumptions of Lemma 3.1 are fulfilled.

Theorem 4.1 ([Stev08])**.** *Suppose that $\mathcal{T}_0$ satisfies (R). Then every uniform refinement of $\mathcal{T}_0$ is conforming.*

5. Numerical Experiments

For our numerical experiments we employ triangulations that are assembled from scaled and translated copies of a partition $\mathcal{T}_{Kuhn}$ of the cube $C = (0,1)^d$ which satisfies condition (R).

Definition 5.1. *Given a permutation π of $\{1, ..., d\}$ the associated* tagged Kuhn simplex *is the tagged element $(z_0^\pi, ..., z_d^\pi)_0$, where $z_0^\pi = (0, ..., 0)$ and $z_k^\pi = \sum_{j=1}^k e_{\pi(j)}$ for $k = 1, ..., d$ and $\{e_1, ..., e_d\}$ denotes the canonical basis of $\mathbb{R}^d$. The triangulation $\mathcal{T}_{Kuhn}$ consists of all $n!$ tagged Kuhn simplices.*

Since $z_0^\pi = \underline{0} = (0, ..., 0)$ and $z_d^\pi = \underline{1} = (1, ..., 1)$ for every permutation π of $\{1, ..., d\}$ we have that the line segment $\overline{\underline{0}, \underline{1}}$ is the refinement edge of every element in $\mathcal{T}_{Kuhn}$. Hence, the triangulation $\mathcal{T}_{Kuhn}$ enables us to define triangulations $\mathcal{T}_0$ which satisfy condition (R) of domains $\Omega \subset \mathbb{R}^d$ that can be partitioned into transformed rectangles or parallelepipeds for $d = 2$ and $d = 3$, respectively. The triangulations $\mathcal{T}_{Kuhn}$ for $d = 2$ and $d = 3$ are shown in Figure 5.

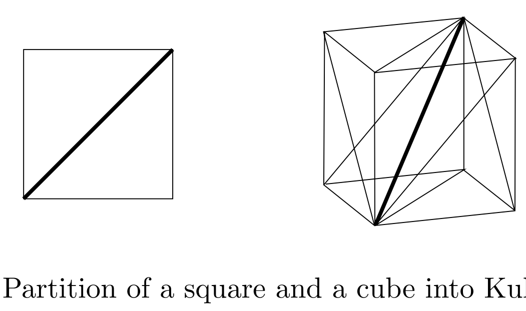

Figure 5. Partition of a square and a cube into Kuhn simplices.

Given a triangulation $\mathcal{T}$ and a set $\mathcal{M} \subseteq \mathcal{T}$ of elements marked for refinement a *refinement step* consists in repeatedly executing the function **refine** with elements in $\mathcal{M}$ until all elements

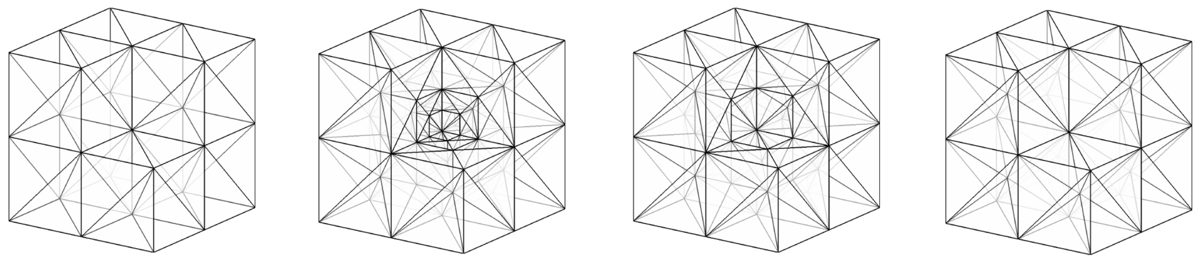

FIGURE 6. Local refinement and coarsening of the Fichera cube: partition into Kuhn simplices (upper left), triangulation after six refinement steps (upper right), and the triangulations after two and five coarsening steps (lower plots). A sixth coarsening step leads to the initial triangulation.

in $\mathcal{M}$ have been refined. This produces the smallest conforming refinement of $\mathcal{T}$ in which all elements in $\mathcal{M}$ are refined, cf. [Stev08]. For a subset $\mathcal{C} \subseteq \mathcal{N}(\mathcal{T})$ of nodes marked for coarsening, a *coarsening step* consists in coarsening all nodes in $\mathcal{C}$ which can be coarsened locally. For an efficient implementation of the coarsening step it turned out to be useful to store the sons $s_1(T)$ and $s_2(T)$ one after another in the list of elements at the position of their father T, cf. Figure 3.

5.1. **Local resolution of a corner singularity.** We choose $\Omega := (-1,1)^3 \setminus [0,1]^3$ and let $\mathcal{T}_0$ be the triangulation consisting of seven translated copies of $\mathcal{T}_{Kuhn}$, cf. the left plot of Figure 6. A sequence $\mathcal{T}_0, ..., \mathcal{T}_6$ is created by marking all elements in the triangulation $\mathcal{T}_j$ that contain the node $z = 0$ for refinement and carrying out a refinement step. The resulting triangulation $\mathcal{T}_6$ is shown in the second plot of Figure 6. We then repeatedly mark all nodes for coarsening and carry out a coarsening step. The resulting triangulations after two and five coarsening steps are displayed in the third and fourth plot of Figure 6, respectively. Six coarsening steps result in the initial triangulation, i.e., the entire local refinement can be reversed by repeatedly applying the coarsening criterion. In this example the refinement steps do not lead to recursions. Therefore, the same number of coarsening steps reverses the entire refinement. This is not the case in general.

5.2. **Local resolution of a moving interface.** Let $\Omega := (0,5) \times (0,1)^{d-1}$ and let $\mathcal{T}_0$ be the triangulation of Ω consisting of five translated copies of $\mathcal{T}_{Kuhn}$. Given $\Delta t > 0$ we inductively generate a sequence of triangulations $\mathcal{T}_0, \mathcal{T}_1, ...$ by defining $\mathcal{T}_j$ through coarsening and refinement of $\mathcal{T}_{j-1}$ as follows:

(i) repeatedly mark those nodes in $\mathcal{T}_{j-1}$ for coarsening which belong to elements that do not intersect the plane $\{x \in \mathbb{R}^d : x_1 = j\Delta t\}$ until no such nodes exist or none of them can be coarsened locally
(ii) carry out four refinement steps where the set of marked elements consists of those elements that intersect the plane $\{x \in \mathbb{R}^d : x_1 = j\Delta t\}$

Figure 7 shows the triangulations with $d = 2$ and $d = 3$ for $\Delta t = 1/4$ and $j = 2, 3, 6, 10$. We see that only a region close to the moving plane $\{x \in \mathbb{R}^d : x_1 = j\Delta t\}$ is refined and this refinement is entirely coarsened subsequently.

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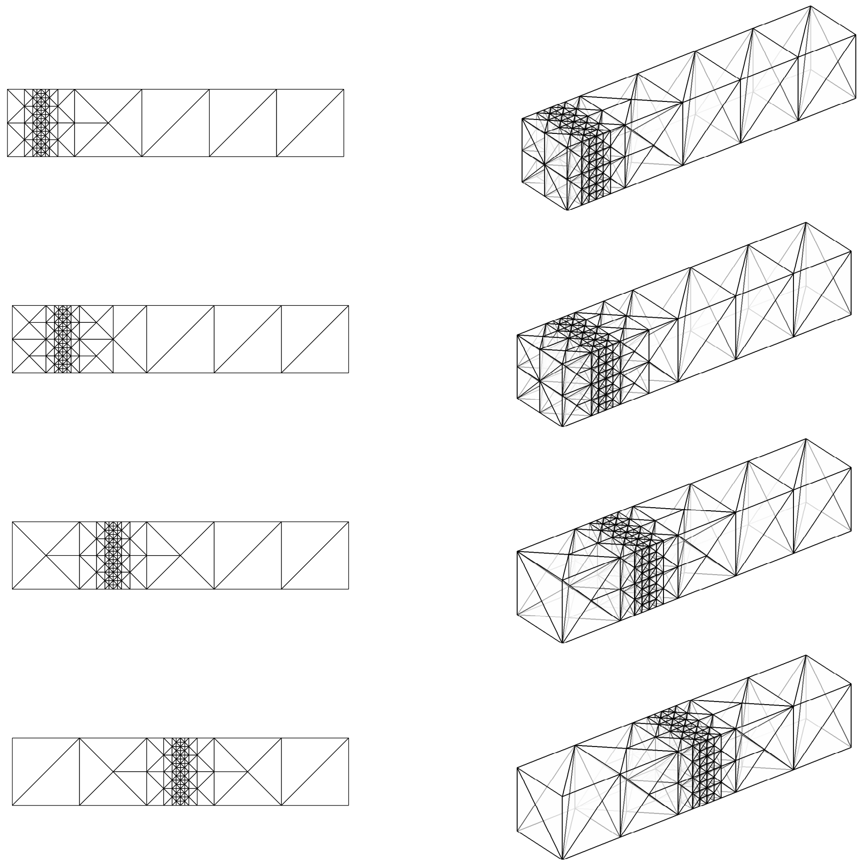

FIGURE 7. Refinement and coarsening in two and three dimensions for a moving interface. The plots show the locally refined and coarsened triangulations for the interface at $x_1 = j/4$ with $j = 2, 3, 6, 10$.

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INSTITUT FÜR NUMERISCHE SIMULATION, RHEINISCHE FRIEDRICH-WILHELMS-UNIVERSITÄT BONN, WEGELERSTRASSE 6, 53115 BONN, GERMANY

E-mail address: `bartels@ins.uni-bonn.de`

INSTITUT FÜR NUMERISCHE SIMULATION, RHEINISCHE FRIEDRICH-WILHELMS-UNIVERSITÄT BONN, WEGELERSTRASSE 6, 53115 BONN, GERMANY

E-mail address: `schreier@uni-bonn.de`