SEMI-IMPLICIT APPROXIMATION OF WAVE MAPS INTO SMOOTH OR CONVEX SURFACES

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ABSTRACT. A semi-implicit, lowest order finite element scheme for the approximation of wave maps into smooth or convex surfaces is devised and its stability is analyzed. Convergence is established for the case of the unit sphere as a target manifold, which is unconditional in case of $(2+1)$ Minkowski space. Numerical experiments illustrate the theoretical results.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^m$ be a bounded Lipschitz domain, $N \subset \mathbb{R}^\ell$ a compact, n-dimensional submanifold without boundary, and $T > 0$. A mapping $u : (0,T) \times \Omega \to N$ is called a *wave map into* N subject to given initial data u_0 and u_1 such that $u_0 \in N$ and $u_1 \in T_{u_0}N$ in Ω if u satisfies

$$\Box u := \partial_t^2 u - \Delta u \perp T_u N \quad \text{in } (0,T) \times \Omega \tag{1.1}$$

and

$$\partial_n u(t,\cdot) = 0 \quad \text{on } \partial\Omega, \qquad u(0,\cdot) = u_0, \qquad \partial_t u(0,\cdot) = u_1. \tag{1.2}$$

The initial boundary value problem (1.1)-(1.2) occurs as a simplified model problem in general relativity and physics, e.g., the Einstein vacuum equations with cylindrical symmetry reduce to a wave map system on (2+1) Minkowski space, cf. [AH02], and wave maps arise as nonlinear σ models in particle physics, cf. [Car04]. Characteristic properties of (smooth) solutions are that they are energy conserving in the sense that the identity

$$E[u(t), \partial_t u(t)] := \frac{1}{2} \int_\Omega |\partial_t u|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx = E[u_0, u_1]$$

is satisfied for all $t \geq 0$ and that (1.1) is invariant with respect to dimensionless scaling. Various properties of solutions to (1.1)-(1.2) such as long time existence for large initial data or occurrence of finite-time blow-up for smooth initial data are still not entirely understood and a lot of research has been carried out in recent years studying these questions. We refer the reader to the survey article [Tat04], the monograph [SS98], and the recent results in [KST08] for related details.

The occurrence of singular solutions for (1.1)-(1.2) has first been studied numerically in [BCT01, IL02] by employing reduced models and direct discretizations. While those numerical results provide strong evidence for the possibility of finite-time blow-up, convergence of approximations to solutions of (1.1)-(1.2) has not been investigated and numerical artifacts cannot be entirely ruled out. Weak accumulation of finite element solutions obtained by projection and penalization methods and their stability has been established in [BFP08] for the case that $N = S^n$. The schemes either converge under restrictive conditions on the discretization parameters or require the solution of nonlinear problems in each time step. A method based on approximate Lagrange multipliers for the computation of wave maps into fixed and varying spheres but which also leads to nonlinear problems in each time step has been studied respectively in [BLP07] and [BPS08]. This article aims at devising a numerical method that leads to linear systems of equations in each time step, is

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unconditionally stable for a large class of target manifolds N, and is convergent to weak solutions of (1.1)-(1.2) if N is the unit sphere under mild conditions on the discretization parameters.

Our proposed scheme is a projection method that linearizes the constraint $U^{j+1}(z) \in N$ about the previous approximation $U^j(z)$ at the nodes z of the underlying triangulation $\mathcal{T}_h$ and then computes the update in the corresponding tangent space of N. This leads to a semi-implicit iteration that is unconditionally stable if $\mathcal{T}_h$ is weakly acute and N is convex. Otherwise, if N is nonconvex or $\mathcal{T}_h$ fails to be weakly acute, certain conditions on the time-step size and the mesh-size are required to guarantee well-posedness and stability of the scheme. Weak accumulation of approximations at weak solutions of (1.1)-(1.2) in case of the unit sphere as the target manifold is established unconditionally if $m \le 2$ and under mild constraints otherwise. This discrepancy is due to the lack of an appropriate multiplicative Sobolev inequality for $m \ge 3$. In the development and the analysis of our scheme we greatly benefit from results on the approximation of harmonic maps in [Alo97, Bar05, BBFP07, BFP08] and in particular [Alo08, Bar08]. In future work we aim at establishing convergence of our algorithm also for general, smooth target manifolds following the ideas in [FMS98, MS98].

The rest of this paper is organized as follows. In Section 2 we introduce the notion of a weak solution of (1.1)-(1.2), state the approximation scheme together with its main properties, and recall some basic facts about lowest order finite element methods. Section 3 is devoted to the stability and well-posedness of our proposed algorithm under various assumptions on N and the underlying triangulation. For the case that N is the n-dimensional unit sphere we prove in Section 4 that approximations obtained with our scheme weakly accumulate at weak solutions of (1.1)-(1.2). Numerical experiments supporting and illustrating our theoretical findings are reported in Section 5.

2. Preliminaries

In this section we introduce the notion of a weak solution of (1.1)-(1.2), define the approximation scheme, summarize its most important properties, and recall basic facts of lowest order finite element methods.

2.1. **Weak solutions.** We let $(\cdot,\cdot)$ denote the inner product in $L^2(\Omega;\mathbb{R}^{\ell_1\times\ell_2})$ with corresponding norm $\|\cdot\|$ and use standard notation for Sobolev and Bochner spaces. For $T>0$ we define $\Omega_T := (0,T)\times\Omega$ and for functions $u \in H^1(\Omega;\mathbb{R}^\ell)$ and $v\in L^2(\Omega;\mathbb{R}^\ell)$ we set

$$E[u,v] := \frac{1}{2}\|v\|^2 + \frac{1}{2}\|\nabla u\|^2.$$

Definition 2.1. *Given $T>0$, $u_0 \in H^1(\Omega;\mathbb{R}^\ell)$, and $u_1 \in L^2(\Omega;\mathbb{R}^\ell)$ such that $u_0\in N$ and $u_1 \in T_{u_0}N$ almost everywhere in Ω we call a mapping $u:\Omega_T\to N$ a* weak solution of (1.1)-(1.2) *if*

(1) *$u \in H^1(0,T;L^2(\Omega;\mathbb{R}^\ell)) \cap L^2(0,T;H^1(\Omega;\mathbb{R}^\ell))$,*
(2) *$u(t,x)\in N$ for almost every $(t,x)\in\Omega_T$,*
(3) *for all $w\in C_0^\infty([0,T);C^\infty(\overline{\Omega};\mathbb{R}^\ell))$ such that $w(t,x)\in T_{u(t,x)}N$ for every $(t,x)\in\Omega_T$ we have*
$$-\int_0^T \big(\partial_t u,\partial_t w\big)\,\mathrm{d}t + \int_0^T \big(\nabla u,\nabla w\big)\,\mathrm{d}t = \big(u_1,w(0)\big),$$
(4) *the initial data u_0 is attained by u in the sense of traces,*
(5) *for almost every $t\in(0,T)$ we have*
$$E[u(t,\cdot),\partial_t u(t,\cdot)] \le E[u_0,u_1].$$

Remarks 2.1. *(i) Global existence of weak solutions of* (1.1)-(1.2)*, i.e., existence of weak solutions for all $T>0$, in case $m=2$ and for parallelizable target manifolds N has been established in* [MS96]*.*
(ii) Weak solutions of (1.1)-(1.2) *in the sense of Definition 2.1 attain the initial data u_0 and u_1*

continuously in $H^1(\Omega;\mathbb{R}^\ell)$ and $L^2(\Omega;\mathbb{R}^\ell)$, respectively, as $t \to 0$, cf. [MS96, SS98]. *For this it is essential that $u_1(x) \in T_{u_0(x)}N$ for almost every $x \in \Omega$.*

2.2. **Approximation scheme and main results.** For a regular triangulation $\mathcal{T}_h$ of the polyhedral domain Ω into simplices with vertices contained in $\mathcal{N}_h$ and whose maximal diameters are bounded by h we define

$$\mathbb{V}_h := \big\{v_h \in C(\overline{\Omega}) : v_h|_K \text{ affine for all } K \in \mathcal{T}_h\big\}.$$

Given a time-step size $\tau > 0$, the backward difference operator d_t is for a sequence $(a^j)_{j\geq 0}$ and $j \geq 0$ defined through

$$d_t a^{j+1} := \tau^{-1}\big(a^{j+1} - a^j\big).$$

For an integer J_T such that $J_T\tau \geq T$ and an arbitrary parameter $\theta \in [0,1]$, we propose the following approximation scheme for solutions of (1.1). It is motivated by ideas in [Bar08] and [Alo08].

Algorithm (A). *Let $\big(U^0, V^0\big) \in \big[\mathbb{V}_h^\ell\big]^2$ such that $U^0(z) \in N$ and $V^0(z) \in T_{U^0(z)}N$ for all $z \in \mathcal{N}_h$. For $j = 0, 1, ..., J_T - 1$ let $\big(U^{j+1}, V^{j+1}\big) \in \big[\mathbb{V}_h^\ell\big]^2$ be defined as follows:*

(1) *Let $V^{j+1} \in \mathbb{V}_h^\ell$ be such that $V^{j+1}(z) \in T_{U^j(z)}N$ for all $z \in \mathcal{N}_h$ and*

$$\big(d_t V^{j+1}, W\big) + \big(\nabla\big[U^j + \theta\tau V^{j+1}\big], \nabla W\big) = 0$$

for all $W \in \mathbb{V}_h^\ell$ satisfying $W(z) \in T_{U^j(z)}N$ for all $z \in \mathcal{N}_h$.

(2) *Define $U^{j+1} \in \mathbb{V}_h^\ell$ by setting*

$$U^{j+1}(z) = \pi_N\big(U^j(z) + \tau V^{j+1}(z)\big)$$

for all $z \in \mathcal{N}_h$.

Here, $\pi_N : U_{\delta_N}(N) \to N$ is the nearest-neighbor projection onto N defined in the tubular neighborhood $U_{\delta_N}(N) := \{q \in \mathbb{R}^\ell : \operatorname{dist}(q, N) \leq \delta_N\}$ of N for some positive number δ_N that depends on the curvature of N. Our main results are summarized in the following theorem and we refer the reader to the subsequent sections for proofs, details, and more general assertions. In particular, the definition of a weakly acute triangulation is found in Subsection 2.3 and the parameter σ_N is introduced in Remark 2.2 (i) below. We say that N is convex if $N = \partial\mathcal{C}$ for an open, convex set $\mathcal{C} \subset \mathbb{R}^\ell$. Throughout this article, C denotes a generic, positive constant that does not depend on h and τ.

Theorem 2.1. *(i) If N is convex, $\mathcal{T}_h$ weakly acute, and $\theta \geq 1/2$ then the iteration of Algorithm (A) is well defined and for all $0 \leq J \leq J_T - 1$ we have*

$$E\big[U^{J+1}, V^{J+1}\big] + \tau \sum_{j=0}^{J} \Big\{\frac{\tau}{2}\big\|d_t V^{j+1}\big\|^2 + \Big(\theta - \frac{1}{2}\Big)\frac{\tau}{2}\big\|\nabla V^{j+1}\big\|^2\Big\} \leq E\big[U^0, V^0\big].$$

(ii) If N is C^3, $\mathcal{T}_h$ quasiuniform, and $\tau \leq C h^{\max\{1+m/2,2\}}\sigma_N^{-1}$ then the iteration of Algorithm (A) is well defined and for all $0 \leq J \leq J_T - 1$ we have

$$\begin{aligned}\big(1 - C'\tau\big)E\big[U^{J+1}, V^{J+1}\big] + \tau \sum_{j=0}^{J} \Big\{\frac{\tau}{2}\big\|d_t V^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_t \nabla U^{j+1}\big\|^2 + \theta\tau\big\|\nabla V^{j+1}\big\|^2\Big\}& \\ \leq E\big[U^0, V^0\big] \exp\big(C''(\tau h^{-2} + \tau^3 h^{-4-m})T\big)&.\end{aligned}$$

(iii) If $m \leq 3$, $N = S^n$, and either $\theta > 1/2$ and $\tau = o\big(h_{min}^{\max\{0,m-2\}}\big)$ or $\theta = 1/2$ and $\tau = o\big(h_{min}^{m/2}\big)$ or $\theta < 1/2$ and $\tau = o\big(h_{min}^2\big)$ then every weak accumulation point of the sequence $\big(u_h\big)_{h>0} \subset L^\infty(0,T;H^1(\Omega;\mathbb{R}^\ell))$ that is obtained by piecewise constant interpolation of $\big(U^j\big)_{j=0,1,...,J_T}$ in time for each $h > 0$ is a weak solution of (1.1)-(1.2).

Remarks 2.2. *(i) Owing to the Lax-Milgram Lemma, Step (1) of Algorithm (A) is always well defined. Step (2) is well defined if* N *is convex or* $\tau\big|V^{j+1}(z)\big| \le \sigma_N$, *where* σ_N *is a number such that* $\operatorname{dist}(p+s,N) \le \delta_N$ *whenever* $p \in N$ *and* $s \in T_pN$ *with* $|s| \le \sigma_N$. *It can be shown that* σ_N *is positive if* N *is* C^2, *cf., e.g.* [Bar08]. *If* $N = \partial\mathcal{C}$ *is convex then, since* $p+s \in \mathbb{R}^\ell \setminus \mathcal{C}$ *the projection* $\pi_N(p+s)$ *is well defined for all* $p \in N$ *and* $s \in T_pN$ *and we may set* $\sigma_N := \infty$.
(ii) The domain Ω *may be replaced by a smooth* m*-dimensional submanifold* $M \subset \mathbb{R}^k$. *In this case the triangulation* $\mathcal{T}_h$ *defines an approximate submanifold* M_h *and the elementwise defined tangential gradient* ∇_{M_h} *has to be employed. We refer the reader to* [DDE05] *for related definitions and estimates.*
(iii) The practical computation of the nearest-neighbor projection π_N *can be realized by solving for given* $p \in \mathbb{R}^\ell$ *the saddle-point formulation*

$$\inf_{q\in\mathbb{R}^\ell}\sup_{\lambda\in\mathbb{R}} |p-q|^2 + \lambda f(q),$$

where $f : \mathbb{R}^\ell \to \mathbb{R}$ *is an arbitrary function such that* $f^{-1}(0) = N$. *Convergence of related iterative methods has been investigated in* [DD07] *and* [Bar08].
(iv) Although not unconditionally convergent, in practice it may be preferable to employ $\theta = 1/2$ *to limit the effect of numerical dissipation.*

2.3. **Auxiliary results.** We next collect a few elementary results about finite element spaces that will be required in the analysis below. We refer the reader to [Cia02, BS02] for detailed information and proofs. Letting $\big(\varphi_z : z \in \mathcal{N}_h\big)$ denote the nodal basis of $\mathbb{V}_h$, the nodal interpolation operator $\mathcal{I}_h : C(\overline{\Omega}) \to \mathbb{V}_h$ is for $v \in C(\overline{\Omega})$ defined by

$$\mathcal{I}_h v = \sum_{z\in\mathcal{N}_h} v(z)\varphi_z.$$

If $m \le 3$ then standard techniques imply that for $v \in C(\overline{\Omega})$ such that $v|_K \in H^2(K)$ for all $K \in \mathcal{T}_h$ we have

$$\big\|v - \mathcal{I}_h v\big\| + \big\|\nabla\big[v - \mathcal{I}_h v\big]\big\| \le C\big\|h_{\mathcal{T}_h}^2 D_{\mathcal{T}_h}^2 v\big\|, \tag{2.1}$$

where $h_{\mathcal{T}_h} \in L^\infty(\Omega)$ satisfies $h_{\mathcal{T}_h}|_K = \operatorname{diam}(K)$ for all $K \in \mathcal{T}_h$ and $D^2_{\mathcal{T}_h} v$ denotes the elementwise evaluation of the Hessian of v. We will make repeated use of inverse estimates of the form

$$\big\|V\big\|_{L^q(\Omega)} \le C h_{min}^{m(1/q-1/p)}\big\|V\big\|_{L^p(\Omega)} \tag{2.2}$$

as well as

$$\big\|h_{\mathcal{T}_h}^{-1}\nabla V\big\|_{L^p(\Omega)} \le C\big\|V\big\|_{L^p(\Omega)} \tag{2.3}$$

which hold for all $V \in \mathbb{V}_h$, real numbers $1 \le p \le q \le \infty$ (where $1/\infty := 0$), and $h_{min} := \min_{K\in\mathcal{T}_h}\operatorname{diam}(K)$. It is well known that for every $1 \le p < \infty$ the equivalence

$$C^{-1}\big\|V\big\|^p_{L^p(\Omega)} \le \sum_{z\in\mathcal{N}_h} h_z^m\big|V(z)\big|^p \le C\big\|V\big\|^p_{L^p(\Omega)} \tag{2.4}$$

is satisfied for all $V \in \mathbb{V}_h$. Here, we set $h_z := \operatorname{diam}\operatorname{supp}\varphi_z$ for every $z \in \mathcal{N}_h$. A triangulation $\mathcal{T}_h$ is called *weakly acute* if for all distinct $z, y \in \mathcal{N}_h$ we have

$$\big(\nabla\varphi_z, \nabla\varphi_y\big) \le 0.$$

The following basic observation is essential for unconditional stability of our algorithm and has first been employed in the context of geometric partial differential equations in [Bar05].

Lemma 2.1. *Let $P : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be a Lipschitz continuous operator with Lipschitz constant $|P|_{Lip}$. Given any $V \in \mathbb{V}_h^\ell$ let $T_P^h V \in \mathbb{V}_h^\ell$ be defined through $T_P^h V(z) = P\big(V(z)\big)$ for all $z \in \mathcal{N}_h$. Then, we have*

$$\|\nabla T_P^h V\| \le |P|_{Lip} \|\nabla V\|.$$

Proof. Under the assumptions of the lemma we have for the finite element stiffness matrix $K = (k_{zy})_{z,y\in\mathcal{N}_h}$ that $k_{zy} = \big(\nabla\varphi_z, \nabla\varphi_y\big) \le 0$ whenever $z \ne y$. Using that the sum of every row of K vanishes and that K is symmetric we infer for every $W \in \mathbb{V}_h^\ell$ that

$$\begin{aligned}\|\nabla W\|^2 = \sum_{z,y\in\mathcal{N}_h} k_{zy} W(z)\cdot W(y) &= \sum_{z,y\in\mathcal{N}_h} k_{zy} W(z)\cdot\big(W(y)-W(z)\big)\\ &= \frac{1}{2}\sum_{z,y\in\mathcal{N}_h} k_{zy} W(z)\cdot\big(W(y)-W(z)\big) + \frac{1}{2}\sum_{z,y\in\mathcal{N}_h} k_{zy} W(y)\cdot\big(W(z)-W(y)\big)\\ &= -\frac{1}{2}\sum_{z,y\in\mathcal{N}_h} k_{zy}\big|W(z)-W(y)\big|^2.\end{aligned}$$

On applying this identity twice we find that

$$\|\nabla T_P^h V\|^2 = -\frac{1}{2}\sum_{z,y\in\mathcal{N}_h} k_{zy}\big|P\big(V(z)\big) - P\big(V(y)\big)\big|^2 \le |P|_{Lip}^2 \|\nabla V\|^2,$$

where we used that $k_{zy} \le 0$ for $z \ne y$ and all contributions to the sum with $z = y$ vanish. ☐

Remark 2.1. *If $m = 2$ then $\mathcal{T}_h$ is weakly acute if and only if every sum of angles opposite to an inner edge is bounded by π and every angle opposite to an edge on the boundary of Ω is bounded by $\pi/2$. A sufficient condition for $m = 3$ is that every angle between two faces that belong to the same element is bounded by $\pi/2$.*

3. Stability of Algorithm (A)

This section discusses stability of the iteration of Algorithm (A) under various assumptions on N, $\mathcal{T}_h$, θ, and τ.

3.1. **Convex targets.**

If $N = \partial\mathcal{C}$ is convex then the projection π_N coincides in the exterior of $\mathcal{C}$ with the restriction of the orthogonal projection $\pi_\mathcal{C} : \mathbb{R}^\ell \to \mathcal{C}$ to $\mathbb{R}^\ell \setminus \mathcal{C}$. Since $\pi_\mathcal{C}$ is Lipschitz continuous with constant 1 we obtain unconditional convergence of Algorithm (A) in case that $\mathcal{T}_h$ is weakly acute and $\theta \ge 1/2$. Otherwise, if $\theta < 1/2$ the iteration is stable if $\tau \le Ch_{min}^2$. The proof of the following proposition partially follows recent work in [Alo08].

Proposition 3.1. *Suppose that N is convex and $\mathcal{T}_h$ weakly acute. If $\theta \ge 1/2$ then the iterates $\big(U^j, V^j\big)_{j=0,1,..,J_T}$ of Algorithm (A) satisfy for every $0 \le J \le J_T - 1$,*

$$E\big[U^{J+1}, V^{J+1}\big] + \tau\sum_{j=0}^{J}\Big\{\frac{\tau}{2}\|d_t V^{j+1}\|^2 + \Big(\theta - \frac{1}{2}\Big)\frac{\tau}{2}\|\nabla V^{j+1}\|^2\Big\} \le E\big[U^0, V^0\big].$$

If $\theta < 1/2$ then for every $0 \le J \le J_T - 1$ we have

$$\big(1 - C'\tau^2 h_{min}^{-2}\big)E\big[U^{J+1}, V^{J+1}\big] + \tau\sum_{j=0}^{J}\frac{\tau}{2}\|d_t V^{j+1}\|^2 \le E\big[U^0, V^0\big]\exp\big(C''\tau h_{min}^{-2} T\big).$$

Proof. We choose $W = V^{j+1}$ in the first step of Algorithm (A) and employ the identities $(a-b)a = (a^2-b^2)/2 + (a-b)^2/2$ as well as $(a+\theta b)b = ((a+b)^2 - a^2)/2 + (\theta - 1/2)b^2$ to verify that

$$\frac{1}{2}d_t\big\|V^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_tV^{j+1}\big\|^2 + \frac{1}{2\tau}\big(\big\|\nabla\big[U^j + \tau V^{j+1}\big]\big\|^2 - \big\|\nabla U^j\big\|^2\big) + \Big(\theta - \frac{1}{2}\Big)\tau\big\|\nabla V^{j+1}\big\|^2 = 0.$$

Since $p + s \in \mathbb{R}^\ell \setminus \mathcal{C}$ for all $p \in N$ and $s \in T_pN$, $U^{j+1}(z) = \pi_N\big(U^j(z) + \tau V^{j+1}(z)\big)$ for all $z \in \mathcal{N}_h$, and $\big|\pi_N|_{\mathbb{R}^\ell\setminus\mathcal{C}}\big|_{Lip} \le 1$, Lemma 3.1 implies that

$$\big\|\nabla U^{j+1}\big\| \le \big\|\nabla\big[U^j + \tau V^{j+1}\big]\big\|.$$

A combination of the last two estimates, multiplication by τ, and summation over $j = 0, 1, ..., J$ implies the first assertion. The second assertion follows upon employing the inverse estimate (2.3) and using a discrete Gronwall inequality. □

3.2. **General targets.** Significantly weaker results are available for the case that N is not convex. In particular, we are not able to benefit from a special structure of a triangulation or the semi-implicit nature of Algorithm (A) for $\theta \ge 1/2$. We notice that the results of this subsection also apply if N is convex but $\mathcal{T}_h$ fails to be weakly acute.

Proposition 3.2. *Suppose that N is C^3 regular and assume that $\mathcal{T}_h$ is quasiuniform, i.e., $Ch \le h_{min}$. If*

$$\tau \le Ch^{\max\{2,1+m/2\}}\sigma_N^{-1}$$

then the iteration of Algorithm (A) is well defined and the iterates $\big(U^j, V^j\big)_{j=0,1,..,J_T}$ satisfy for every $0 \le J \le J_T - 1$,

$$\begin{aligned}\big(1 - C'\tau\big)E\big[U^{J+1}, V^{J+1}\big] + \tau\sum_{j=0}^{J}\Big\{\frac{\tau}{2}\big\|d_tV^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_t\nabla U^{j+1}\big\|^2 + \theta\tau\big\|\nabla V^{j+1}\big\|^2\Big\}&\\ \le E\big[U^0, V^0\big]\exp\big(C''(\tau h^{-2} + \tau^3h^{-m+4})T\big)&.\end{aligned}$$

Remark 3.1. *If N is convex then the assertion of the proposition holds true if $\tau \le Ch^{\max\{2,(m+4)/3\}}$.*

Proof. We adopt arguments of [BBFP07, BFP08, Bar08] and divide the proof into several steps.
Step 1: rough bound and well posedness. Suppose that the first $J' \ge 0$ iterations of Algorithm (A) are well defined and assume that $\big\|\nabla U^j\big\| \le C_0$ for all $0 \le j \le J'$. This holds, e.g., for $J' = 0$. Then, upon choosing $W = V^{j+1}$ in Step (1) of Algorithm (A) and employing the inverse estimate (2.3) we find

$$\frac{1}{2}d_t\big\|V^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_tV^{j+1}\big\|^2 + \theta\tau\big\|\nabla V^{j+1}\big\|^2 = -\big(\nabla U^j, \nabla V^{j+1}\big) \le \frac{1}{2}h^{-2}CC_0^2 + \frac{1}{2}\big\|V^{j+1}\big\|^2.$$

An application of the discrete Gronwall lemma implies that for $j = 0, 1, ..., J'$ we have

$$\big\|V^{j+1}\big\| \le Ch^{-1} \tag{3.1}$$

and hence, by application of the inverse estimate (2.2),

$$\big\|V^{j+1}\big\|_{L^\infty(\Omega)} \le Ch^{-m/2}\big\|V^{j+1}\big\| \le Ch^{-1-m/2}.$$

Thus, under the condition $\tau \le C\sigma_N^{-1}h^{1+m/2}$, cf. Remark 2.2 (i), we have

$$\operatorname{dist}\big(U^j(z) + \tau V^{j+1}(z), N\big) \le \delta_N$$

for $j = J'$ which implies that Step (2) of Algorithm (A) and hence $U^{J'+1}$ is well defined.
Step 2: estimates for $V^{j+1} - d_tU^{j+1}$. For $j = 0, 1, ..., J'$ we define $R^{j+1} := d_tU^{j+1} - V^{j+1}$. Using

the definition of U^{j+1}, noting that $U^j(z) = \pi_N\big(U^j(z)\big)$, and employing a Taylor expansion of the C^2 regular mapping π_N about $U^j(z)$, we have for every $z \in \mathcal{N}_h$ that

$$
\begin{aligned}
\big|R^{j+1}(z)\big| &= \tau^{-1}\big|\pi_N\big(U^{j+1}(z) + \tau V^{j+1}(z)\big) - U^j(z) - \tau V^{j+1}(z)\big| \\
&= \tau^{-1}\big|\pi_N\big(U^{j+1}(z) + \tau V^{j+1}(z)\big) - \pi_N\big(U^j(z)\big) - \tau V^{j+1}(z)\big| \qquad (3.2)\\
&\le C\tau\big|V^{j+1}(z)\big|^2,
\end{aligned}
$$

where we used that $D\pi_N(p)s = s$ for every $p \in N$ and $s \in T_pN$, i.e., $D\pi_N\big(U^j(z)\big)V^{j+1}(z) = V^{j+1}(z)$. The equivalence (2.4) implies

$$
\big\|R^{j+1}\big\|_{L^p(\Omega)} \le C\tau\big\|V^{j+1}\big\|^2_{L^{2p}(\Omega)} \tag{3.3}
$$

for all $1 \le p \le \infty$.

Step 3: intermediate energy inequality. Upon choosing $W = V^{j+1} = d_tU^{j+1} - R^{j+1}$ in Step (1) of Algorithm (A) we find

$$
\begin{aligned}
\frac{1}{2}d_t\big\|V^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_tV^{j+1}\big\|^2 + \frac{1}{2}d_t\big\|\nabla U^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_t\nabla U^{j+1}\big\|^2 \\
= -\big(\nabla\big[U^j + \tau\theta V^{j+1}\big], \nabla V^{j+1}\big) + \big(\nabla U^{j+1}, \nabla d_tU^{j+1}\big) \\
= -\theta\tau\big\|\nabla V^{j+1}\big\|^2 + \tau\big\|d_t\nabla U^{j+1}\big\|^2 + \big(\nabla U^j, \nabla R^{j+1}\big).
\end{aligned}
$$

Inverse estimates, the definition of R^{j+1}, the bound $\big\|U^j\big\|_{L^\infty(\Omega)} \le C$, and (3.3) show

$$
\begin{aligned}
\frac{1}{2}d_t\big\|V^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_tV^{j+1}\big\|^2 + \frac{1}{2}d_t\big\|\nabla U^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_t\nabla U^{j+1}\big\|^2 + \theta\tau\big\|\nabla V^{j+1}\big\|^2 \\
\le C\tau h^{-2}\big\|d_tU^{j+1}\big\|^2 + Ch^{-2}\big\|U^j\big\|_{L^\infty(\Omega)}\big\|R^{j+1}\big\|_{L^1(\Omega)} \\
\le C\tau h^{-2}\big\|V^{j+1}\big\|^2 + C\tau h^{-2}\big\|R^{j+1}\big\|^2 \\
\le C\tau h^{-2}\big\|V^{j+1}\big\|^2 + C\tau^3 h^{-2}\big\|V^{j+1}\big\|^4_{L^4(\Omega)}.
\end{aligned}
$$

We use (2.2) to verify

$$
\big\|V^{j+1}\big\|^4_{L^4(\Omega)} \le \big\|V^{j+1}\big\|^2_{L^\infty(\Omega)}\big\|V^{j+1}\big\|^2 \le Ch^{-m}\big\|V^{j+1}\big\|^4
$$

and to deduce that

$$
\begin{aligned}
\frac{1}{2}d_t\big\|V^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_tV^{j+1}\big\|^2 + \frac{1}{2}d_t\big\|\nabla U^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_t\nabla U^{j+1}\big\|^2 + \theta\tau\big\|\nabla V^{j+1}\big\|^2 \\
\le C\tau h^{-2}\big\|V^{j+1}\big\|^2 + C\tau^3 h^{-2-m}\big\|V^{j+1}\big\|^4
\end{aligned}
$$

is satisfied for all $0 \le j \le J'$. Multiplication by τ, summation over $j = 0, 1, ..., J$ with $0 \le J \le J'$, and the estimate $\big\|V^{j+1}\big\|^2 \le Ch^{-2}$, cf. (3.1), imply

$$
\begin{aligned}
\big(1 - C\tau^2h^{-2} - C\tau^4h^{-4-m}\big)E\big[U^{J+1}, V^{J+1}\big] + \tau\sum_{j=0}^{J}\Big\{\frac{\tau}{2}\big\|d_tV^{j+1}\big\|^2 + \frac{\tau}{2}\big\|d_t\nabla U^{j+1}\big\|^2 + \theta\tau\big\|\nabla V^{j+1}\big\|^2\Big\} \\
\le E\big[U^0, V^0\big] + C\big(\tau h^{-2} + \tau^3h^{-4-m}\big)\tau\sum_{j=0}^{J-1}\big\|V^{j+1}\big\|^2.
\end{aligned}
$$

Step 4: energy inequality up to $J'+1$. An application of the discrete Gronwall lemma shows that the assertion of the proposition holds up to $J \le J'$. From that estimate we find that $\big\|\nabla U^{J'+1}\big\| \le C_0$ if, e.g., $C_0 = 4E\big[U^0, V^0\big]$ and τ and h are such that $1 - C'\tau \geq 1/2$ and $\exp\big(C''(\tau h^{-2} + \tau^3 h^{-4-m})\big) \leq 2$. This permits us to continue the inductive argumentation started in Step 1. $\square$

4. Weak accumulation at wave maps into the unit sphere

For the symmetric situation $N = S^n = \big\{q \in \mathbb{R}^{n+1} : |q| = 1\big\}$ we show that iterates provided by Algorithm (A) weakly accumulate at weak solutions of (1.1) in the sense of Definition 2.1. Throughout this section we assume that $\mathcal{T}_h$ is weakly acute so that we are in the situation of Proposition 3.1. For a convergence analysis on general triangulations but under more restrictive conditions on the time-step size τ we refer the reader to [BFP08]. We note that throughout this section we have $\ell = n + 1$.

Piecewise affine and constant interpolations of the sequences $\big(U^j\big)_{j=0,1,...,J_T}$ and $\big(V^j\big)_{j=0,1,...,J_T}$ are for $t \in [0, T]$ such that $t \in [t_j, t_{j+1})$ for some $j = 0, 1, ..., J_T - 1$ and $t_j := j\tau$ defined by

$$\widehat{U}(t) := U^j + (t - t_j) d_t U^{j+1}, \qquad \widehat{V}(t) := V^j + (t - t_j) d_t V^{j+1},$$

and

$$V^+(t) := V^{j+1}, \qquad U^-(t) := U^j, \qquad U^+(t) := U^{j+1}.$$

To verify that $\partial_t \widehat{U}$ and V^+ have the same accumulation points we need the following bound which is also valid if $N \neq S^n$.

Definition 4.1. *For $\theta \in [0, 1]$ and $h_{min} > 0$ let $\Upsilon_\theta(h_{min})$ be defined through*

$$\Upsilon_\theta(h_{min}) := \begin{cases} 1 & \text{for } m \leq 2 \text{ and } \theta > 1/2, \\ h_{min}^{-m/2} & \text{for } m \geq 1 \text{ and } \theta = 1/2, \\ h_{min}^{2-m} & \text{for } m \geq 3 \text{ or } \theta < 1/2. \end{cases}$$

Lemma 4.1. *Suppose that $\tau \leq C h_{min}^{-2}$ if $\theta < 1/2$ and $\tau \leq C h_{min}^{m/2}$ if $\theta = 1/2$. Then we have*

$$\int_0^T \big\|\partial_t \widehat{U} - V^+\big\|^2 \, \mathrm{d}t \leq C C_\theta \tau \Upsilon_\theta(h_{min}) E\big[U^0, V^0\big]^2,$$

where $C_\theta = (\theta - 1/2)^{-2}$ if $\theta > 1/2$ and $C_\theta = 1$ otherwise.

Proof. As in (3.2)-(3.3) we find

$$\big\|\partial_t \widehat{U}(t, \cdot) - V^+(t, \cdot)\big\|^2 \leq C\tau^2 \big\|V^+(t, \cdot)\big\|_{L^4(\Omega)}^4.$$

If $\theta > 1/2$ and $m \leq 2$ we employ the multiplicative Sobolev inequality

$$\big\|V^+(t, \cdot)\big\|_{L^4(\Omega)}^4 \leq C\big\|V^+(t, \cdot)\|_{H^1(\Omega)}^2 \big\|V^+(t, \cdot)\big\|^2,$$

cf. [LU68, Str00], to verify the assertion of the lemma with the bounds of Proposition 3.1. If $\theta > 1/2$ and $m \geq 3$ we let $p^* := 2m/(m-2)$ denote the Sobolev conjugate exponent of 2 and use $r = p^*/2$ in Hölder's inequality to estimate

$$\big\|V^+(t, \cdot)\big\|_{L^4(\Omega)}^4 \leq \big\|V^+(t, \cdot)\big\|_{L^{2r}(\Omega)}^2 \big\|V^+(t, \cdot)\big\|_{L^{2r'}(\Omega)}^2 = \big\|V^+(t, \cdot)\big\|_{L^{p^*}(\Omega)}^2 \big\|V^+(t, \cdot)\big\|_{L^m(\Omega)}^2$$

where we used $2r' = 2r/(r-1) = 2p^*/(p^* - 2) = m$. The Sobolev inequality $\|V^+(t, \cdot)\|_{L^{p^*}(\Omega)} \leq C\|V^+(t, \cdot)\|_{H^1(\Omega)}$, the inverse estimate (2.2) with $q = m$ and $p = 2$, and the bounds of Proposition 3.1 then imply the assertion. In all remaining cases $\theta \leq 1/2$ we use the inverse estimate (2.2) to verify

$$\big\|V^+(t, \cdot)\big\|_{L^4(\Omega)}^4 \leq \big\|V^+(t, \cdot)\big\|_{L^\infty(\Omega)}^2 \big\|V^+(t, \cdot)\big\|^2 \leq C h_{min}^{-m} \big\|V^+(t, \cdot)\big\|^4,$$

employ the estimates of Proposition 3.1, and incorporate the assumed bounds $\tau \leq C h_{min}^2$ or $\tau \leq C h_{min}^{m/2}$ if respectively $\theta < 1/2$ or $\theta = 1/2$ to verify the assertion. $\square$

Lemma 4.2. *Assume that $\mathcal{T}_h$ is weakly acute and $\tau \le Ch_{min}^2$ if $\theta < 1/2$ and $\tau \le Ch_{min}^{m/2}$ if $\theta = 1/2$. If $\tau \Upsilon_\theta(h_{min}) \to 0$ as $h \to 0$ then every weak accumulation point $u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^\ell))$ of the bounded sequence $\big(U^+\big)_{h>0} \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^\ell))$ satisfies $u \in H^1(0,T;L^2(\Omega;\mathbb{R}^\ell))$ and, after extraction of an appropriate subsequence, as $h \to 0$,*

$$\begin{aligned} V^+ &\rightharpoonup^* \partial_t u && \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^\ell)),\\ \partial_t \widehat{U} &\rightharpoonup \partial_t u && \text{in } L^2(\Omega_T;\mathbb{R}^\ell),\\ U^+,\, \widehat{U} &\rightharpoonup^* u && \text{in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^\ell)),\\ U^+,\, U^-,\, \widehat{U} &\to u && \text{in } L^2(\Omega_T;\mathbb{R}^\ell). \end{aligned}$$

Moreover, we have $|u(t,x)| = 1$ for almost every $(t,x) \in \Omega_T$.

Proof. Proposition 3.1 implies that there exists $u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^\ell))$ such that $U^+ \rightharpoonup^* u$ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^\ell))$ for an appropriate subsequence as $h \to 0$. Owing to boundedness of $\partial_t \widehat{U}$ in $L^2(\Omega_T)$ guaranteed by Proposition 4.1 and after extraction of another subsequence, we also have $U^-,\, \widehat{U} \rightharpoonup^* u$ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^\ell))$ as $h \to 0$. Proposition 3.1 also ensures the existence of some $v \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^\ell))$ such that $V^+ \rightharpoonup^* v$ as $h \to 0$ and owing to Lemma 4.1 we have $\partial_t \widehat{U} \rightharpoonup v$ in $L^2(\Omega_T;\mathbb{R}^\ell)$ as $h \to 0$. An integration by parts in time then proves that $v = \partial_t u$. Thus, we have $\widehat{U} \rightharpoonup u$ in $H^1(\Omega_T;\mathbb{R}^\ell)$ and hence $\widehat{U} \to u$ in $L^2(\Omega_T;\mathbb{R}^\ell)$. Boundedness of $\partial_t \widehat{U}$ in $L^2(\Omega_T;\mathbb{R}^\ell)$ then implies that also $U^- \to u$ in $L^2(\Omega_T;\mathbb{R}^\ell)$. Since for all $t \in (0,T)$ we have $\mathcal{I}_h \big|U^-(t,\cdot)\big|^2 \equiv 1$ we verify with (2.1), (2.3), and $\big|U^-(t,x)\big| \le 1$ for all $x \in \Omega$ that

$$\begin{aligned} \big\| |U^-(t,\cdot)|^2 - 1 \big\| &\le C \big\| h_{\mathcal{T}_h}^2 D_{\mathcal{T}_h}^2 |U^-(t,\cdot)|^2 \big\| \le Ch \big\| h_{\mathcal{T}_h} \nabla U^-(t,\cdot) \big\|_{L^\infty(\Omega)} \big\| \nabla U^-(t,\cdot) \big\| \\ &\le Ch \big\| U^-(t,\cdot) \big\|_{L^\infty(\Omega)} \big\| \nabla U^-(t,\cdot) \big\| \le Ch \big\| \nabla U^-(t,\cdot) \big\|. \end{aligned}$$

Together with the pointwise convergence $U^- \to u$ almost everywhere in Ω_T this proves $|u| = 1$ almost everywhere in Ω_T. □

Proposition 4.1. *Assume that $m \le 3$, $\mathcal{T}_h$ is weakly acute and $\tau \le Ch_{min}^2$ if $\theta < 1/2$ and $\tau \le Ch_{min}^{m/2}$ if $\theta = 1/2$. Suppose that $U^0 \to u_0$ and $V^0 \to u_1$ in $L^2(\Omega;\mathbb{R}^\ell)$ as $h \to 0$. If $\tau \Upsilon_\theta(h_{min}) \to 0$ as $h \to 0$ then every weak accumulation point $u \in H^1(\Omega_T)$ of the sequence $\big(U^+\big)_{h>0}$ as in Lemma 4.2 satisfies*

$$-\int_0^T \big(\partial_t u, \partial_t [\phi u]\big)\, \mathrm{d}t + \int_0^T \big(\nabla u, \nabla [\phi u]\big)\, \mathrm{d}t = \big(u_1, \phi(0) u_0\big)$$

for all $\phi \in C_0^\infty([0,T);C^\infty(\overline{\Omega};so(n+1)))$, where $so(n+1)$ denotes the space of all skew-symmetric matrices in $\mathbb{R}^{(n+1)\times(n+1)}$.

Proof. In terms of $\widehat{V}$, V^+, and U^- we may recast the equation in the first step of Algorithm (A) as

$$\int_0^T \big(\partial_t \widehat{V}, w\big) + \big(\nabla U^-, \nabla w\big) + \theta\tau\big(\nabla V^+, \nabla w\big)\, \mathrm{d}t = 0 \tag{4.1}$$

for every $w \in L^2(0,T;\mathbb{V}_h^\ell)$ such that $w(t,z) \cdot U^-(t,z) = 0$ for almost every $t \in [0,T]$ and all $z \in \mathcal{N}_h$. Given a mapping $\phi \in C_0^\infty\big([0,T);C^\infty(\overline{\Omega};so(n+1))\big)$ we choose $w(t,\cdot) = \mathcal{I}_h\big[\phi(t,\cdot)U^-(t,\cdot)\big]$. In the following three steps we identify respectively the limits of the three terms on the left-hand side of (4.1) to verify the assertion of the proposition.

First term. Using integration by parts in time, $\phi(T) \equiv 0$, $\widehat{U}(0) = U^0$, and $\partial_t \widehat{U} \cdot \big(\phi \partial_t \widehat{U}\big) = 0$, we rewrite the first term on the left-hand side of (4.1) as

$$
\begin{aligned}
\int_0^T \big(\partial_t \widehat{V}, \mathcal{I}_h[\phi U^-]\big)\,\mathrm{d}t &= \int_0^T \big(\partial_t \widehat{V}, \mathcal{I}_h[\phi \widehat{U}]\big)\,\mathrm{d}t + \int_0^T \big(\partial_t \widehat{V}, \mathcal{I}_h[\phi\{U^- - \widehat{U}\}]\big)\,\mathrm{d}t \\
&= -\int_0^T \big(\widehat{V}, \partial_t \mathcal{I}_h[\phi \widehat{U}]\big)\,\mathrm{d}t + \big(\widehat{V}(0), \mathcal{I}_h[\phi(0)\widehat{U}(0)]\big) + \int_0^T \big(\partial_t \widehat{V}, \mathcal{I}_h[\phi\{U^- - \widehat{U}\}]\big)\,\mathrm{d}t \\
&= -\int_0^T \big(\widehat{V}, \partial_t[\phi \widehat{U}]\big)\,\mathrm{d}t - \int_0^T \big(\widehat{V}, \partial_t \mathcal{I}_h[\phi \widehat{U}] - \partial_t[\phi \widehat{U}]\big)\,\mathrm{d}t + \big(V^0, \phi(0) U^0\big) \\
&\quad + \big(V^0, \mathcal{I}_h[\phi(0)U^0] - \phi(0)U^0\big) + \int_0^T \big(\partial_t \widehat{V}, \mathcal{I}_h[\phi\{U^- - \widehat{U}\}]\big)\,\mathrm{d}t \\
&= -\int_0^T \big(\partial_t \widehat{U}, (\partial_t \phi)\widehat{U}\big)\,\mathrm{d}t - \int_0^T \big(\widehat{V} - \partial_t \widehat{U}, \partial_t[\phi \widehat{U}]\big)\,\mathrm{d}t - \int_0^T \big(\widehat{V}, \partial_t \mathcal{I}_h[\phi \widehat{U}] - \partial_t[\phi \widehat{U}]\big)\,\mathrm{d}t \\
&\quad + \big(V^0, \phi(0)U^0\big) + \big(V^0, \mathcal{I}_h[\phi(0)U^0] - \phi(0)U^0\big) + \int_0^T \big(\partial_t \widehat{V}, \mathcal{I}_h[\phi\{U^- - \widehat{U}\}]\big)\,\mathrm{d}t.
\end{aligned}
\tag{4.2}
$$

For the first and fourth term on the right-hand side of (4.2) we have, owing to $\partial_t \widehat{U} \rightharpoonup \partial_t u$ in $L^2(\Omega_T;\mathbb{R}^\ell)$, $\widehat{U} \to u$ in $L^2(\Omega_T;\mathbb{R}^\ell)$ in $L^2(\Omega_T;\mathbb{R}^\ell)$, $V^0 \to u_1$ and $U^0 \to u_0$ in $L^2(\Omega;\mathbb{R}^\ell)$ that, as $h \to 0$,

$$
\begin{aligned}
-\int_0^T \big(\partial_t \widehat{U}, (\partial_t \phi)\widehat{U}\big)\,\mathrm{d}t + \big(V^0, \phi(0)U^0\big) &\to -\int_0^T \big(\partial_t u, (\partial_t\phi) u\big)\,\mathrm{d}t + \big(u_1, \phi(0)u_0\big) \\
&= -\int_0^T \big(\partial_t u, \partial_t[\phi u]\big)\,\mathrm{d}t + \big(u_1, \phi(0)u_0\big),
\end{aligned}
\tag{4.3}
$$

where we used $\partial_t u \cdot \big(\phi \partial_t u\big) = 0$ almost everywhere in Ω_T. For the second term on the right-hand side of (4.2) we use $\widehat{V} = V^+ + (t - t_{j+1})\partial_t \widehat{V}$ to verify that

$$
\begin{aligned}
\Big|\int_0^T \big(\widehat{V} - \partial_t \widehat{U}, \partial_t[\phi \widehat{U}]\big)\,\mathrm{d}t\Big| &= \Big|\int_0^T \big(V^+ - \partial_t \widehat{U}, \partial_t[\phi \widehat{U}]\big)\,\mathrm{d}t + \int_0^T \big(\widehat{V} - V^+, \partial_t[\phi \widehat{U}]\big)\,\mathrm{d}t\Big| \\
&\le \Big\{\Big(\int_0^T \|V^+ - \partial_t \widehat{U}\|^2\,\mathrm{d}t\Big)^{1/2} + \tau^{1/2}\Big(\tau \int_0^T \|\partial_t \widehat{V}\|^2\,\mathrm{d}t\Big)^{1/2}\Big\} \\
&\quad \times \Big(\int_0^T \big(\|\partial_t \phi\| \|\widehat{U}\|_{L^\infty(\Omega)} + \|\phi\|_{L^\infty(\Omega)} \|\partial_t \widehat{U}\|\big)^2\,\mathrm{d}t\Big)^{1/2}.
\end{aligned}
\tag{4.4}
$$

The nodal interpolation estimate (2.1) allows us to control the fifth term on the right-hand side of (4.2) by

$$
\begin{aligned}
\big|\big(V^0, \mathcal{I}_h[\phi(0)U^0] - \phi(0)U^0\big)\big| &\le C\|V^0\| \|h_{\mathcal{T}_h}^2 D_{\mathcal{T}_h}^2[\phi(0)U^0]\| \\
&\le Ch^2 \|V^0\| \big(\|D^2\phi(0)\|\, \|U^0\|_{L^\infty(\Omega)} + \|\nabla\phi(0)\|_{L^\infty(\Omega)} \|\nabla U^0\|\big),
\end{aligned}
\tag{4.5}
$$

where we used that U^0 is elementwise affine. Similarly, but incorporating also the inverse estimate (2.3) to bound $\|h_{\mathcal{T}_h} \nabla \partial_t \widehat{U}\| \le C\|\partial_t \widehat{U}\|$, we control the third term on the right-hand side of (4.2), i.e.,

$$
\begin{aligned}
\Big|\int_0^T \big(\widehat{V}, \partial_t \mathcal{I}_h[\phi\widehat{U}] - \partial_t[\phi\widehat{U}]\big)\,\mathrm{d}t\Big| &\le C\int_0^T \|\widehat{V}\|\big(\|h_{\mathcal{T}_h}^2 D_{\mathcal{T}_h}^2[(\partial_t\phi)\widehat{U}]\| + \|h_{\mathcal{T}_h}^2 D_{\mathcal{T}_h}^2[\phi(\partial_t\widehat{U})]\|\big) \\
&\le C\int_0^T \|\widehat{V}\|\big(h^2\|D^2\partial_t\phi\|\,\|\widehat{U}\|_{L^\infty(\Omega)} + h^2\|\nabla\partial_t\phi\|_{L^\infty(\Omega)}\|\nabla\widehat{U}\|\big)\,\mathrm{d}t \\
&\quad + C\int_0^T \|\widehat{V}\|\big(h^2\|D^2\phi\|_{L^\infty(\Omega)}\|\partial_t\widehat{U}\| + h\|\nabla\phi\|_{L^\infty(\Omega)}\|\partial_t\widehat{U}\|\big)\,\mathrm{d}t.
\end{aligned}
\tag{4.6}
$$

We use $\widehat{U}(t,\cdot) - U^-(t,\cdot) = (t-t_j)\partial_t\widehat{U}(t,\cdot)$ for $t\in[t_j,t_{j+1})$ and $j=0,1,\dots,J_T-1$ to verify that for the last term on the right-hand side of (4.2) we have

$$
\begin{aligned}
\Big|\int_0^T \big(\partial_t\widehat{V}, \mathcal{I}_h[\phi\{U^- - \widehat{U}\}]\big)\,\mathrm{d}t\Big| &\le \tau\int_0^T \|\partial_t\widehat{V}\|\,\|\partial_t\widehat{U}\|\,\|\phi\|_{L^\infty(\Omega)}\,\mathrm{d}t \\
&\le \tau^{1/2}\Big(\tau\int_0^T \|\partial_t\widehat{V}\|^2\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \|\partial_t\widehat{U}\|^2\,\mathrm{d}t\Big)^{1/2}\|\phi\|_{L^\infty(\Omega_T)}.
\end{aligned}
\tag{4.7}
$$

Here, we also employed that $\|v_h\|^2 \le \int_\Omega \mathcal{I}_h[v_h^2]\,\mathrm{d}x \le C\|v_h\|^2$ for all $v_h\in\mathbb{V}_h$. On combining (4.2)-(4.7) and using Proposition 3.1 as well as Lemmas 4.1 and 4.2 we verify that, as $h\to 0$,

$$
\int_0^T \big(\partial_t\widehat{V}, \mathcal{I}_h[\phi U^-]\big)\,\mathrm{d}t \to -\int_0^T \big(\partial_t u, \partial_t[\phi u]\big)\,\mathrm{d}t + \big(u_1, \phi(0)u_0\big).
\tag{4.8}
$$

Second term. The treatment of the second term of (4.1) is similar to the analysis carried out above. Using that $\partial_i U^-\cdot\big(\phi\partial_i U^-\big) = 0$, $i=1,2,\dots,m$, we notice that

$$
\begin{aligned}
&\int_0^T \big(\nabla U^-, \nabla\mathcal{I}_h[\phi U^-]\big)\,\mathrm{d}t \\
&\quad = \sum_{i=1}^m \int_0^T \big(\partial_i U^-, (\partial_i\phi)U^-\big)\,\mathrm{d}t + \int_0^T \big(\nabla U^-, \nabla\{\mathcal{I}_h[\phi U^-] - [\phi U^-]\}\big)\,\mathrm{d}t.
\end{aligned}
\tag{4.9}
$$

Since $\nabla U^- \rightharpoonup \nabla u$ in $L^2(\Omega_T;\mathbb{R}^{m\times\ell})$ and $U^-\to u$ in $L^2(\Omega_T;\mathbb{R}^\ell)$ we verify that for the first term on the right-hand side of (4.9) we have, as $h\to 0$,

$$
\sum_{i=1}^m \int_0^T \big(\partial_i U^-, (\partial_i\phi)U^-\big)\,\mathrm{d}t \to \sum_{i=1}^m \int_0^T \big(\partial_i u, (\partial_i\phi)u\big)\,\mathrm{d}t.
\tag{4.10}
$$

The second term on the right-hand side of (4.9) is estimated as

$$
\begin{aligned}
\Big|\int_0^T \big(\nabla U^-, \nabla\{\mathcal{I}_h[\phi U^-] - [\phi U^-]\}\big)\,\mathrm{d}t\Big| &\le Ch\int_0^T \|\nabla U^-\|\,\|D_{\mathcal{T}_h}^2[\phi U^-]\|\,\mathrm{d}t \\
&\le h\int_0^T \|\nabla U^-\|\big(\|D^2\phi\|\,\|U^-\|_{L^\infty(\Omega)} + \|\nabla\phi\|_{L^\infty(\Omega)}\|\nabla U^-\|\big)\,\mathrm{d}t.
\end{aligned}
\tag{4.11}
$$

A combination of (4.9)-(4.11) shows that, as $h\to 0$, we have

$$
\int_0^T \big(\nabla U^-, \nabla\mathcal{I}_h[\phi U^-]\big)\,\mathrm{d}t \to \int_0^T \big(\nabla u, \nabla[\phi u]\big)\,\mathrm{d}t,
\tag{4.12}
$$

where we used $\partial_i u \cdot \big(\phi \partial_i u\big) = 0$, $i = 1, 2, ..., m$, almost everywhere in Ω_T.
Third term. To control the third term on the left-hand side of (4.1) we employ

$$\begin{aligned}
\big\|\nabla \mathcal{I}_h\big[\phi U^-\big]\big\| &\le \big\|\nabla\big[\phi U^-\big]\big\| + \big\|\nabla\big\{\mathcal{I}_h\big[\phi U^-\big] - \big[\phi U^-\big]\big\}\big\| \\
&\le \big\|\nabla\big[\phi U^-\big]\big\| + Ch\big\|D^2_{\mathcal{T}_h}\big[\phi U^-\big]\big\| \\
&\le \big\|\nabla\phi\big\|\big\|U^-\big\|_{L^\infty(\Omega)} + \big\|\phi\big\|_{L^\infty(\Omega)}\big\|\nabla U^-\big\| + Ch\big\|D^2\phi\big\|\big\|U^-\big\|_{L^\infty(\Omega)} + Ch\big\|\nabla\phi\big\|_{L^\infty(\Omega)}\big\|\nabla U^-\big\|
\end{aligned}$$

to verify with $\big\|U^-\big\|_{L^\infty(\Omega)} \le 1$ that

$$\begin{aligned}
\Big|\theta\tau\int_0^T \big(\nabla V^+, \nabla\mathcal{I}_h\big[\phi U^-\big]\big)\,\mathrm{d}t\Big| &\le C\theta\tau^{1/2}\Big(\tau\int_0^T \big\|\nabla V^+\big\|^2\,\mathrm{d}t\Big)^{1/2} \\
&\quad\times\Big(\int_0^T \big(\big\|\nabla\phi\big\| + \big\|\phi\big\|_{L^\infty(\Omega)}\big\|\nabla U^-\big\| + Ch\big\|D^2\phi\big\| + Ch\big\|\nabla\phi\big\|_{L^\infty(\Omega)}\big\|\nabla U^-\big\|\big)^2\,\mathrm{d}t\Big)
\end{aligned} \tag{4.13}$$

The right-hand side tends to 0 as $h \to 0$ if $\theta > 1/2$. Otherwise, we employ the inverse estimate $\big\|\nabla V^+\big\| \le Ch_{min}^{-1}\big\|V^+\big\|$, cf. (2.3), to conclude that the right-hand side of (4.13) tends to 0 as $h \to 0$ if $\theta \le 1/2$. □

The previous results enable us to establish the convergence result claimed in part (iii) of Theorem 2.1.

Proposition 4.2. *Assume that $\mathcal{T}_h$ is weakly acute and $\tau = o\big(h_{min}^2\big)$ if $\theta < 1/2$. Suppose that $U^0 \to u_0$ and $V^0 \to u_1$ in $L^2(\Omega;\mathbb{R}^\ell)$ as $h \to 0$. If $\tau\Upsilon_\theta(h_{min}) \to 0$ as $h \to 0$ then every weak accumulation point $u \in H^1(\Omega_T)$ of the sequence $\big(U^+\big)_{h>0}$ as in Lemma 4.2 is a weak solution of* (1.1)-(1.2).

Proof. Items (1) and (2) of Definition 2.1 have been verified in Lemma 4.2. The energy inequality of item (5) is a direct consequence of Proposition 3.1 and weak lower semicontinuity of norms. Since every $w \in C_0^\infty([0,T); C^\infty(\overline{\Omega};\mathbb{R}^{n+1}))$ with $w(t,x) \in T_{u(t,x)}S^n$, i.e., $w(t,x)\cdot u(t,x) = 0$, almost everywhere in Ω_T can be written as $w = \phi u$ with $\phi \in H^1(0,T;H^1(\Omega;so(n+1)))$ given by $\phi = u \wedge w := \big(u_i w_j - u_j w_i\big)_{i,j=1,2,\dots,n+1}$ and since the identity of Proposition 4.1 holds in fact for all $\phi \in H^1(0,T;H^1(\Omega;so(n{+}1)))$ with $\phi(T) = 0$ we verify item (3) of Definition 2.1. Weak continuity of the trace operator $v \mapsto v(0)$ as a mapping from $H^1(\Omega_T;\mathbb{R}^\ell)$ into $L^2(\Omega;\mathbb{R}^\ell)$ and $U^0 \to u_0$ in $L^2(\Omega;\mathbb{R}^\ell)$ as $h \to 0$ prove that $u(0) = u_0$ in the sense of traces and thus yield item (4) of Definition 2.1. □

5. Numerical Experiments

To test the practical performance of Algorithm (A) we employ initial data which were first used in [BKP08] to experimentally study the existence of singular solutions for Landau-Lifshitz-Gilbert equations.

Example 5.1. *Set $N := S^2$, $\Omega := (-1/2, 1/2)^2$, $T := 1$, and*

$$u_0(x) := \begin{cases} \dfrac{\big(2ax_1, 2ax_2, a^2 - |x|^2\big)}{a^2 + |x|^2} & \text{for } |x| \le 1/2, \\ (0,0,-1) & \text{for } |x| \ge 1/2, \end{cases} \qquad u_1(x) := 0$$

for $x = (x_1, x_2) \in \Omega$, $r := |x|$, and $a := (1-2r)^4$.

Throughout this section, we use triangulations $\mathcal{T}_\ell$ of $(-1/2, 1/2)^2$ which are obtained by ℓ uniform red refinements of the triangulation $\mathcal{T}_0 = \{K_1, K_2\}$ which consists of the two triangles

$$K_1 = \mathrm{conv}\big\{(-1/2,-1/2), (1/2,-1/2), (1/2,1/2)\big\},$$
$$K_2 = \mathrm{conv}\big\{(-1/2,-1/2), (1/2,1/2), (1/2,-1/2)\big\}.$$

In the following we refer to the maximal diameter of elements in $\mathcal{T}_\ell$ through $h = h_\ell = 2^{-\ell}$, i.e., we omit the factor $\sqrt{2}$. The time-step size is always chosen as $\tau = h_\ell/4$.

The iterative scheme was implemented in Matlab and all systems of linear equations were solved with Matlab's backslash operator. The CPU-time for 1024 time steps of Algorithm (A) on the triangulation $\mathcal{T}_8$ with 131,072 triangles was less than 4 hours on an Intel Dual-Core Xeon E7220 (2.93GHz) processor with 8MB cache.

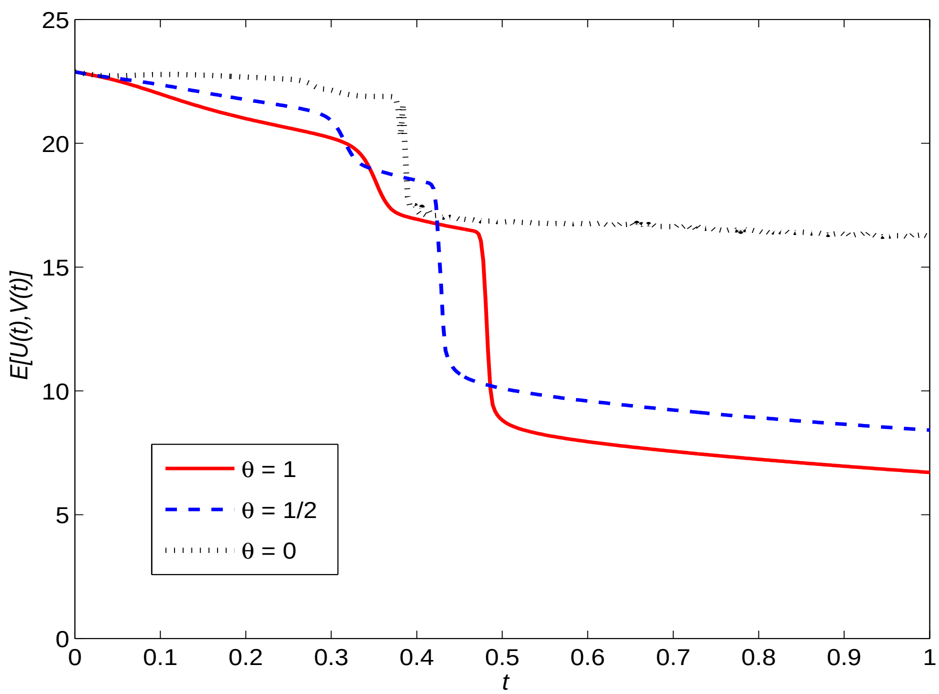

FIGURE 1. Discrete energy $E\big[U^j, V^j\big]$ for approximations obtained with Algorithm (A) on triangulation $\mathcal{T}_7$ with $\tau = h_7/4$ and parameters $\theta = 0, 1/2, 1$.

5.1. **Instability of the scheme for $\theta = 0$.** We ran Algorithm (A) with $\theta = 0,\ 1/2,\ 1$ on triangulation $\mathcal{T}_7$. Figure 1 displays the discrete energy $E\big[U^j, V^j\big]$ as a function of t_j, $j = 0, 1, ..., J_T$. We observe that for the semi-implicit schemes defined through $\theta = 1/2$ and $\theta = 1$ the energy is monotonically decreasing as predicted by Proposition 3.1. For $\theta = 0$ the condition that guarantees stability of the iteration, i.e., $\tau \le h^2$, is not satisfied and we see in Figure 1 that the energy is not monotonically decreasing indicating an unstable behavior of the iteration in the explicit case.

5.2. **Effect of numerical dissipation.** The good stability properties of Algorithm (A) for $\theta \ge 1$ are accompanied by strong damping of the iteration and large numerical dissipation. In Figure 2 we plotted for the triangulations $\mathcal{T}_6$, $\mathcal{T}_7$, and $\mathcal{T}_8$ the relative dissipation defined through the quantity

$$\frac{E\big[U^j, V^j\big] - E\big[U^0, V^0\big]}{E\big[U^0, V^0\big]}$$

in dependence of the discrete time-steps t_j, $j = 0, 1, ..., J_T$. The graphs show that in the time interval $(0, 3/10)$ the relative dissipation decays linearly as $h \to 0$ which is in correct agreement with the estimate of Proposition 3.1. At later times, the relative dissipation does not decay as the mesh- and time-step size become smaller and this is assumed to be related to the occurrence of a singularity indicated by large, maximal spatial gradients, i.e., energy dissipation is not just a numerical artifact when topological changes take place.

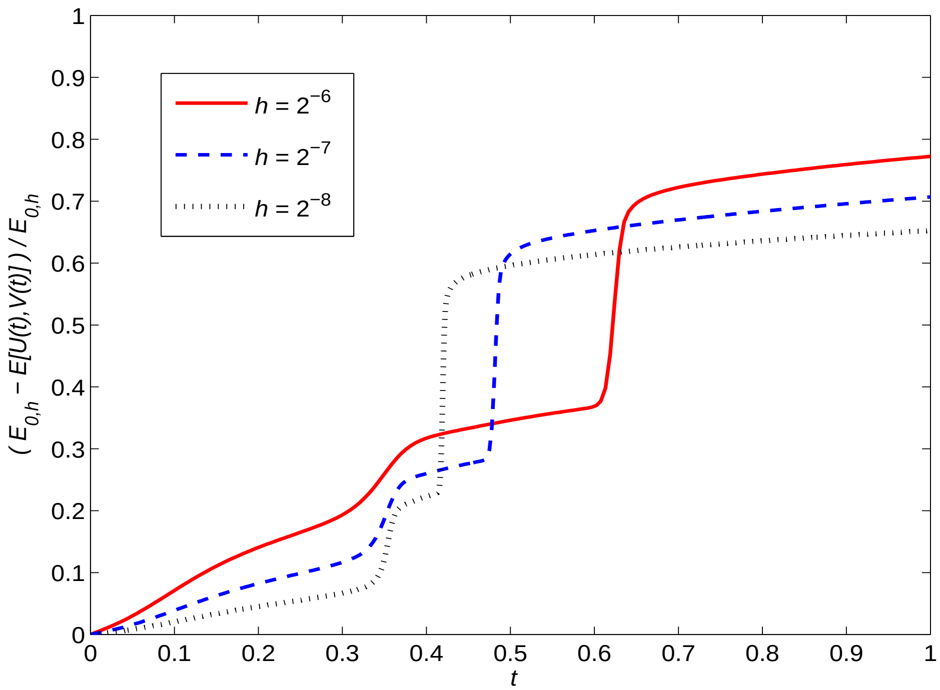

FIGURE 2. Relative dissipation of the approximations obtained with Algorithm (A) for $\theta = 1$ on triangulations $\mathcal{T}_6$, $\mathcal{T}_7$, $\mathcal{T}_8$ and with $\tau = h_\ell/4$.

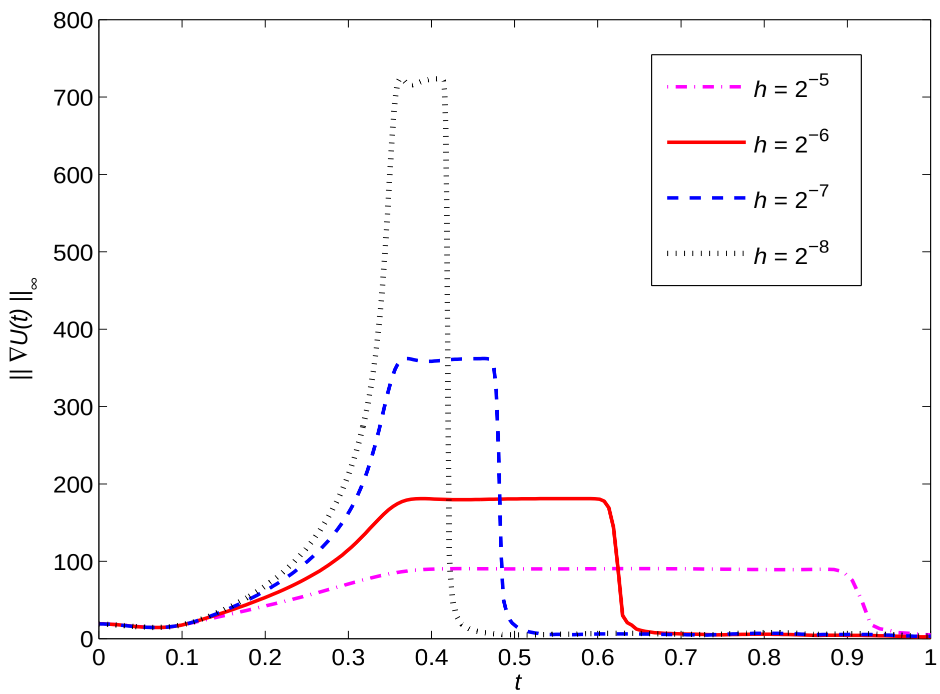

FIGURE 3. $W^{1,\infty}$ semi-norm as a function of $t \in (0, 1)$ for various approximations obtained with $\theta = 1$.

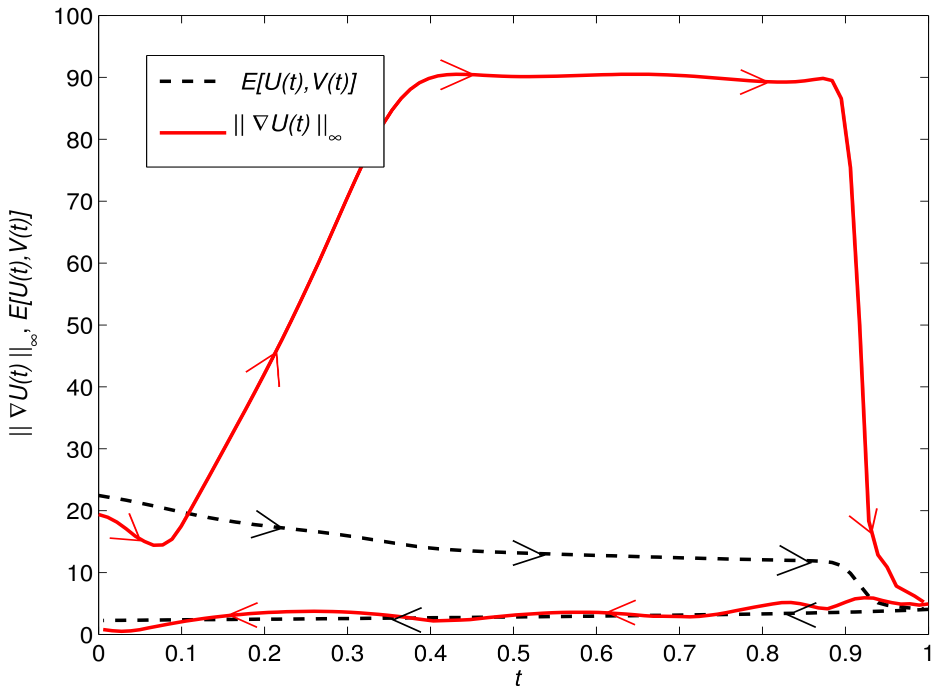

FIGURE 4. Energy and $W^{1,\infty}$ semi-norm for an approximation obtained with $\mathcal{T}_5$ and $\theta = 1$. The discrete evolution was reversed at $t = 1$ by employing the initial data U^{J_T} and $-V^{J_T}$.

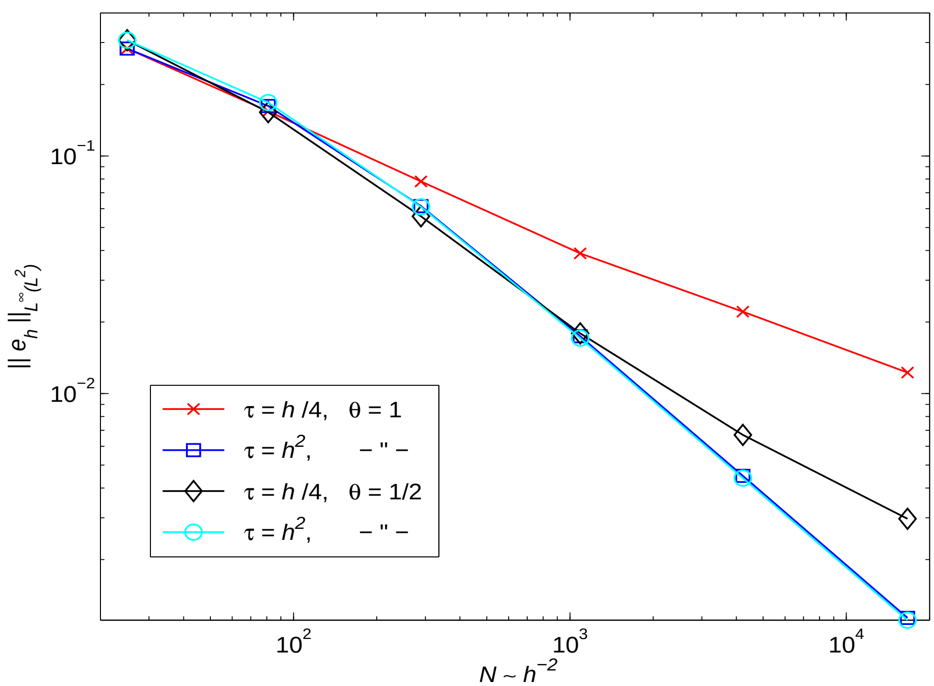

FIGURE 5. Error $\max_{j=0,1,\ldots,J_T} \left\| U^j - \mathcal{I}_h u(t_j,\cdot) \right\|$ in dependence of the maximal mesh-size h of distorted triangulations $\widetilde{\mathcal{T}}_\ell$ for different relations of τ and h and $\theta = 1,\, 1/2$.

5.3. **Finite-time blow-up of weak solutions.** The loss of energy reported in the previous subsection occurs together with changes of the topological properties of the numerical approximations. Figure 3 shows that at the time when the energy drops by approximately 2π, the numerical solution attains the maximal $W^{1,\infty}(\Omega)$ semi-norm among functions in $\mathbb{V}_h^3$ whose nodal values belong to S^2, i.e.,

$$\max_{\substack{w_h \in \mathbb{V}_\ell^3 \\ \forall z \in \mathcal{N}_h,\, |w_h(z)|=1}} \big\|\nabla w_h\big\|_{L^\infty(\Omega)} = 2\sqrt{2}\, h_\ell^{-1},$$

where $\mathbb{V}_\ell$ is the lowest order finite element space defined through the triangulation $\mathcal{T}_\ell$. Figures 6 and 7 illustrate the behavior of the solution when finite-time blow-up occurs. For $\ell = 5$ we plotted the scaled discrete vector field and a zoom to the origin for various time steps. We observe that at $t_j \approx 0.375$ the unit length vector $U(t_j, 0)$ points into the opposite direction than the surrounding vectors. Within a short time interval, the numerical approximation at the origin changes its direction and this is accompanied by a drop of the maximal $W^{1,\infty}$ semi-norm to an h-independent value. Although this series of experiments indicates existence of singular solutions for (1.1)-(1.2), it cannot be ruled out that this discrete finite-time blow-up may break down for very fine mesh-sizes and Neumann boundary conditions as in the case of the Landau-Lifshitz-Gilbert equations, cf. [BKP08]

5.4. **Irreversibility of the discrete evolution.** Owing to the semi-implicit, non-symmetric discretization, the discrete evolution defined by Algorithm (A) is in general not reversible. To illustrate this effect we ran the algorithm with $\ell = 5$ and $\theta = 1$ and used the final output U^{J_T} and V^{J_T} of Algorithm (A) to restart the algorithm with the discrete initial vector fields $U^0 = U^{J_T}$ and $V^0 = -V^{J_T}$. Figure 4 displays the energy and the $W^{1,\infty}$ semi-norms for the forward evolution with t running from 0 to 1 and the backward evolution for t decreasing from 1 to 0.

5.5. **Experimental convergence rates.** To experimentally study the rate of convergence that we can get for smooth solutions we introduce a source term $f \in C(0,T; L^2(\Omega;\mathbb{R}^\ell))$ in (1.1), i.e., we approximate the problem of finding a map $u : \Omega_T \to N$ satisfying

$$-\int_0^T \big(\partial_t u, \partial_t w\big)\,\mathrm{d}t + \int_0^T \big(\nabla u, \nabla w\big)\,\mathrm{d}t = \big(u_1, w(0)\big) + \int_0^T \big(f, w\big)\,\mathrm{d}t \tag{5.1}$$

for all $w \in C_0^\infty([0,T); C^\infty(\overline{\Omega};\mathbb{R}^\ell))$ such that $w(t,x) \in T_{u(t,x)}N$ for every $(t,x) \in \Omega_T$. Setting $t_{j+\theta} := (j+\theta)\tau$, the appropriate modification of Step (1) in Algorithm (A) then reads: Find $V^{j+1} \in \mathbb{V}_h^\ell$ such that $V^{j+1}(z) \in T_{U^j(z)}N$ for all $z \in \mathcal{N}_h$ and

$$\big(d_t V^{j+1}, W\big) + \big(\nabla\big[U^j + \theta\tau V^{j+1}\big], \nabla W\big) = \big(f(t_{j+\theta}), W\big)$$

for all $W \in \mathbb{V}_h^\ell$ satisfying $W(z) \in T_{U^j(z)}N$ for all $z \in \mathcal{N}_h$. The second step of Algorithm (A) remains unchanged. We choose $\Omega := (0,1)^2$, $T := 1$, and $N := S^1$ and identify $\mathbb{R}^2$ with $\mathbb{C}$ in the following. The function $u : \Omega_T \to S^1$ is for $(t,x) \in \Omega_T$ defined through

$$u(t,x) := e^{i\phi(t,x)}$$

where $\phi : \Omega_T \to \mathbb{R}$ satisfies

$$\phi(t,(x_1,x_2)) = \pi/3 + \sin(2\pi t)\cos(2\pi x_1)\cos(2\pi x_2).$$

To obtain u as an exact solution of (5.1) we employ

$$f := \partial_t^2 u - \Delta u. \tag{5.2}$$

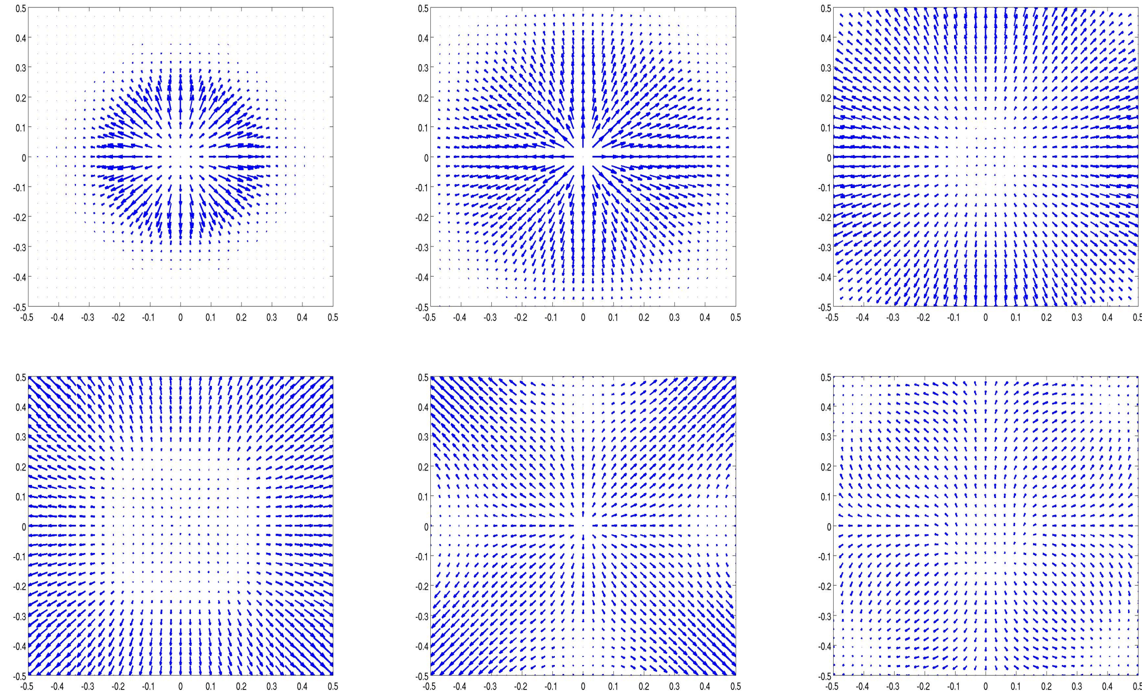

FIGURE 6. Approximations $U(t)$ on $\mathcal{T}_5$ for $t = 0.0,\ 0.1875,\ 0.375,\ 0.5625,\ 0.75,\ 0.9375$ (vectors are scaled by a constant factor).

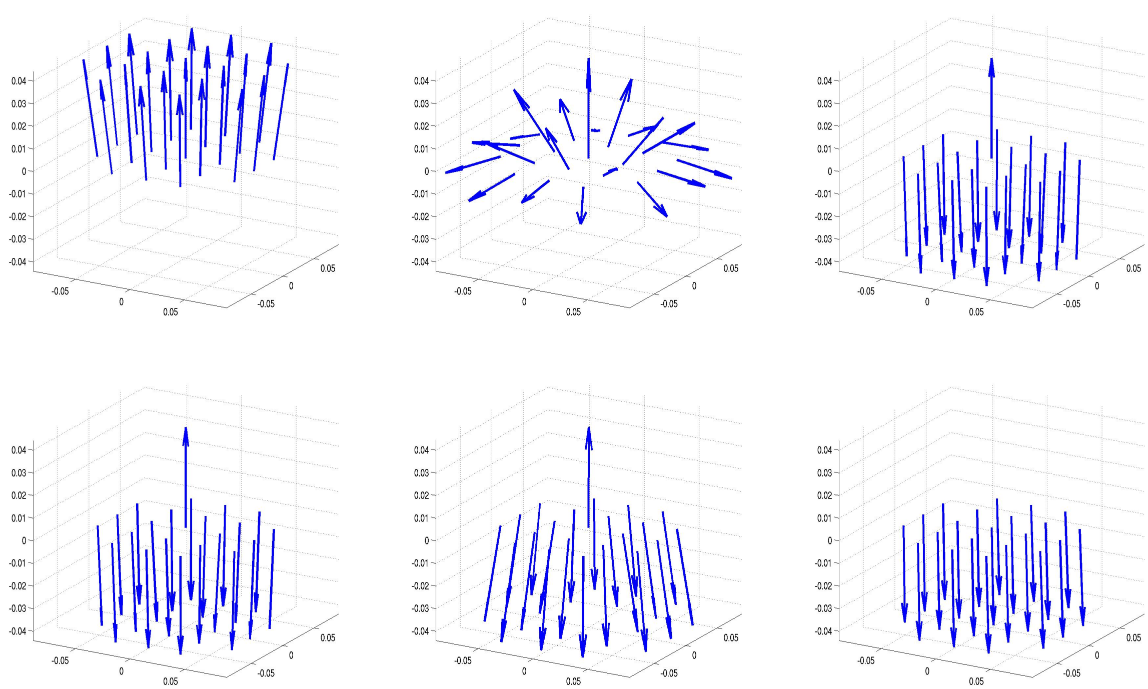

FIGURE 7. Approximations $U(t)$ on $\mathcal{T}_5$ in a neighborhood of the origin for $t = 0.0,\ 0.1875,\ 0.375,\ 0.5625,\ 0.75,\ 0.9375$ (vectors are scaled by a constant factor).

For triangulations $\widetilde{\mathcal{T}}_\ell$, $\ell = 2, 3, ..., 7$, which are obtained by perturbing the inner nodes of the triangulations $\mathcal{T}_\ell$ by random vectors of lengths at most $h_\ell/4$, we employed the modification of Algorithm (A) with the pairs

$$(\tau, \theta) = \Big(\frac{h_\ell}{4}, 1\Big),\ \big(h_\ell^2, 1\big),\ \Big(\frac{h_\ell}{4}, \frac{1}{2}\Big),\ \Big(h_\ell^2, \frac{1}{2}\Big)$$

to obtain approximations of the expected unique solution u of (5.1) with f from (5.2). For each numerical approximation we computed the error

$$\|e_h\|_{L^\infty(L^2)} := \max_{j=0,1,\dots,J_T} \big\|U^j - \mathcal{I}_h u(t_j, \cdot)\big\|.$$

The distorted triangulations avoid additional error reductions due to a possible superconvergence property of highly symmetric triangulations. For the four choices of pairs (τ, θ) as above, Figure 5 displays the error $\|e_h\|_{L^\infty(L^2)}$ as a function of the maximal mesh-size h_ℓ. Nearly linear convergence of the error can be observed if $\tau = h_\ell/4$ and the choice $\theta = 1/2$ significantly reduces the error but does not lead to a higher rate of convergence. If $\tau = h_\ell^2$ then the error converges quadratically with respect to h and the lines in Figure 5 almost coincide for $\theta = 1$ and $\theta = 1/2$.

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