A SIMPLE SCHEME FOR THE APPROXIMATION OF THE ELASTIC FLOW OF INEXTENSIBLE CURVES

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Abstract. A numerical scheme for the approximation of the elastic flow of inextensible curves is devised and convergence of approximations to exact solutions of the nonlinear time-dependent partial differential equation is proved. The nonlinear, pointwise constraint of local length preservation is linearized about a previous solution in each time step which leads to a sequence of linear saddle-point problems. The spatial discretization is based on piecewise Bézier curves and the resulting semi-implicit scheme is unconditionally stable and convergent.

1. Introduction

The elastic energy of an arc-length parametrized curve $z : I \to \mathbb{R}^\ell$ with a closed interval $I = [\alpha, \beta] \subset \mathbb{R}$ and $\ell = 2, 3$ is given by

$$E(z) = \frac{1}{2} \int_I \kappa^2 \, \mathrm{d}x.$$

Since z is arc-length parametrized, i.e., $|z'(x)| = 1$ for all $x \in I$, we have that $|z''| = \kappa$ for the curvature κ of the curve parametrized by z. The elastic flow of inextensible curves is the L^2 gradient flow of the energy functional E on the set of admissible curves

$$\mathcal{A} = \{y \in H^2(I; \mathbb{R}^\ell) : |y'| = 1 \text{ in } I\}.$$

The functions in $\mathcal{A}$ are isometric deformations of the reference interval I. Boundary conditions can be included in the definition of $\mathcal{A}$, e.g., that the curves are clamped at one end of the interval, i.e., that $y(\alpha)$ and $y'(\alpha)$ are prescribed if $I = [\alpha, \beta]$, or that the curve is closed. Unless otherwise stated we assume that $\mathcal{A}$ is defined as above and that no explicit boundary conditions are imposed. This represents a worst case scenario since certain Sobolev estimates are not available in the absence of essential boundary conditions. Necessary modifications of our proposed method for clamped or periodic conditions will be discussed below.

Different versions of the elastic flow of curves have been investigated analytically in [Lin98, DKS02, Ölz11]. The pointwise constraint $|z'(x)| = 1$ in I models the inextensibility of an elastic rod and is motivated by rigorous derivations of lower dimensional theories from three-dimensional hyperelasticity in [FJM02, MM03]. Numerical methods for various versions of the flow have been proposed and analyzed in [DD09, BGN11, BGN12]. In particular, error estimates for a semi-discrete scheme in space and smooth solutions are derived in [DD09] and schemes with equidistribution properties of grid points are discussed in [BGN11, BGN12]. All of those schemes work with the curve directly rather than with an explicit parametrization. Moreover, the methods employ a global length constraint. In this paper we use an approach based on the identity $|z''| = \kappa$ for arc-length parametrized curves. This will lead to a new scheme that only requires the solution of linear systems of equations in every time step, is unconditionally stable, and allows to prove convergence to an exact solution.

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To our knowledge it is the first scheme with these properties. Our approach uses techniques recently developed for the approximation of harmonic maps [Alo97, Bar05, BBFP07] and for the computation of large bending isometries [Bar11b, Bar11a].
Given $z_0 \in \mathcal{A}$ the nonlinear evolution equation seeks $z \in H^1(0,T;L^2(I;\mathbb{R}^\ell)) \cap L^\infty(0,T;H^2(I;\mathbb{R}^\ell))$ such that

$$(\partial_t z, y) + (z'', y'') + (\lambda z', y') = 0, \quad z(0) = z_0, \quad |z'| = 1 \text{ in } I$$

for all $t \in I$, all $y \in H^2(I;\mathbb{R}^\ell)$, and with $(\cdot,\cdot)$ denoting the L^2 inner product on I. Here, λ is a Lagrange multiplier associated to the pointwise constraint $|z'| = 1$ in I.
For the motivation of our numerical scheme we linearize the constraint $|z'| = 1$ about a previous approximation z^n, i.e., the update z^{n+1} is required to satisfy the constraint

$$[z^n]' \cdot [z^{n+1} - z^n]' = 0 \quad \text{in } I.$$

For a time-step size $\tau > 0$ the difference quotient in time is defined by $d_t z^{n+1} = (z^{n+1} - z^n)/\tau$ so that the linearized constraint can be recast as

$$[z^n]' \cdot [d_t z^{n+1}]' = 0 \quad \text{in } I.$$

Imposing the same linear constraint on the test functions y, i.e., $[z^n]' \cdot y' = 0$ in I, leads to the following time-stepping scheme:

$$\begin{aligned}
&\text{Given } z^0, z^1, ..., z^n \in H^2(I;\mathbb{R}^\ell) \text{ compute } z^{n+1} \in H^2(I;\mathbb{R}^\ell) \text{ such that}\\
&[z^n]' \cdot [d_t z^{n+1}]' = 0 \text{ in } I \text{ and}\\
&(d_t z^{n+1}, y) + ([z^n + \tau d_t z^{n+1}]'', y'') = 0\\
&\text{for all } y \in H^2(I;\mathbb{R}^\ell) \text{ satisfying } [z^n]' \cdot [y]' = 0 \text{ in } I.
\end{aligned}$$

The constraint defines a closed linear subspace of $H^2(I;\mathbb{R}^\ell)$ so that the Lax-Milgram lemma yields the existence of a unique solution $d_t z^{n+1}$ in every time step which defines the new approximation $z^{n+1} = z^n + \tau d_t z^{n+1}$. The approximations $(z^n)_{n\geq 0}$ will in general not satisfy the pointwise constraint $|z^n| = 1$ exactly. For a fully discrete version of this scheme we show below that the violation of this identity is of order $O(\tau)$ at the nodes of the underlying triangulation.
For the spatial discretization of the semi-discrete scheme above we use conforming subspaces $\mathbb{V}_h \subset H^2(I;\mathbb{R}^\ell)$ that are subordinated to a partition $\mathcal{T}_h$ of I into subintervals of maximal length h. We identify the partition $\mathcal{T}_h$ with a sequence of nodes $x_0 < x_1 < ... < x_M$ and the space $\mathbb{V}_h$ is given by cubic splines on $\mathcal{T}_h$, i.e.,

$$\mathbb{V}_h = \{v_h \in C^1(I;\mathbb{R}^\ell) : v_h|_{I_i} \in \mathcal{P}_3(I_i)^\ell,\ i = 1,2,...,M\},$$

where $I_i = [x_{i-1}, x_i]$ and $\mathcal{P}_k(I_i)$ denotes the set of polynomials of degree $k \geq 0$ restricted to I_i. The restriction of $v_h \in \mathbb{V}_h$ to an interval I_i is entirely determined by the values of v_h and v_h' at the endpoints of I_i, i.e., by the four vectors $v_h(x_{i-1})$, $v_h(x_i)$, $v_h'(x_{i-1})$, and $v_h'(x_i)$ that define the positions and tangents of the nodes of the discrete curve.
The arc-length condition and its linearized version cannot be satisfied by piecewise polynomial functions unless they are globally affine. Therefore, in the fully discrete scheme we only impose the constraint at the nodes of the partition $\mathcal{T}_h$. The scheme we propose thus reads:

$$\begin{aligned}
&\text{Given } z_h^0, z_h^1, ..., z_h^n \in \mathbb{V}_h \text{ compute } z_h^{n+1} \in \mathbb{V}_h \text{ such that}\\
&[z_h^n]'(x_i) \cdot [d_t z_h^{n+1}]'(x_i) = 0 \text{ for } i = 0,1,...,M \text{ and}\\
&(d_t z_h^{n+1}, y_h) + ([z_h^n + \tau d_t z_h^{n+1}]'', y_h'') = 0\\
&\text{for all } y_h \in \mathbb{V}_h \text{ satisfying } [z_h^n]'(x_i) \cdot [y_h]'(x_i) = 0 \text{ for } i = 0,1,...,M.
\end{aligned}$$

As above, existence and uniqueness of the corrections $d_t z_h^{n+1}$ in each time step follow from the Lax-Milgram lemma and this defines the updates $z_h^{n+1} = z_h^n + \tau d_t z_h^{n+1}$. We remark that only a time-independent mass and fourth order stiffness matrix have to be computed.
Unconditional stability of the iteration follows immediately from testing the formulation with $y_h = d_t z_h^{n+1}$, i.e., incorporating a binomial formula we obtain,

$$\|d_t z_h^{n+1}\|^2 + \frac{d_t}{2}\|[z_h^{n+1}]''\|^2 + \frac{\tau}{2}\|[d_t z_h^{n+1}]''\|^2 = 0 \tag{1.1}$$

for all $n \geq 0$ and with $\|\cdot\|$ denoting the L^2 norm on I. The dissipative character of the semi-implicit scheme, i.e., the occurrence of the term $(\tau/2)\|[d_t z_h^{n+1}]''\|^2$, is important for our convergence analysis below. To bound the error in the constraint we note that owing to the orthogonality $[z_h^m]'(x_i) \cdot [d_t z_h^{m+1}]'(x_i) = 0$ for $i = 0, 1, ..., M$ we have

$$|[z_h^{m+1}]'(x_i)|^2 = |[z_h^m + \tau d_t z_h^{m+1}]'(x_i)|^2 = |[z_h^m]'(x_i)|^2 + \tau^2 |[d_t z_h^{m+1}]'(x_i)|^2 \tag{1.2}$$

so that a summation over $m = 0, 1, ..., n$ leads to

$$|[z_h^{n+1}]'(x_i)|^2 - 1 = \tau^2 \sum_{m=0}^{n} |[d_t z_h^{m+1}]'(x_i)|^2$$

provided that the initial discrete curve $z_h^0 \in \mathbb{V}_h$ satisfies $|[z_h^0]'(x_i)| = 1$ for $i = 0, 1, ..., M$.
Boundary conditions that fix the curve and/or its tangent are easily included in the numerical scheme, i.e., by directly imposing combinations of the constraints

$$d_t z_h^{n+1}(x_0) = 0,\ [d_t z_h^{n+1}]'(x_0) = 0,\ d_t z_h^{n+1}(x_M) = 0,\ [d_t z_h^{n+1}]'(x_M) = 0$$

and requiring that the initial curve z_0 satisfies the corresponding, possibly inhomogeneous conditions. Similarly, a closed curve is modeled with the periodicity conditions $d_t z_h^{n+1}(x_0) = d_t z_h^{n+1}(x_M)$ and $[d_t z_h^{n+1}]'(x_0) = [d_t z_h^{n+1}]'(x_M)$.
The remainder of this paper is organized as follows. In Section 2 we prove that the approximations accumulate at exact solutions as the discretization parameters tend to zero. Numerical experiments reported in Section 3 illustrate the qualitative behaviour of the evolution problem and confirm the theoretically predicted properties of the algorithm.

2. Convergence

Given the sequence $(z_h^n)_{n=0,...,N}$ obtained with our approximation scheme we set $I_T = [0, T] \times I$ and define the continuous, piecewise affine interpolant $\widehat{Z} : I_T \to \mathbb{R}^\ell$ and the piecewise constant interpolants $Z^\pm : I_T \to \mathbb{R}^\ell$ by setting for $t \in (t_{n-1}, t_n]$ with $t_n = n\tau$

$$\widehat{Z}(t, x) = z_h^{n-1}(x) + (t - t_{n-1}) d_t z_h^n(x), \quad Z^+(t, x) = z_h^n(x), \quad Z^-(t, x) = z_h^{n-1}(x).$$

For the convergence analysis we need the interpolation operators $\mathcal{I}_{3,h} : C^1(I; \mathbb{R}^\ell) \to \mathbb{V}_h$ and $\mathcal{I}_{1,h} : C(I) \to \mathcal{S}^1(\mathcal{T}_h)$, where $\mathcal{S}^1(\mathcal{T}_h) = \{v_h \in C(I) : v_h|_{I_i} \in \mathcal{P}_1(I_i),\ i = 1, ..., M\}$ is the space of continuous, elementwise affine functions. The operators are entirely defined by the identities

$$\mathcal{I}_{3,h} v(x_i) = v(x_i), \quad [\mathcal{I}_{3,h} v]'(x_i) = v'(x_i), \quad \mathcal{I}_{1,h} w(x_i) = w(x_i) \tag{2.1}$$

for $i = 0, 1, ..., M$, $v \in C^1(I; \mathbb{R}^\ell)$, and $w \in C(I)$. Throughout the following we let $c > 0$ denote a generic constant and we define $Z_0 = z_h^0$.

Lemma 2.1. *For all $t \in (0, T)$ we have*

$$\frac{1}{2}\|[Z^+]''(t)\|^2 + \int_0^t \|\partial_t \widehat{Z}\|^2 \, \mathrm{d}r + \tau \int_0^t \|[\partial_t \widehat{Z}]''\|^2 \, \mathrm{d}r \leq \frac{1}{2}\|Z_0''\|^2$$

and $\|\mathcal{I}_{1,h}\{|[Z^+]'(t, \cdot)|^2\} - 1\|_{L^1(I)} \leq c\tau^{1/2} \|Z_0''\|^2$.

Proof. The energy estimate follows from (1.1) by summation over $n = 0, 1, ..., N$. Elementary norm equivalences, (1.2), continuity of the interpolation operator $\mathcal{I}_{1,h}$ on polynomials, and the estimate $\|v'\|^2 \le c\|v\|(\|v\| + \|v''\|)$ for all $v \in H^2(I)$ show

$$\|\mathcal{I}_{1,h}\{|[Z^+(t,\cdot)]'|^2 - 1\}\|_{L^1(I)} \le c\tau \int_0^T \|\mathcal{I}_{1,h}\{[\partial_t \widehat{Z}]'(t,\cdot)\}\|^2 \,\mathrm{d}t \le c\tau \int_0^T \|[\partial_t \widehat{Z}]'(t,\cdot)\|^2 \,\mathrm{d}t$$

$$\le c\tau^{1/2}\Big(\int_0^T \|[\partial_t \widehat{Z}](t_m,\cdot)\|^2 \,\mathrm{d}t\Big)^{1/2}\Big(\tau\int_0^T \|[\partial_t \widehat{Z}]''(t_m,\cdot)\|^2 \,\mathrm{d}t\Big)^{1/2} + \tau \int_0^T \|[\partial_t \widehat{Z}](t_m,\cdot)\|^2 \,\mathrm{d}t.$$

The energy estimate implies the assertion. □

Remark 2.1. *The bounds of Lemma 2.1 are uniform in t and independent of T. For clamped or periodic boundary conditions the estimate $\|v'\| \le c\|v''\|$ for $v \in H^2(I)$ leads to the improved bound $\|\mathcal{I}_{1,h}\{|[Z^+]'(t,\cdot)|^2\} - 1\|_{L^1(I)} \le c\tau\|Z_0''\|^2$.*

The a priori bounds of Lemma 2.1 allow us to identify the accumulation points of the sequence of approximations.

Lemma 2.2. *Suppose that $\tau = o(1)$ as $h \to 0$ and $\|Z_0\|_{H^2(I)}$ is uniformly bounded. Every accumulation point $z \in L^\infty(0,T;H^2(I;\mathbb{R}^\ell))$ of the bounded sequence $(Z^+)_{h>0}$ satisfies $z \in H^1(0,T;L^2(I;\mathbb{R}^\ell))$ and after extraction of an appropriate subsequence we have*

$$\begin{aligned} \partial_t \widehat{Z} &\rightharpoonup \partial_t z \quad &&\text{in } L^2(I_T;\mathbb{R}^\ell),\\ Z^+,\ \widehat{Z} &\rightharpoonup^* z &&\text{in } L^\infty(0,T;H^2(I;\mathbb{R}^\ell)),\\ Z^+,\ Z^-,\ \widehat{Z} &\to z &&\text{in } L^2(I_T;\mathbb{R}^\ell). \end{aligned}$$

Moreover, we have $|z'| = 1$ in I_T and if $Z^0 \to z_0$ in $L^2(I;\mathbb{R}^\ell)$ then $z(0,\cdot) = z_0$.

Proof. The claimed convergences to a limit z follow from the estimates of Lemma 2.1. To show that the limits satisfy $|z'| = 1$ in I_T we first notice that $[Z^+](t,\cdot)$ is uniformly bounded, i.e.,

$$\|Z^+(t,\cdot)\| \le \|Z_0\| + \int_0^T \|\partial_t \widehat{Z}\| \,\mathrm{d}t \le \|Z_0\| + T^{1/2}\Big(\int_0^T \|\partial_t \widehat{Z}\|^2 \,\mathrm{d}t\Big)^{1/2}.$$

We then verify that for $t \in (0,T)$ we have

$$\|\mathcal{I}_{h,1}\{|[Z^+]'|^2\} - \{|[Z^+]'|^2\}\|_{L^1(I)} \le ch\|(|[Z^+]'|^2)'\|_{L^1(I)} \le ch\|[Z^+]'\|_{L^2(I)}\|[Z^+]''\|_{L^2(I)}.$$

The triangle inequality, the estimate $\|v'\| \le c(\|v\| + \|v''\|)$ for $v \in H^2(I)$, and the bound of Lemma 2.1 show that $\||[Z^+]'|^2 - 1\|_{L^1(I)} \le c(h + \tau^{1/2})$ which implies that $|z'| = 1$ in I_T. □

We next show that the limits satisfy the right evolution equation in the limit $h \to 0$. For ease of presentation we restrict to the most relevant case $\ell = 3$ and refer to Remark 2.1 (ii) below for generalizations.

Theorem 2.1. *Under the conditions of Lemma 2.2 each accumulation point $z \in L^\infty(0,T;H^2(I;\mathbb{R}^3))$ of the sequence $(Z^+)_{h>0}$ satisfies $|z'| = 1$ in I_T, $z(0,\cdot) = z_0$,*

$$\frac{1}{2}\|z''(t)\|^2 + \int_0^t \|\partial_t z\|^2 \,\mathrm{d}r \le \frac{1}{2}\|z_0''\|^2$$

for $t \in [0,T]$, and

$$\int_0^T (\partial_t z, y) + (z'', [z' \times \phi]') \,\mathrm{d}t = 0$$

for all pairs $y, \phi \in C^\infty(I_T;\mathbb{R}^3)$ with $y' = z' \times \phi$.

Remarks 2.1. *(i) The pairs y, ϕ are related by $y(t,x) = y(t,\alpha) + \int_\alpha^x z' \times \phi \,\mathrm{d}s$, i.e., y is entirely determined by its values at $x = \alpha$ and the function ϕ.*
(ii) Since every function $y \in L^\infty(0,T;H^2(I;\mathbb{R}^\ell))$ with $y' \cdot z' = 0$ in I_T can be written in the above form the identity of the theorem is equivalent to

$$\int_0^T (\partial_t z, y) + (z'', y'')\,\mathrm{d}t = 0$$

for all $y \in L^\infty(0,T;H^2(I;\mathbb{R}^\ell))$ with $y' \cdot z' = 0$ in I_T. More generally, for such functions y there exists $\phi \in L^\infty(0,T;H^2(I;so(\ell)))$ with $y' = \phi z'$ in I_T, where $so(\ell) = \{X \in \mathbb{R}^{\ell\times\ell} : X^\top = -X\}$.
(iii) Our notion of solution respects the geometric nature of the problem, i.e., that the gradient flow is defined on the tangent space of the manifold $\mathcal{A}$. Alternatively, the constraint included in $\mathcal{A}$ can be treated with a Lagrange multiplier [Ölz11].

Proof. The energy bound follows from Lemmas 2.1 and 2.2. With the above definitions our approximation scheme can be written as

$$\int_0^T (\partial_t \widehat{Z}, y_h) + ([Z^+]'', [y_h]'')\,\mathrm{d}t = 0 \tag{2.2}$$

for all functions $y_h \in L^\infty(0,T;\mathbb{V}_h)$ with $[Z^-]'(t,x_i) \cdot y_h'(t,x_i) = 0$ for $i = 0,1,...,M$ and $t \in [0,T]$. Given any $\phi \in C^\infty(I_T;\mathbb{R}^3)$ and $y_\alpha \in C^\infty([0,T];\mathbb{R}^3)$ we define an admissible test function y_h by setting $y_h(t,\cdot) = \mathcal{I}_{3,h}\widetilde{y}(t,\cdot)$ for

$$\widetilde{y}(t,x) = y_\alpha(t) + \int_\alpha^x [Z^-]'(s,t) \times \phi_h(s,t)\,\mathrm{d}s.$$

Owing to (2.1) we then have for $i = 0,1,...,M$ and $t \in [0,T]$ that

$$y_h'(t,x_i) = [Z^-]'(t,x_i) \times \phi(t,x_i).$$

Using $\widetilde{y}'' = [Z^-]'' \times \phi + [Z^-]' \times \phi'$ and $[Z^+]'' \cdot ([Z^+]'' \times \phi)) = 0$ we see that

$$\begin{aligned}
([Z^+]'', [y_h]'') &= ([Z^+]'', [\widetilde{y}]'') + ([Z^+]'', [y_h - \widetilde{y}]'') \\
&= ([Z^+]'', [Z^- - Z^+]'' \times \phi) + ([Z^+]'', [Z^-]' \times \phi') + ([Z^+]'', [y_h - \widetilde{y}]'') \\
&= -\tau([Z^+]'', [\partial_t \widehat{Z}]'' \times \phi) + ([Z^+]'', [Z^-]' \times \phi') + ([Z^+]'', [y_h - \widetilde{y}]'') = I + II + III.
\end{aligned}$$

After integration in time the first term on the right-hand side is bounded by

$$\begin{aligned}
\int_0^T I\,\mathrm{d}t &\le \tau \int_0^T \|[Z^+]''\|\|[\partial_t \widehat{Z}]''\|\|\phi\|_{L^\infty(I)}\,\mathrm{d}t \\
&\le \tau^{1/2} T \sup_{t\in(0,T)} \|[Z^+(t,\cdot)]''\| \Big(\tau \int_0^T \|[\partial_t \widehat{Z}]''\|^2\,\mathrm{d}t\Big)^{1/2} \|\phi\|_{L^\infty(0,T;L^\infty(I))} \le c\tau^{1/2}\|\phi\|_{L^\infty(I_T)}
\end{aligned}$$

and converges to 0 as $h \to 0$. The second term gives a contribution in the limit $h \to 0$, i.e., owing to the assertions of Lemma 2.2 we have

$$\int_0^T II\,\mathrm{d}t = \int_0^T ([Z^+]'', [Z^-]' \times \phi')\,\mathrm{d}t \to \int_0^T (z'', z' \times \phi')\,\mathrm{d}t.$$

The third term disappears for $h \to 0$ since on every subinterval I_i we have $(Z^-|_{I_i})^{(4)} = 0$ and thus, with interpolation and inverse estimates,

$$\begin{aligned}
\|[y_h - \widetilde{y}]''\|_{L^2(I_i)} \le ch_i^2 \|\widetilde{y}^{(4)}\|_{L^2(I_i)} &\le ch_i^2 (\|[Z^-]'''\|_{L^2(I_i)} + \|[Z^-]''\|_{L^2(I_i)})\|\phi\|_{W^{3,\infty}(I_i)} \\
&\le ch_i \|[Z^-]''\|_{L^2(I_i)} \|\phi\|_{W^{3,\infty}(I_i)}.
\end{aligned}$$

This shows that

$$\int_0^T III \,\mathrm{d}t = \int_0^T ([Z^+]'', [y_h - \widetilde{y}]'') \,\mathrm{d}t \to 0.$$

It remains to prove convergence for the term involving the time derivative in (2.2). Setting

$$y(t,x) = y_\alpha(t) + \int_\alpha^x z'(s,t) \times \phi \,\mathrm{d}s$$

we have, upon inserting $\widetilde{y}$ that

$$y(t,x) - y_h(t,x) = \int_\alpha^x [z - Z^-]'(s,t) \times \phi(s,t) \,\mathrm{d}s + \widetilde{y}(t,x) - \mathcal{I}_{3,h}\widetilde{y}(t,x)$$

and this implies $\|y(t,x) - y_h(t,x)\|_{L^2(0,T;L^\infty(I))} \to 0$. It follows that

$$\int_0^T (\partial_t \widehat{Z}, y_h) \,\mathrm{d}t \to \int_0^T (\partial_t z, y) \,\mathrm{d}t$$

as $h \to 0$ and this finishes the proof of the theorem. □

3. Numerical experiments

The implementation of our numerical scheme leads to linear systems of equations of the form

$$\begin{bmatrix} M + \tau S & B_n^\top \\ B_n & 0 \end{bmatrix} \begin{bmatrix} d_t Z^{n+1} \\ \Lambda^{n+1} \end{bmatrix} = \begin{bmatrix} -SZ^n \\ 0 \end{bmatrix},$$

where M and S are the matrices that represent the inner products (v_h, w_h) and (v_h'', w_h'') for $v_h, w_h \in \mathbb{V}_h$ in a suitable basis, respectively, and B_n encodes the constraints $[z_h^n]'(x_i) \cdot [d_t z_h^{n+1}]'(x_i) = 0$ for $i = 0, 1, ..., M$ together with possible boundary conditions. Our implementation was realized in Matlab with a direct solution of the linear systems of equations.

Our first numerical experiment employs as initial data an arc-length parametrized helix that is clamped at one end.

Example 3.1. *Set $\ell = 3$, $T = 6400$, $I = [0, 4\pi]$, $\alpha^2 = 99/100$, $\beta^2 = 1/100$, and*

$$z_0(x) = \big(\sin(\alpha x), \cos(\alpha x), \beta x\big)$$

for $x \in I$. Clamped boundary conditions $z(t,0) = (0,1,0)$ and $z'(t,0) = (\alpha, 0, \beta)$ are used at $x = 0$.

The employed uniform partitions are defined through the integer $M > 0$ and the nodes $x_i = i4\pi/M$, $i = 0, 1, ..., M$. Figure 1 shows the piecewise cubic interpolant of the initial curve z_0 defined through the partition with $M = 40$ intervals and snapshots of the evolution to a straight line segment. We see that the initial helix immediately unwinds while nearly preserving its length and converges to a straight line as t increases. Qualitatively very similar results were obtained for coarse partitions with 5 intervals.

Table 1 shows the violation of the length preservation constraint for the times $t = 2^k \cdot 100$, $k = 0, 1, ..., 5$ and mesh-sizes $M = 2^m \cdot 10$, $m = 0, 1, ..., 5$ with corresponding time-step size $\tau = h/(4\pi) = 1/M$, i.e., the quotients

$$\ell_h(t) = \frac{\big|h \sum_{i=0}^M |z_h'(t, x_i)| - |z_h'(0, x_i)|\big|}{h \sum_{i=0}^M |z_h'(0, x_i)|}.$$

We see that the error decays approximately linearly as M increases which is in good agreement with our estimates, cf. Remark 2.1 and the proof of Lemma 2.2. The error in the constraint seems to occur in the first time steps and remains nearly constant afterwards which confirms the validity of the time-independent bound of Lemma 2.1. It is expected that the initial error is related to incompatibilities of the initial data.

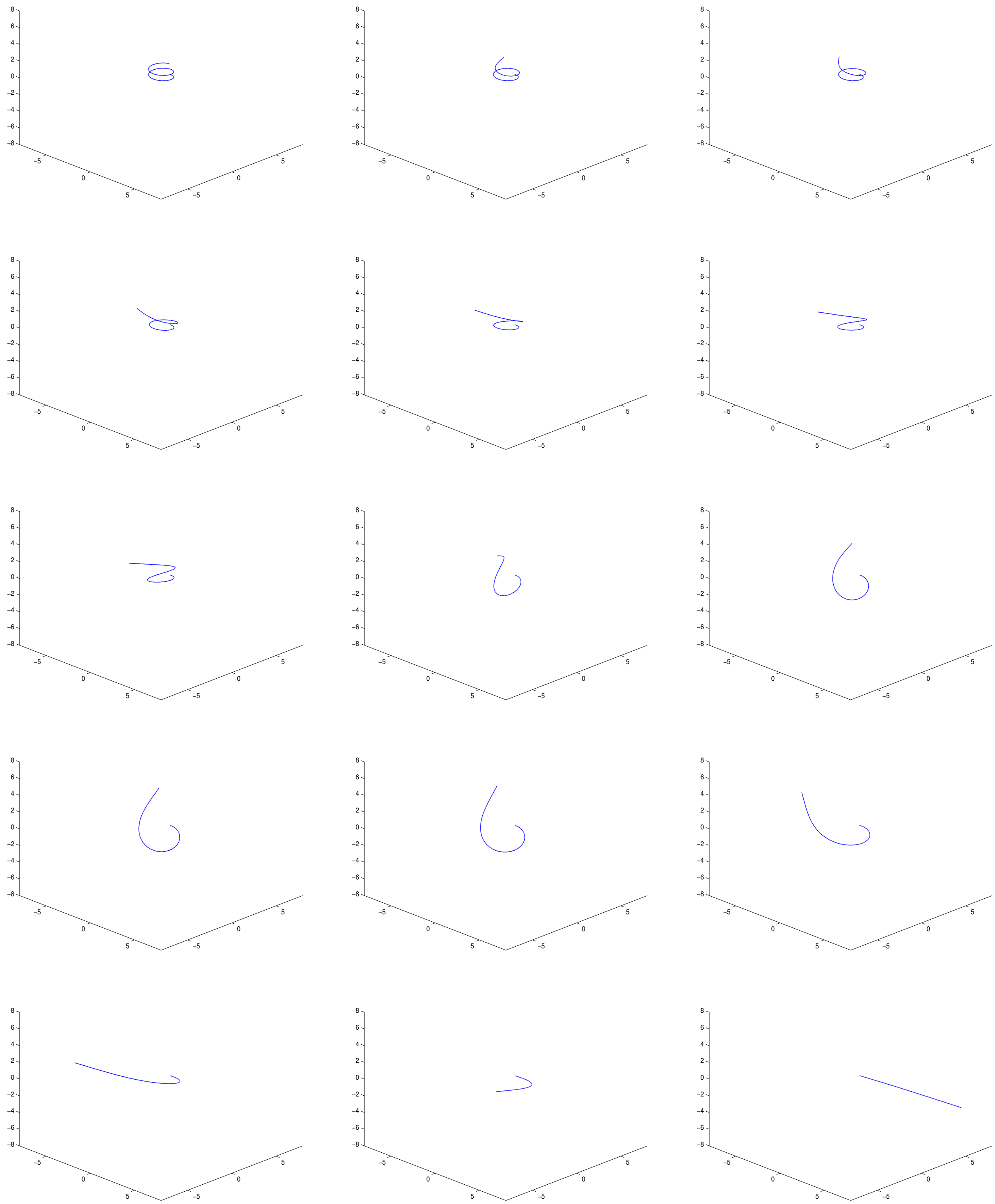

FIGURE 1. Deformed clamped curve $z_h(t,\cdot)$ for $t = 0$, 2, 4, 8, 12, 16, 20, 40, 60, 80, 100, 200, 400, 1600, 6400 (from left to right and top to bottom) and a partition of the parameter domain $I = [0, 4\pi]$ into 40 subintervals and $\tau = 1/40$ in Example 3.1.

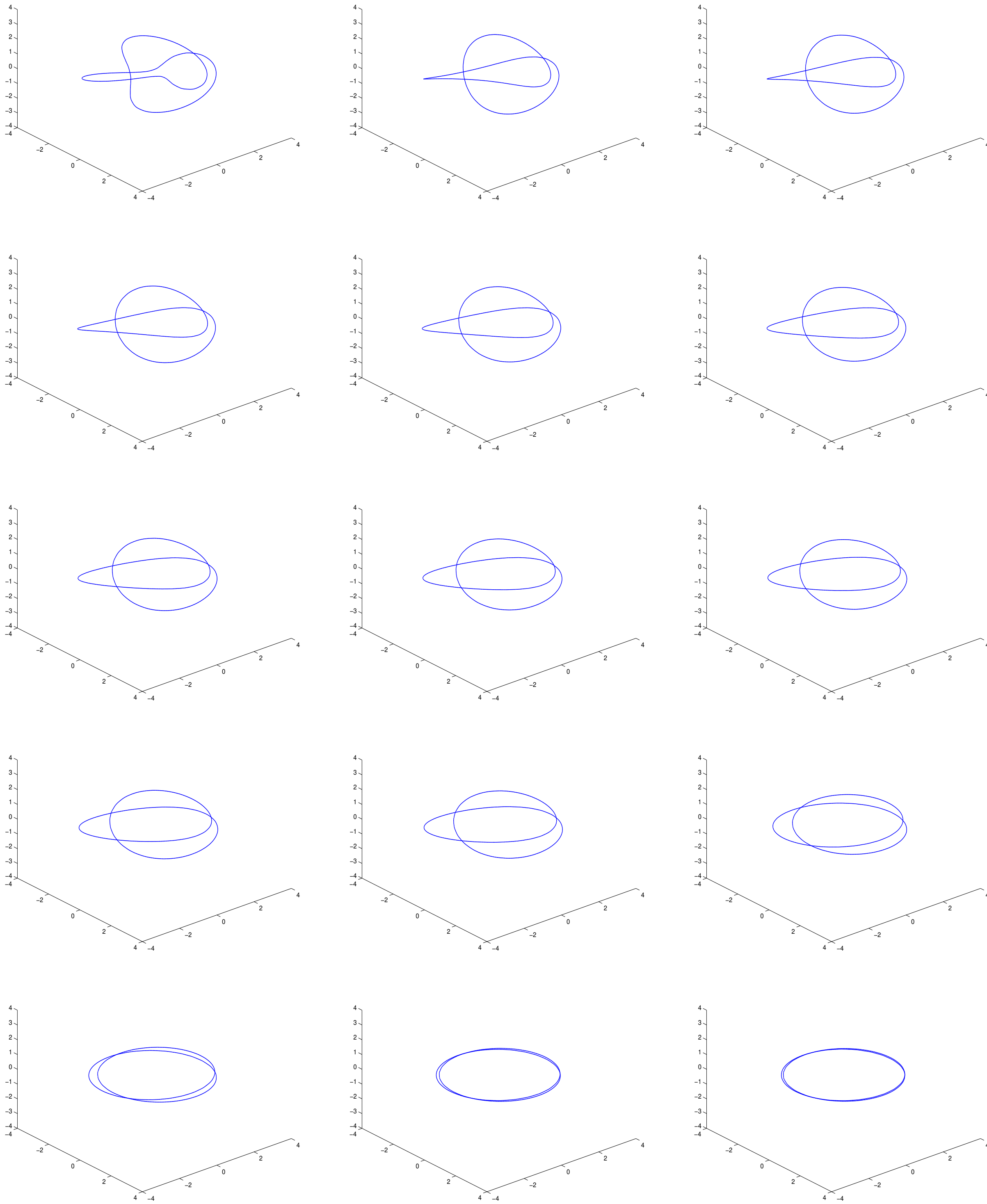

FIGURE 2. Deformed closed curve $z_h(t, \cdot)$ for $t = 0$, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 40, 80, 160, 320 (from left to right and top to bottom) and a nonuniform partition of the parameter domain $I = [0, 32]$ into 40 subintervals with $\tau = 1/40$ in Example 3.2.

M	$t = 100$	$t = 200$	$t = 400$	$t = 800$	$t = 1600$	$t = 3200$	$t = 6400$
10	0.036086	0.036320	0.036447	0.036487	0.036509	0.036521	0.036523
20	0.015775	0.015904	0.015964	0.015982	0.015993	0.015998	0.015999
40	0.007500	0.007565	0.007594	0.007603	0.007608	0.007611	0.007611
80	0.003880	0.003912	0.003926	0.003931	0.003933	0.003935	0.003935
160	0.002117	0.002134	0.002141	0.002143	0.002144	0.002145	0.002145
320	0.001195	0.001203	0.001207	0.001208	0.001208	0.001209	0.001209

TABLE 1. Relative error $\ell_h(t)$ in the violation of the length preservation constraint for different times t and discretizations defined by $h = 4\pi/M$ and $\tau = h/(4\pi) = 1/M$ in Example 3.1. The error decays approximately linearly as $h \sim \tau \to 0$ and is nearly constant for $t \geq 100$.

Our second numerical experiment studies an example from [BGN12] with a closed initial curve that remains closed during the evolution, i.e., we incorporate periodic boundary conditions.

Example 3.2. *Set* $\ell = 3$, $T = 320$, $\widetilde{I} = [0, 1]$, *and for* $\widetilde{x} \in \widetilde{I}$

$$\widetilde{z}_0(\widetilde{x}) = \big((2 + \cos(6\pi\widetilde{x}))\cos(4\pi\widetilde{x}), (2 + \cos(6\pi\widetilde{x}))\sin(4\pi\widetilde{x}), \sin(6\pi\widetilde{x})\big).$$

With the strictly increasing function $\psi(r) = \int_0^r |\widetilde{z}_0'(\widetilde{x})| \,\mathrm{d}\widetilde{x}$ *and its inverse* $\psi^{-1} : I \to \widetilde{I}$ *for* $I = [0, 32]$ *we define* $z_0 = \widetilde{z}_0 \circ \psi^{-1}$ *so that* $|z_0'(x)| = 1$ *for all* $x \in I$.

Snapshots of the evolution for a partition of I into $M = 40$ subintervals, and $\tau = 1/M$ are displayed in Figure 2. We observe that the initial trefoil knot smoothly deforms into a double circle while preserving its length. The relative error in the length-preservation condition was violated by less than 0.1% throughout the evolution.

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