QUASI-OPTIMAL ERROR ESTIMATES FOR IMPLICIT DISCRETIZATIONS OF RATE-INDEPENDENT EVOLUTIONS

SÖREN BARTELS

Abstract. We derive quasi-optimal error bounds for implicit discretizations of a class of rate-independent evolution problems. No regularity assumptions on the exact solutions are made but involved load functionals are assumed to be twice continuously differentiable in time.

1. Introduction

Rate-independent evolution problems provide an attractive framework to describe the deformation of solids that undergo internal changes and develop hysteresis effects, e.g., in plasticity or damage when inertial effects are neglected. The reader is referred to the survey article [Mie05] for details on related analytical developments. Existence theories and numerical approximation schemes are often based on time-discretizations. In particular, implicit schemes allow to construct the evolutions by time incremental minimization problems. A key step in the analysis is then to prove convergence of the semi-discrete approximations to solutions of the original formulation. The rate of convergence is of major interest in the development of efficient numerical schemes. It is the aim of this article to show that implicit time discretizations are quasi-optimally convergent even in the absence of higher regularity properties of the solution but under a moderate regularity condition on the forcing term and a compatibility requirement on the initial configuration.

We will show that a transformation of the unknown in a class of rate-independent evolution problems allows to carry out an error analysis that employs techniques developed in [Rul96, NSV00]. These techniques control the consistency error by discrete dissipation quantities and exploit a cyclic property of subdifferentials. The resulting analysis controls the time-discretization error and allows to analyze fully discrete schemes via a splitting of the error sources as in, e.g., [MPPS10]. As opposed to the temporal discretization the determination of convergence rates with respect to the spatial discretization parameter seems to require regularity results unless the dissipation functional and the forcing terms are of lower order compared to the order of the involved spaces, e.g., as is the case in gradient plasticity in the absence of surface traction terms.

Rigorous but suboptimal convergence rates for time discretizations of elastoplastic evolution problems with C^3 regular loads have been proved in [HR13, MT04]. Optimal rates under strong regularity assumptions that exclude typical hysteresis effects

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were derived in [AC00]. On the basis of a regularity result that requires C^3 regular load functionals and C^2 regular yield surfaces a quasi-optimal error estimate was proved in [Mie05]. These requirements exclude certain practically relevant cases such as the Tresca yield criterion. We will show that the optimal convergence rate can be proved without smoothness assumptions on the yield surface and for loads that are merely C^2 regular. This setting allows solutions that are only Lipschitz continuous with respect to the time variable and therefore to describe hysteresis effects. As a byproduct of the analysis we obtain a posteriori error estimates that allow to control the time-discretization error by computable quantities and to adjust the step size by adaptive refinement procedures.

The outline of this article is as follows. In Section 2 we define the class of considered evolution problems and their discretization in time. The error analysis that leads to quasi-optimal a priori and a posteriori error estimates is carried out in Section 3. In Section 4 the application of the result to fully discrete settings is discussed.

2. Model problem and discretization

In the mathematical description of elastoplastic material behaviour intertial terms are often neglected in the equilibrium equation leading to a quasi-stationary evolution problem. This induces a rate-independence of the problem in the sense that a reparametrization of the considered time-interval does not change the solution. In the elastoplastic model problems this property is true since dissipation functionals are homogeneous of degree one. We consider the following class of rate-independent evolution problems with quadratic energy.

Definition 2.1. *Let $T > 0$, Y be a Hilbert space, $y_0 \in Y$, $\mathcal{A} : Y \times Y \to \mathbb{R}$ a continuous, symmetric, and coercive bilinear form, and $\ell \in W^{1,\infty}([0,T];Y')$ and for $t \in [0,T]$ and $y \in Y$ define*

$$\mathcal{E}(t,y) = \frac{1}{2}\mathcal{A}(y,y) - \langle \ell(t), y\rangle.$$

Let $\psi : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous functional that is degree-one homogeneous. The rate-independent evolution problem *seeks $y : [0,T] \to Y$ such that $y(0) = y_0$ and*

$$\mathcal{A}\big(y(t), w - \dot{y}(t)\big) - \langle \ell(t), w - \dot{y}(t)\rangle + \psi(w) - \psi\big(\dot{y}(t)\big) \geq 0 \tag{2.1}$$

for all $w \in Y$ and $t \in [0,T]$.

Remark 2.1. *For the results of Section 3 it suffices to assume that $\mathcal{A}$ is coercive on $\operatorname{dom}\psi$.*

Associating the operator $A : Y \to Y'$ to the bilinear form $\mathcal{A}$ the evolution problem is equivalent to the evolutionary inclusion

$$-Ay + \ell \in \partial\psi\big(\dot{y}\big).$$

The degree-one homogeneity of ψ implies that $\psi = I^*_{C_*}$, i.e., ψ is the Legendre transform of the indicator functional I_{C_*} of the set $C_* = \partial\psi(0)$. Convex duality relations yield that we have the equivalent inclusion

$$\dot{y} \in \partial I_{C_*}(-Ay + \ell).$$

We refer the reader to [MT04, Mie05] for detailed proofs of existence and uniqueness results in more general settings and only provide a sketch of a proof for the following result.

Theorem 2.1 (Existence and uniqueness)**.** *If $-Ay_0 + \ell(0) \in C_*$ then the rate-independent evolution problem has a unique solution $y \in W^{1,\infty}([0,T];Y)$.*

Sketch of the proof. Introducing the variable $z = -y + A^{-1}\ell$ the problem is equivalent to $-\dot{z} \in \partial I^*_{C_*}(Az) - A^{-1}\dot{\ell}$. Since the operator $v \mapsto \partial I_{C_*}(Av)$ is maximally monotone classical results based on [Bré73] imply the existence of a unique solution $z \in W^{1,\infty}([0,T];Y)$ provided $\partial I_{C_*}(Az_0) \neq \emptyset$, i.e., $-Ay_0 + \ell(0) \in C_*$. □

Remark 2.2. *Omitting the condition $-Ay_0 + \ell(0) \in C_*$ leads to a weaker notion of solution.*

The solution of the evolution problem can be constructed by considering a sequence of minimization problems. This is equivalent to an implicit discretization of the time-dependent variational inequality.

Algorithm 1 (Implicit discretization)**.** *Given $y^0 \in Y$ and $\tau > 0$ set $t_k = k\tau$, $k = 0, 1, ..., K$, $K = \lceil T/\tau \rceil$, and let $(y^k)_{k=1,...,K} \subset Y$ be a sequence of minimizers for the functionals*

$$I^k_\tau(w) = \psi(w - y^{k-1}) + \frac{1}{2}\mathcal{A}(w,w) - \langle \ell(t_k), w\rangle.$$

The iterates of the algorithm are uniquely defined.

Proposition 2.1 (Existence of semi-discrete iterates)**.** *For $k = 1, 2, ..., K$ there exists a unique minimizer $y^k \in Y$ for I^k_τ and we have*

$$\mathcal{A}(y^k, v - d_t y^k) - \langle \ell(t_k), v - d_t y^k\rangle + \psi(v) - \psi(d_t y^k) \geq 0$$

for all $v \in Y$. In particular, we have $-Ay^k + \ell(t_k) \in C_*$ for $k = 1, 2, ..., K$.

Proof. The existence of a minimizer in every time step follows from the direct method in the calculus of variations and we have $0 \in \partial I^k_\tau(y^k)$, i.e.,

$$0 \in Ay^k - \ell(t_k) + \partial\psi(y^k - y^{k-1})$$

This implies the variational inequality by incorporating the degree-one homogeneity of ψ. The relation $\partial\psi(w) \subset \partial\psi(0)$ for all $w \in Y$ implies the asserted inclusion. The uniqueness follows from the convexity of I^k_τ and the coercivity of $\mathcal{A}$. □

3. Error analysis

Recalling that $\psi = I^*_{C_*}$ with $C_* = \partial\psi(0)$ the transformation $z = -y + A^{-1}\ell$ shows that the evolution problem is equivalent to the inclusion

$$-\dot{z} \in \partial I_{C_*}(Az) - r, \quad r = A^{-1}\dot{\ell}. \tag{3.1}$$

Similarly, the transformation $z^k = -y^k + A^{-1}\ell^k$ shows that the variational inequalities of Proposition 2.1 are equivalent to the inclusions

$$-d_t z^k \in \partial I_{C_*}(Az^k) - r^k, \quad r^k = A^{-1}d_t\ell^k. \tag{3.2}$$

We denote $\|v\|^2_{\mathcal{A}} = \mathcal{A}(v,v)$ and assume $K\tau = T$ in the following.

Theorem 3.1 (Error estimates). *Assume that $r \in W^{1,1}([0,T];Y)$ and suppose that $z \in W^{1,\infty}([0,T];Y)$ satisfies (3.1) and the sequence $(z^k)_{k=0,...,K} \subset Y$ satisfies (3.2) for $k = 1,2,...,K$ with $z^0 = z(0)$ and $Az^0 \in C_*$. Let $\widehat{z}_\tau : [0,T] \to Y$ denote the piecewise affine interpolant of the sequence $(z^k)_{k=0,...,K}$.*
(i) With the non-negative quantities $\mathcal{E}_k = -\tau\|d_t z^k\|^2 + \tau\langle r^k, Az^k\rangle$ for $k = 1,2,...,K$ we have the a posteriori error estimate

$$\sup_{t\in[0,T]} \|z - \widehat{z}\|_{\mathcal{A}} \le \Big(2\tau\sum_{k=1}^K \mathcal{E}_k\Big)^{1/2} + \tau\sum_{k=1}^K \int_{t_{k-1}}^{t_k} \|\dot{r}\|_{\mathcal{A}}\,\mathrm{d}t.$$

(ii) Let $\partial^0 I_{C_*}(Az^0)$ denote the element of minimal norm in $\partial I_{C_*}(Az^0)$. We have the a priori error estimate

$$\sup_{t\in[0,T]} \|z - \widehat{z}_\tau\|_{\mathcal{A}} \le \tau\Big(\|r^0 - \partial^0 I_{C_*}(Az^0)\|_{\mathcal{A}} + 2\sup_{t\in[0,T]}\|r\|_{\mathcal{A}} + 2\int_0^T \|\dot{r}\|_{\mathcal{A}}\,\mathrm{d}t\Big).$$

Proof. The discrete evolution equation is equivalent to the variational inequalities

$$\langle -d_t z^k + r^k, v - Az^k\rangle \le 0$$

for all $v \in C_*$ and $k = 1,2,...,K$. With the choice $v = Az^{k-1}$ we define

$$-\mathcal{E}_k = \tau\|d_t z^k\|^2_{\mathcal{A}} - \tau\langle r^k, Ad_t z^k\rangle \le 0$$

and note

$$\|d_t z^k\|_{\mathcal{A}} \le \|r^k\|_{\mathcal{A}}, \quad \mathcal{E}_k \le \frac{\tau}{2}\|r^k\|_{\mathcal{A}}.$$

We let $z_\tau^+ : [0,T] \to Y$ and $r_\tau^+ : [0,T] \to Y'$ denote the piecewise constant functions satisfying $z_\tau^+(t) = z^k$ and $r_\tau^+(t) = r^k$ for $t_{k-1} < t \le t_k$, respectively. We have

$$\langle -\partial_t \widehat{z}_\tau + r^+, v - Az_\tau^+\rangle \le 0$$

for all $v \in C_*$. Defining

$$\mathcal{C}_\tau(t) = \langle -\partial_t \widehat{z}_\tau + r^+, Az_\tau^+ - A\widehat{z}_\tau\rangle$$

we find

$$\langle -\partial_t \widehat{z}_\tau + r, v - A\widehat{z}_\tau\rangle \le \mathcal{C}_\tau(t) + \langle r - r_\tau^+, v - A\widehat{z}_\tau\rangle.$$

Incorporating the equation for z, i.e.,

$$\langle -\partial_t z + r, v - Az\rangle \le 0$$

and choosing $v = Az$ and $v = A\widehat{z}_\tau$ in the equations for $\widehat{z}_\tau$ and z, respectively, and adding the inequalities we deduce that

$$\frac{1}{2}\frac{d}{dt}\|z - \widehat{z}_\tau\|^2_{\mathcal{A}} \le \mathcal{C}_\tau(t) + \|r - r_\tau^+\|_{\mathcal{A}}\|z - \widehat{z}_\tau\|_{\mathcal{A}}.$$

Using that $z_\tau^+ - \widehat{z}_\tau = -(t - t_k)d_t z^k$ for $t \in [t_{k-1}, t_k]$ we have

$$\mathcal{C}_\tau(t) = (t - t_k)\|d_t z^k\|^2_{\mathcal{A}} - (t - t_k)\langle r^k, Ad_t z^k\rangle = \frac{t - t_k}{\tau}(-\mathcal{E}_k) \le \mathcal{E}_k.$$

A version of the Gronwall lemma, cf. [NSV00, Lemma 3.7], which asserts

$$\frac{1}{2}\frac{d}{dt}a^2 \le \frac{1}{2}c + ba,\ t \in [0,T] \quad \Longrightarrow \quad \sup_{t\in[0,T]} a \le \Big(\int_0^T c\Big)^{1/2} + \int_0^T b\,\mathrm{d}s$$

for appropriate non-negative functions $a, b, c : [0, T] \to \mathbb{R}$ with $a(0) = 0$ implies

$$\sup_{t\in[0,T]} \|z - \widehat{z}_\tau\|_{\mathcal{A}} \leq \Big(2\tau \sum_{k=1}^{K} \mathcal{E}_k\Big)^{1/2} + \int_0^T \|r - r_\tau^+\|_{\mathcal{A}} \,\mathrm{d}t$$

Using that $\int_{t_{k-1}}^{t_k} \|r - r_\tau^+\|_{\mathcal{A}} \,\mathrm{d}t \leq \tau \int_{t_{k-1}}^{t_k} \|\dot{r}\|_{\mathcal{A}} \,\mathrm{d}t$ proves the asserted a posteriori error estimate. To prove the a priori error estimate we derive an improved bound for $\mathcal{E}_k$. For this we note that the equation for z^{k-1} with $k \geq 2$ reads

$$\langle -d_t z^{k-1} + r^{k-1}, v - Az^{k-1}\rangle \leq 0.$$

By defining z^{-1} so that $-d_t z^0 + r^0 = \partial^0 I_{C_*}(Az^0)$ this variational inequality also holds for $k = 1$. The choice $v = Az^k$ yields for $k = 1, 2, ..., K$ that

$$\langle -d_t z^{k-1} + r^{k-1}, A d_t z^k\rangle \leq 0.$$

Together with the definition of $\mathcal{E}_k$ we find

$$\begin{aligned}
\mathcal{E}_k &\leq -\tau\langle d_t z^k - r_k, A d_t z^k\rangle + \tau\langle d_t z^{k-1} - r^{k-1}, A d_t z^k\rangle \\
&= -\tau^2\langle d_t^2 z^k - d_t r^k, A d_t z^k\rangle \\
&= -\tau^2 \frac{d_t}{2}\|d_t z^k\|_{\mathcal{A}}^2 - \frac{\tau^3}{2}\|d_t^2 z^k\|_{\mathcal{A}}^2 + \tau^2\langle d_t r^k, A d_t z^k\rangle \\
&\leq -\tau^2 \frac{d_t}{2}\|d_t z^k\|_{\mathcal{A}}^2 + \tau^2\|d_t r^k\|_{\mathcal{A}}\|d_t z^k\|_{\mathcal{A}} \\
&\leq -\tau^2 \frac{d_t}{2}\|d_t z^k\|_{\mathcal{A}}^2 + \tau^2\|d_t r^k\|_{\mathcal{A}}\|r^k\|_{\mathcal{A}},
\end{aligned}$$

where we used $\|d_t z^k\|_{\mathcal{A}} \leq \|r^k\|_{\mathcal{A}}$. A summation of $\mathcal{E}_k$ over $k = 1, 2, ..., K$ combined with the estimate $\tau\|d_t r^k\|_{\mathcal{A}} \leq \int_{t_{k-1}}^{t_k} \|\dot{r}\|_{\mathcal{A}} \,\mathrm{d}t$ yields that

$$\tau \sum_{k=1}^{K} \mathcal{E}_k \leq \frac{\tau^2}{2}\|d_t z^0\|_{\mathcal{A}}^2 + \tau^2 \sup_{t\in[0,T]} \|r\|_{\mathcal{A}} \int_0^T \|\dot{r}\|_{\mathcal{A}} \,\mathrm{d}t.$$

The choice of z^{-1} implies the assertion. □

The theorem implies an error estimate for the approximation of the original formulation.

Corollary 3.1 (Time discretization)**.** *Assume that* $y_0 \in Y$ *satisfies* $-Ay_0 + \ell(0) \in \partial\psi(0)$ *and* $\ell \in W^{2,1}([0, T]; Y')$. *For the solution* $y \in W^{1,\infty}([0, T]; Y)$ *of the rate-independent evolution problem and the piecewise affine interpolant* $\widehat{y}_\tau \in W^{1,\infty}([0, T]; Y)$ *of the iterates* $(y^k)_{k=0,...,K}$ *of Algorithm 1 with* $y^0 = y_0$ *we have*

$$\sup_{t\in[0,T]} \|y - \widehat{y}_\tau\| \leq c\tau.$$

Remark 3.1. *The error estimate also holds for a spatially discrete version of the problem, i.e.,* $\sup_{t\in[0,T]} \|y_h - \widehat{y}_{h,\tau}\| \leq c\tau$. *In this case* Y *is replaced by a finite-dimensional subspace* $Y_h \subset Y$ *and the subdifferential is defined with respect to this space.*

Example 3.1. *The error estimate applies to linearized elastoplasticity with kinematic hardening for which*

$$\begin{aligned}\mathcal{A}(y,w) &= \int_\Omega \mathbb{C}(\varepsilon(u)-p):(\varepsilon(v)-q)+\mathbb{H}_{\rm kin}p:q\,{\rm d}x,\\ \ell(t)[w] &= \int_\Omega f(t)\cdot v\,{\rm d}x+\int_{\Gamma_{\rm N}} g(t)\cdot v\,{\rm d}s,\\ \psi(y) &= \sigma_y\int_\Omega |p|_0\,{\rm d}x\end{aligned}$$

for $y=(u,p)$ *and* $w=(v,q)$ *and with* $|p|_0=|p|$ *if* $\operatorname{tr} p=0$ *and* $|p|_0=+\infty$ *otherwise. Hence, with* $\sigma=\mathbb{C}(\varepsilon(u)-p)$ *we have* $Ay=(-\operatorname{div}\sigma,-\sigma+\mathbb{H}_{\rm kin}p)$*. With* $\sigma_0=\sigma(0)$ *the compatibility condition* $-Ay_0+\ell(0)\in\partial\psi(0)$ *is thus equivalent to* $\operatorname{div}\sigma_0+f(0)=0$, $\sigma_0\nu=g(0)$ *on* $\Gamma_{\rm N}$*, and* $\sigma_0-\mathbb{H}_{\rm kin}p_0\in\sigma_y\partial|\cdot|_0$*, i.e.,* $|\operatorname{dev}(\sigma_0-\mathbb{H}_{\rm kin}p_0)|\le\sigma_y$*.*

4. Discretization in space

We briefly discuss the error introduced by a spatial discretization. For this we assume that we are given a finite-dimensional subspace $Y_h\subset Y$ and let $P_{\mathcal{A},h}:Y\to Y_h$ denote the orthogonal projection onto Y_h with respect to $\mathcal{A}$, i.e., for $z\in Y$ we let $P_{\mathcal{A},h}z\in Y_h$ be such that

$$\mathcal{A}(P_{\mathcal{A},h}z-z,v_h)=0$$

for all $v_h\in Y_h$. We assume that there exists a bounded linear operator $\mathcal{J}_h:Y\to Y_h$ be such that $\mathcal{J}_h z\in\operatorname{dom}\psi$ whenever $z\in\operatorname{dom}\psi$. We adopt arguments from [MPPS10] in the following.

Proposition 4.1 (Space discretization)**.** *Let* $y\in W^{1,\infty}([0,T];Y)$ *be the solution of the rate-independent problem and let* $y_h\in W^{1,\infty}([0,T];Y)$ *be the uniquely defined function* $y_h:[0,T]\to Y_h$ *satisfying* $y_h(0)=y_h^0=P_{\mathcal{A},h}y_0$ *and*

$$\mathcal{A}(y_h,v_h-\dot y_h)-\langle\ell(t),v_h-\dot y_h\rangle+\psi(v_h)-\psi(\dot y_h)\ge 0 \tag{4.1}$$

for all $v_h\in Y_h$ *and* $t\in[0,T]$*. Assume that there exists* c_ψ *such that*

$$|\psi(v)-\psi(w)|\le c_\psi\|v-w\|$$

for all $v,w\in\operatorname{dom}\psi$*. We then have*

$$\sup_{t\in[0,T]}\|y-y_h\|^2\le c\int_0^T\|(1-\mathcal{J}_h)\dot y\|\,{\rm d}t+\|(1-P_{\mathcal{A},h})y_0\|^2.$$

Proof. The existence of the spatially discrete solution follows as in the continuous case. We choose $v=\dot y_h$ in (2.1) and add the resulting inequality to the discrete evolution equation (4.1) to verify

$$\mathcal{A}\big(y_h,\dot y-\dot y_h)+\mathcal{A}(y,\dot y_h-\dot y)+\mathcal{A}(y_h,v_h-\dot y)-\langle\ell(t),v_h-\dot y\rangle+\psi(v_h)-\psi(\dot y)\ge 0.$$

The choice $v_h=\mathcal{J}_h\dot y$ leads to

$$\begin{aligned}\mathcal{A}(y_h-y,\dot y_h-\dot y)&\le-\langle\ell(t),\mathcal{J}_h\dot y-\dot y\rangle+\psi(\mathcal{J}_h\dot y)-\psi(\dot y)+\mathcal{A}(y_h,\mathcal{J}_h\dot y-\dot y)\\ &\le\big(\|\ell(t)\|+c_\psi+c_{\mathcal{A}}\|y_h\|\big)\|\dot y-\mathcal{J}_h\dot y\|.\end{aligned}$$

The estimate $\sup_{t\in[0,T]}\|y_h\|\le c_\ell$ implies the asserted bound. □

Remark 4.1. *In the case of kinematic hardening and lowest order finite elements we may employ* $\mathcal{J}_h(v,q) = (P_1 v, P_0 q)$*, where* P_1 *and* P_0 *denote the orthogonal projections in* $H^1(\Omega;\mathbb{R}^{d\times d})$ *and* $L^2(\Omega;\mathbb{R}^{d\times d})$ *onto the spaces of continuous, piecewise affine and elementwise constant finite element functions subordinate to a triangulation of* Ω*, respectively. In particular, if* $\operatorname{tr} q = 0$ *then we have* $\operatorname{tr} P_0 q = 0$ *so that* $\mathcal{J}_h z \in \operatorname{dom}\psi$ *whenever* $z \in \operatorname{dom}\psi$.

The combination of the estimates for the semi-discrete schemes allows us to derive an error estimate for fully discrete approximations. These are obtained with the following algorithm.

Algorithm 2 (Fully discrete iteration)**.** *Given* $y_h^0 \in Y_h$ *and* $\tau > 0$ *let* $(y_h^k)_{k=1,\dots,K}$ *be a sequence of minimizers* $y_h^k \in Y_h$ *for the functionals*

$$I_{\tau,h}^k(w_h) = \psi\big(w_h - y_h^k\big) + \frac{1}{2}\mathcal{A}(w_h,w_h) - \langle \ell(t_k), w_h\rangle.$$

The iterates of the algorithm are uniquely defined and satisfy a discrete variational inequality.

Proposition 4.2 (Existence of fully discrete approximations)**.** *There exists a unique discrete solution* $(y_h^k)_{k=0,\dots,K}$ *and we have*

$$\mathcal{A}(y_h^k, v_h - d_t y_h^k) - \langle \ell(t_k), v_h - d_t y_h^k\rangle + \psi(v_h) - \psi(d_t y_h^k) \ge 0 \tag{4.2}$$

for $k = 1,2,\dots,K$ *and all* $v_h \in Y_h$*. If* $-Ay_0 + \ell(0) \in \partial\psi(0)$ *and* $y_h^0 = P_{\mathcal{A},h} y_0$ *then we have*

$$\max_{k=1,\dots,K} \|d_t y_h^k\| \le c\|\ell\|_{W^{1,\infty}([0,T];Y')}.$$

Proof. The derivation of the variational inequality is analogous to the proof of Proposition 2.1. The assumption on y_0 and the definition of y_h^0 imply

$$-\mathcal{A}(y_h^0, v_h) + \langle \ell(0), v_h\rangle = -\mathcal{A}(y_0,v_h) + \langle \ell(0), v_h\rangle \le \psi(v_h)$$

and by setting $y_h^{-1} = y_h^0$, i.e., $d_t y_h^0 = 0$ the variational inequality (4.2) also holds for $k = 0$. To prove the estimate we note that the choice $v_h = 0$ yields

$$\mathcal{A}(y_h^k, d_t y_h^k) \le \langle \ell(t_k), d_t y_h^k\rangle - \psi(d_t y_h^k)$$

while the choice $v_h = d_t y_h^k + d_t y_h^{k-1}$ in the equation for y_h^{k-1} leads to

$$-\mathcal{A}(y_h^{k-1}, d_t y_h^k) \le -\langle \ell(t_{k-1}), d_t y_h^k\rangle + \psi(d_t y_h^k + d_t y_h^{k-1}) - \psi(d_t y_h^{k-1}).$$

Adding the two inequalities shows

$$\begin{aligned}\tau\mathcal{A}(d_t y_h^k, d_t y_h^k) &\le \langle \tau d_t \ell(t_k), d_t y_h^k\rangle + \psi(d_t y_h^k + d_t y_h^{k-1}) - \psi(d_t y_h^k) - \psi(d_t y_h^{k-1}) \\ &\le \tau \sup_{t\in[t_k,t_{k-1}]} \|\dot{\ell}\|_{Y'} \|d_t y_h^k\|,\end{aligned}$$

where we used the convexity and degree-one homogeneity of ψ. □

Given the sequence of approximations $(y_h^k)_{k=0,\dots,K}$ and assuming that $K\tau = T$ we let $\widehat{y}_{h,\tau} \in W^{1,\infty}([0,T];Y)$ denote its piecewise affine interpolant. The combination of the estimates for the semi-discrete schemes implies the following error estimate.

Theorem 4.1 (Fully discrete approximations)**.** *Given the sequence of approximations* $(y_h^k)_{k=0,\dots,K} \in Y_h$ *such that* $y_h^0 = P_{\mathcal{A},h} y_0$ *we have*

$$\sup_{t\in[0,T]} \|y - \widehat{y}_{h,\tau}\|^2 \le c\Big(\tau^2 + \int_0^T \|(1-\mathcal{J}_h)\dot{y}\| \,\mathrm{d}t + \|(1-P_{\mathcal{A},h})y_0\|^2\Big).$$

Proof. We let $y_h \in W^{1,\infty}([0,T];Y)$ be the solution of the semi-discrete approximation in space and notice that according to Proposition 4.1 we have

$$\sup_{t\in[0,T]} \|y - y_h\|^2 \le c \int_0^T \|(1-\mathcal{J}_h)\dot{y}\| \,\mathrm{d}t + \|(1-P_{\mathcal{A},h})y_0\|^2.$$

The fully discrete scheme is interpreted as a temporal discretization of the semi-discrete scheme in space and the arguments of the proof of Theorem 3.1 lead to the estimate

$$\sup_{t\in[0,T]} \|y_h - \widehat{y}_{h,\tau}\| \le c\tau.$$

The combination of the two estimates implies the estimate of the theorem. □

Example 4.1. *For kinematic hardening we have* $Y = H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d})$ *and* $y = (u,p) \in Y$. *Assuming that* $u \in W^{1,1}([0,T];H^2(\Omega;\mathbb{R}^d))$ *and* $p \in W^{1,1}([0,T];H^1(\Omega;\mathbb{R}^{d\times d}))$ *we obtain with the corresponding lowest order finite element spaces the convergence rate* $\mathcal{O}(\tau + h^{1/2})$. *If* ℓ *and* ψ *are of lower order, e.g.,* $\ell \in W^{2,1}([0,T];L^2(\Omega;\mathbb{R}^d)')$ *and* $\psi : L^2(\Omega;\mathbb{R}^{d\times d}) \to \mathbb{R}\cup\{+\infty\}$, *and* $\mathcal{A}$ *is elliptic on* $H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)\times H^1(\Omega;\mathbb{R}^{d\times d})$, *e.g., in the case of gradient plasticity, then no regularity is required to deduce the convergence rate* $\mathcal{O}(\tau + h^{1/2})$ *for lowest order conforming finite elements.*

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Department of Applied Mathematics, Mathematical Institute, University of Freiburg, Hermann-Herder-Str 9, 79104 Freiburg i. Br., Germany

E-mail address: `bartels@mathematik.uni-freiburg.de`