A LOWER BOUND FOR THE SPECTRUM OF THE LINEARIZED ALLEN-CAHN OPERATOR NEAR A SINGULARITY

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ABSTRACT. A lower bound for the principal eigenvalue of the linearized Allen-Cahn operator near a generic singularity is derived. The estimate leads to robust stability estimates past topological changes.

1. Introduction

Spectral estimates have recently been successfully employed to derive robust stability and error estimates for phase field models, cf. [FP03, KNS04, Bar05, BMO09]. In particular, the time-averaged principal eigenvalue of the linearized Allen-Cahn operator about the exact or approximate solution enters such estimates exponentially and thus logarithmic bounds for this quantity lead to useful estimates, cf. [BMO09]. For the smooth evolution of developed interfaces it is known that the eigenvalue remains uniformly bounded [AF93, Che94, dMS95] while for topological changes it attains the square of the inverse of the interface thickness. Numerical experiments in [BMO09] indicate that the principal eigenvalue grows like $1/|t|$, $t < 0$, prior to a topological change at $t = 0$. Hence, an integration of it in time leads to a logarithmic bound.

For the mean curvature flow

$$V = -H$$

with a sphere of radius $\sqrt{2}$ at $t = -1$ as initial data, the evolution is defined through $\dot{R} = -1/R$, i.e., $R(t) = \sqrt{2}|t|^{1/2}$ for $-1 \le t < 0$. The linearization of H in the class of circles is given by

$$H'(t) = -1/R(t)^2 = -(1/2)|t|^{-1}.$$

Since the Allen-Cahn problem approximates the mean curvature flow as the interface thickness tends to zero [Ilm93, BP95] we expect that a similar bound holds for the principal eigenvalue of the linearized Allen-Cahn operator. We adopt the techniques of [Che94] to give a proof of this statement under the following assumption.

Assumption A. *The solution ϕ_ε of the Allen-Cahn problem in $(-T,0) \times B_2$ with $B_2 := B_2(0) \subset \mathbb{R}^2$, i.e., the function ϕ_ε that satisfies*

$$\partial_t \phi_\varepsilon - \Delta \phi_\varepsilon = -\varepsilon^{-2} f(\phi_\varepsilon),$$

with $f(u) := 2(u^2-1)u$ and $0 < \varepsilon < 1$, is for $t \le -\varepsilon^2 \log(\varepsilon^{-1})$ given by

$$\phi_\varepsilon(r,t) = \tanh\big((r - \sqrt{2}|t|^{1/2})/\varepsilon\big) + \varepsilon^2 q_\varepsilon(r,t) \tag{1.1}$$

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with a function q_ε that satisfies

$$|q_\varepsilon(r,t)| \le c_0 |t|^{-1}. \tag{1.2}$$

This assumption is motivated by the expansion $\phi_\varepsilon(x,t) = \tanh(d/\varepsilon) + \varepsilon^2 H^2 \xi(d/\varepsilon) + \mathcal{O}(\varepsilon^3)$ with d denoting the signed distance to the interface, cf. [PV92, BP95], is confirmed by numerical experiments reported in Appendix A below, and is exptected to be justifiable rigorously by an appropriate construction of super- and subsolutions. The assumption enables us to prove the asserted result.

Theorem 1.1. *Suppose that Assumption A holds. Then the estimate*

$$\lambda_{AC}(t) := \inf_{0 \ne \psi \in H^1(B_2)} \frac{\int_{B_2} |\nabla \psi|^2 + \varepsilon^{-2} f'\big(\phi_\varepsilon(r,t)\big)\psi^2 dx}{\|\psi\|^2_{L^2(B_2)}} \ge -C_{AC}|t|^{-1}$$

holds for $t \in [-T, -\varepsilon^2 \log(\varepsilon^{-1})]$ with an ε-independent constant $C_{AC} \ge 0$.

Remarks 1.1. (i) Terms of order ε can be included if one restricts to $t \in [-T, -\varepsilon^2 \log(\varepsilon^{-1})^2]$. This however is not sufficient to prove robust stability estimates.
(ii) The arguments also apply to the linearized Cahn-Hilliard operator, cf. [Che94].
(iii) The same lower bound is expected to hold in three space dimensions.
(iv) Since interfaces become circular in two-dimensional Allen-Cahn evolutions, cf. [GH86], the considered situation is generic.
(v) An equivalent statement is to say that $\lambda_{AC}(t) \ge -cH_m^2(t)$, where $H_m(t)$ is the maximal curvature of the interface.

2. Allen-Cahn profile on a bounded interval

Given $t \le -\varepsilon^2 \log(\varepsilon^{-1})$ we consider the operator

$$\mathcal{L}^0_{\varepsilon,t} := -d^2/dz^2 + f'(\theta_0) \qquad \text{in} \quad I_{\varepsilon,t} := (-|t|^{1/2}/(\sqrt{2}\varepsilon), 1/\varepsilon)$$

subject to homogeneous Neumann boundary conditions on $\partial I_{\varepsilon,t}$ and define

$$L^0_{\varepsilon,t}(\Phi,\Psi) := \int_{I_{\varepsilon,t}} \Phi'\Psi' + f'(\theta_0)\Phi\Psi dz$$

for $\Phi, \Psi \in H^1(I_{\varepsilon,t})$. The function $\theta_0(z) := \tanh(z) = (e^z - e^{-z})/(e^z + e^{-z})$ solves

$$-\theta'' + f(\theta) = 0, \quad \theta(0) = 0, \quad \lim_{z \to \pm\infty} \theta(z) = \pm 1.$$

Moreover, $\theta_0'(z) = 1/\cosh^2(z)$ and $\theta_0''(z) = -2\sinh(z)/\cosh^3(z)$ satisfy

$$0 < \theta_0'(z) \le 4e^{-2|z|} \quad \text{and} \quad |\theta_0''(z)| \le 8e^{-2|z|}. \tag{2.1}$$

Lemma 2.1. *The principal eigenvalue λ_1^0 of $\mathcal{L}^0_{\varepsilon,t}$ satisfies*

$$-c_4 e^{-(1+2\sqrt{2})|t|^{1/2}/\varepsilon} \le \lambda_1^0 = \inf_{0 \ne \Psi \in H^1(I_{\varepsilon,t})} \frac{L^0_{\varepsilon,t}(\Psi,\Psi)}{\|\Psi\|^2_{L^2(I_{\varepsilon,t})}} \le c_1 e^{-2\sqrt{2}|t|^{1/2}/\varepsilon},$$

where $c_1, c_4 > 0$ are ε-independent constants.

Proof. The shifted operator $\mathcal{L}^0_{\varepsilon,t} + \max_{I_{\varepsilon,t}} |f'(\theta_0)|$ is self-adjoint and positive definite so that $-\max_{I_{\varepsilon,t}} |f'(\theta_0)| \le \lambda^0_1$. Integration by parts, $\mathcal{L}^0_{\varepsilon,t}\theta_0' = (-\theta_0'' + f(\theta_0))' = 0$, and (2.1) show

$$\lambda^0_1 \le \beta^2 L^0_{\varepsilon,t}(\theta_0', \theta_0') = \beta^2 \theta_0' \theta_0'' \big|_{-|t|^{1/2}/(\sqrt{2}\varepsilon)}^{1/\varepsilon} \le \beta^2 64 e^{-4|t|^{1/2}/(\sqrt{2}\varepsilon)} =: c_1 e^{-2\sqrt{2}|t|^{1/2}/\varepsilon}, \tag{2.2}$$

where we used that $|t|^{1/2}/(\sqrt{2}\varepsilon) \le 1/\varepsilon$ and defined $\beta^2 := \|\theta_0'\|^{-2}_{L^2(I_{\varepsilon,t})} \le 1$. Set $m := \max\{f'(-1), f'(1)\} = 4$ and let $a_0 > 0$ be such that $f'(\theta_0(z)) \ge 3m/4$ for all $|z| \ge a_0$. Since a_0 is independent of ε we may assume that $\pm(a_0+1) \in I_{\varepsilon,t}$. Owing to (2.2) we may assume that $\lambda^0_1 \le m/4$. Then, the positive eigenfunction Ψ^0_1 satisfies

$$-(\Psi^0_1)'' + (f'(\theta_0) - \lambda^0_1)\Psi^0_1 = 0,$$

with $f'(\theta_0) - \lambda^0_1 \ge m/2$ in $I_{\varepsilon,t} \setminus [-a_0, a_0]$. Since $\|\Psi^0_1\|_{L^2(I_{\varepsilon,t})} = 1$ we may choose $a_0' \in [a_0, a_0+1]$ such that $\Psi^0_1(\pm a_0') \le 1$. A comparison argument with the functions

$$\Phi_+(z) :=: \Psi(a_0') \frac{\cosh\big(c(1/\varepsilon - z)\big)}{\cosh\big(c(1/\varepsilon - a_0')\big)}, \qquad \Phi_-(z) := \Psi(-a_0') \frac{\cosh\big(c(|t|^{1/2}/(\sqrt{2}\varepsilon) + z)\big)}{\cosh\big(c(|t|^{1/2}/(\sqrt{2}\varepsilon) - a_0')\big)},$$

where $c = \sqrt{m/2}$, shows that $\Psi^0_1(z) \le \Phi_+(z)$ for $z \ge a_0'$ and $\Psi^0_1(z) \le \Phi_-(z)$ for $z \le -a_0'$, cf. Lemma B.1 for details. We thus deduce that for $z \in I_{\varepsilon,t} \setminus [-(a_0+1), a_0+1]$ we have

$$\Psi^0_1(z) \le \Phi_\pm(z) \le 2e^{-c|z|} e^{ca_0'} \le 2e^{-c|z|} e^{c(a_0+1)} =: c_2 e^{-\sqrt{2}|z|}.$$

This, integration by parts, $\mathcal{L}^0_{\varepsilon,t}\theta_0' = 0$, and (2.1) imply

$$\lambda^0_1 \int_{I_{\varepsilon,t}} \Psi^0_1 \theta_0' dz = \int_{I_{\varepsilon,t}} (\mathcal{L}^0_{\varepsilon,t}\Psi^0_1)\theta_0' dz = \theta_0'' \Psi^0_1 \big|_{-|t|^{1/2}/(\sqrt{2}\varepsilon)}^{1/\varepsilon} \ge -16 c_2 e^{-(\sqrt{2}+4)|t|^{1/2}/(\sqrt{2}\varepsilon)}.$$

It remains to prove a lower bound for $\int_{I_{\varepsilon,t}} \Psi^0_1 \theta_0' dz$. Since $\theta_0' > 0$ it suffices to show that Ψ^0_1 is uniformly bounded from below in $(-a^*, a^*)$ for some a^* independent of ε. Since $\|\Psi^0_1\|_{L^2(I_{\varepsilon,t})} = 1$ and $\Psi^0_1(z) \le c_2 e^{-\sqrt{2}|z|}$, $|z| \ge a_0 + 1$, there exists an ε-independent number $a^* > 0$ such that

$$\int_{-a^*}^{a^*} |\Psi^0_1(z)|^2 dz \ge 1/2.$$

The coefficients of $\mathcal{L}^0_{\varepsilon,t} - \lambda^0_1$ are uniformly bounded so that an application of Harnack's inequality, cf. , e.g., [GT01], to the identity

$$(\mathcal{L}^0_{\varepsilon,t} - \lambda^0_1)\Psi^0_1 = 0 \qquad \text{in} \quad (-a^* - 1, a^* + 1)$$

implies the existence of a constant $c_3 > 0$ such that

$$\inf_{z \in (-a^*, a^*)} \Psi^0_1(z) \ge c_3 \sup_{z \in (-a^*, a^*)} \Psi^0_1(z) \ge c_3 \Big(\frac{1}{2a^*} \int_{-a^*}^{a^*} (\Psi^0_1)^2 dz \Big)^{1/2} \ge c_3 \frac{1}{(4a^*)^{1/2}}.$$

This proves $\lambda^0_1 \ge -c_4 e^{-(1+2\sqrt{2})|t|^{1/2}/\varepsilon}$ and finishes the proof. □

3. Reduction to the one-dimensional situation

Under the assumptions on ϕ_ε stated in Assumption A, the estimation of λ_{AC} reduces to a one-dimensional problem. For $\psi \in C^1(B_2)$ we have

$$\int_{B_2} \varepsilon|\nabla\psi|^2 + \varepsilon^{-1} f'(\phi_\varepsilon)\psi^2 dx \geq 2\pi \int_{|t|^{1/2}/\sqrt{2}}^{1+\sqrt{2}|t|^{1/2}} \big(\varepsilon|\psi_r|^2 + \varepsilon^{-1} f'(\phi_\varepsilon)\psi^2\big) r dr.$$

The transformation $z = (r - \sqrt{2}|t|^{1/2})/\varepsilon$ and the rescaling $\Psi(z) := \varepsilon^{1/2}\psi(r)$ lead to

$$\begin{aligned}
&\int_{B_2} \varepsilon|\nabla\psi|^2 + \varepsilon^{-1} f'(\phi_\varepsilon)\psi^2 dx \\
&\qquad \geq \frac{2\pi}{\varepsilon} \int_{I_{\varepsilon,t}} \big(|\Psi_z|^2 + f'(\tilde\phi_\varepsilon)\Psi^2\big) \tilde J(z) dz =: \frac{2\pi}{\varepsilon} L_{\varepsilon,t}(\Psi,\Psi),
\end{aligned} \tag{3.1}$$

where $I_{\varepsilon,t} = \big(-|t|^{1/2}/(\sqrt{2}\varepsilon), 1/\varepsilon\big)$, $\tilde J(z) = \varepsilon z + \sqrt{2}|t|^{1/2}$, and $\tilde\phi_\varepsilon(z,t) = \phi_\varepsilon(\varepsilon z + \sqrt{2}|t|^{1/2}) = \theta_0(z) + \varepsilon^2 \tilde q_\varepsilon(z,t)$ with $\tilde q_\varepsilon(z,t) = q_\varepsilon(\varepsilon z + \sqrt{2}|t|^{1/2}, t)$. Since

$$\|\psi\|^2_{L^2(B_2)} \geq 2\pi \int_{|t|^{1/2}/\sqrt{2}}^{1+\sqrt{2}|t|^{1/2}} \psi^2 r dr = 2\pi \int_{I_{\varepsilon,t}} \Psi^2 \tilde J(z) dz$$

Theorem 1.1 follows from the next lemma.

Lemma 3.1. *For $t \in [-T, -\varepsilon^2\log(\varepsilon^{-1})]$ the principal eigenvalue λ_1 of $L_{\varepsilon,t}$ defined in* (3.1) *satisfies*

$$-c_8\varepsilon^2|t|^{-1} \leq \lambda_1 = \inf_{0\neq\Psi\in H^1(I_{\varepsilon,t})} \frac{L_{\varepsilon,t}(\Psi,\Psi)}{\|\Psi\tilde J^{1/2}\|^2_{L^2(I_{\varepsilon,t})}} \leq c_6\varepsilon|t|^{-1},$$

where $c_6, c_8 > 0$ are ε-independent constants.

Proof. Let $\Psi \in H^1(I_{\varepsilon,t})$ and define $\widehat\Psi := \tilde J^{1/2}\Psi$. Noting $\Psi_z \tilde J^{1/2} = \widehat\Psi_z - \varepsilon\tilde J^{-1}\widehat\Psi/2$, where $\tilde J^{-1} := 1/\tilde J$, we deduce

$$\begin{aligned}
L_{\varepsilon,t}(\Psi,\Psi) &= \int_{I_{\varepsilon,t}} \widehat\Psi_z^2 + f'(\tilde\phi_\varepsilon(z,t))\widehat\Psi^2 dz + \frac{\varepsilon^2}{4}\int_{I_{\varepsilon,t}} \tilde J^{-2}\widehat\Psi^2 dz - \varepsilon\int_{I_{\varepsilon,t}} \tilde J^{-1}\widehat\Psi\widehat\Psi_z dz \\
&= L^0_{\varepsilon,t}(\widehat\Psi,\widehat\Psi) + \int_{I_{\varepsilon,t}} \big(f'(\tilde\phi_\varepsilon(z,t)) - f'(\theta_0(z))\big)\widehat\Psi^2 dz + \frac{\varepsilon^2}{4}\int_{I_{\varepsilon,t}} \tilde J^{-2}\widehat\Psi^2 dz - \frac{\varepsilon}{2}\int_{I_{\varepsilon,t}} \tilde J^{-1}(\widehat\Psi^2)_z dz.
\end{aligned}$$

A Taylor expansion of the quadratic function f' about θ_0 shows

$$f'\big(\tilde\phi_\varepsilon(z,t)\big) - f'\big(\theta_0(z)\big) = \varepsilon^2 f''(\theta_0)\tilde q_\varepsilon + f'''(\theta_0)(\varepsilon^2\tilde q_\varepsilon)^2/2 =: \varepsilon^2 r_\varepsilon \tag{3.2}$$

with $|r_\varepsilon| \leq c_5|t|^{-1}$ owing to (1.2). An integration by parts and $(\tilde J^{-1})_z = -\tilde J^{-2}\varepsilon$ lead to

$$-\int_{I_{\varepsilon,t}} \tilde J^{-1}(\widehat\Psi^2)_z dz = -\varepsilon\int_{I_{\varepsilon,t}} \tilde J^{-2}\widehat\Psi^2 dz - \tilde J^{-1}\widehat\Psi^2\big|^{1/\varepsilon}_{-|t|^{1/2}/(\sqrt{2}\varepsilon)}.$$

This implies

$$L_{\varepsilon,t}(\Psi,\Psi) = L^0_{\varepsilon,t}(\widehat\Psi,\widehat\Psi) + \varepsilon^2\int_{I_{\varepsilon,t}} r_\varepsilon\widehat\Psi^2 dz - \frac{\varepsilon^2}{4}\int_{I_{\varepsilon,t}} \tilde J^{-2}\widehat\Psi^2 dz - \frac{\varepsilon}{2}\tilde J^{-1}\widehat\Psi^2\big|^{1/\varepsilon}_{-|t|^{1/2}/(\sqrt{2}\varepsilon)}. \tag{3.3}$$

We conclude with (2.1) and (2.2) that

$$\lambda_1 = \inf_{\|\Psi \tilde{J}^{1/2}\|_{L^2(I_{\varepsilon,t})}=1} L_{\varepsilon,t}(\Psi,\Psi) \le L_{\varepsilon,t}(\beta \tilde{J}^{-1/2}\theta_0', \beta \tilde{J}^{-1/2}\theta_0')$$
$$\le \beta^2 L^0_{\varepsilon,t}(\theta_0',\theta_0') + c_5\varepsilon^2|t|^{-1} + \beta^2 \frac{\varepsilon}{2}\big(\theta_0'(-\sqrt{2}|t|^{1/2}/(2\varepsilon))\big)^2 \le c_6\varepsilon|t|^{-1},$$

where we used that $e^{-2\sqrt{2}|t|^{1/2}/\varepsilon} \le 1$ for ε sufficiently small. Let Ψ_1 be the positive eigenfunction corresponding to λ_1 with $\|\Psi_1 \tilde{J}^{1/2}\|_{L^2(I_{\varepsilon,t})} = 1$ and note that Ψ_1 satisfies

$$-\tilde{J}^{-1}\frac{d}{dz}\big(\tilde{J}\frac{d}{dz}\Psi_1\big) + f'(\tilde{\phi}_\varepsilon(z,t))\Psi_1 = \lambda_1\Psi_1$$

in $I_{\varepsilon,t}$. We may assume that $\lambda_1 \le m/4$, and $f'(\tilde{\phi}_\varepsilon(z,t)) \ge 3m/4$ for $z \ge a_0$ with an ε-independent number $a_0 > 0$ such that $a_0 + 1 \le 1/\varepsilon$. Let $a_0' \in [a_0, a_0+1]$ such that $\Psi_1(a_0')\tilde{J}^{1/2}(a_0') \le 1$. Employing the comparison function

$$\Phi(z) = \Psi_1(a_0')\frac{\cosh\big(c(1/\varepsilon - z)\big)}{\cosh\big(c(1/\varepsilon - a_0')\big)},$$

where $c = \sqrt{m/2}$, we deduce that, cf. Lemma B.2 for details,

$$\Psi_1(z) \le c_7 e^{-\sqrt{2}z}\tilde{J}^{-1/2}(a_0').$$

With $\widehat{\Psi}_1 := \tilde{J}^{1/2}\Psi_1$ we deduce from (3.3) and Lemma 2.1 that

$$\lambda_1 = L_{\varepsilon,t}(\Psi_1,\Psi_1) \ge L^0_{\varepsilon,t}(\widehat{\Psi}_1,\widehat{\Psi}_1) - c_5\varepsilon^2|t|^{-1} - \frac{\varepsilon^2}{4}\sup_{z\in I_{\varepsilon,t}}\tilde{J}^{-2}(z) - \frac{\varepsilon}{2}\big(\Psi_1(1/\varepsilon)\big)^2$$
$$\ge \lambda_1^0 - c_5\varepsilon^2|t|^{-1} - \varepsilon^2|t|^{-1} - c_7\varepsilon e^{-2\sqrt{2}/\varepsilon}\tilde{J}^{-1}(a_0') \ge -c_8\varepsilon^2|t|^{-1},$$

provided that ε is sufficiently small so that $e^{-2\sqrt{2}/\varepsilon}\tilde{J}^{-1}(a_0') \le \varepsilon$. □

Remark 3.1. For $t \le -\varepsilon^2\log(\varepsilon^{-1})^2$ the upper bound $\lambda_1 \le c_6\varepsilon^2|t|^{-1}$.

Appendix A. Experimental verification of Assumption A

For a triangulation $\mathcal{T}$ of B_2 with mesh-size $h \sim 2^{-8}$ we approximated the Allen-Cahn problem with a semi-implicit time-stepping scheme with step-size $\tau = h/10$ for the initial data $u_0(r) = \tanh\big((r-\sqrt{2}|t_0|^{1/2}\big)/\varepsilon$ at $t_0 = -1/4$. In Figure 1 we plotted for $\varepsilon = 2^{-\ell}$, $\ell = 2,3,4,5$ the quantity

$$\eta_\varepsilon(t) := \varepsilon^{-2}\max_{z\in\mathcal{N}}\big|u_h(z,t) - \tanh\big((|z| - \sqrt{2}|t|^{1/2})/\varepsilon\big)\big|,$$

where $\mathcal{N}$ denotes the set of nodes in the triangulation $\mathcal{T}$. The results show that $\eta_\varepsilon(t) \le c|t|^{-1}$ and thus justify Assumption A.

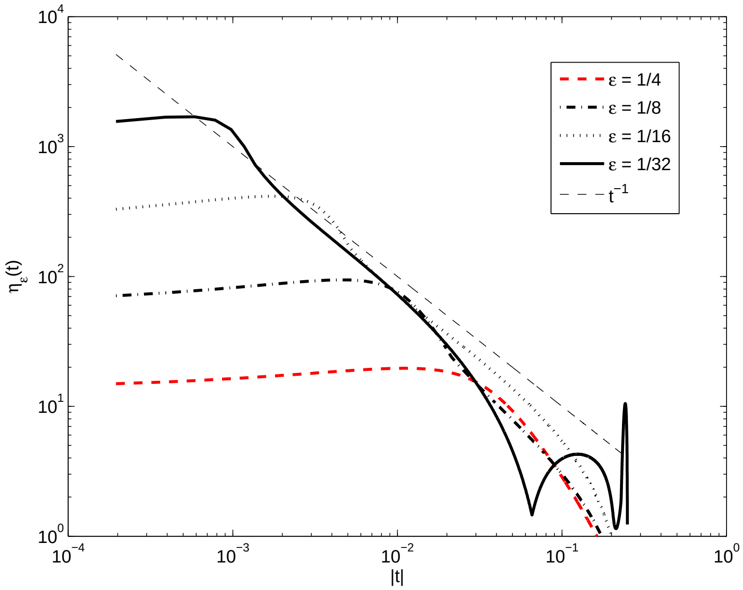

FIGURE 1. Experimental bounds on the second order term in the asymptotic expansion.

APPENDIX B. COMPARISON PRINCIPLES

Lemma B.1. *a) Let $a < b$, set $I := (a, b) \subseteq \mathbb{R}$, let $p, q \in C(I)$, and suppose $p \geq q \geq 0$. Assume $\Psi, \Phi \in C^2(\mathbb{R})$, satisfy $\Psi \geq 0$, $\Psi(a) = \Phi(a)$, $\Psi'(b) = \Phi'(b) = 0$, and*

$$-\Psi'' + p\Psi = 0, \qquad -\Phi'' + q\Phi = 0 \quad \text{in } I.$$

Then $\Psi \leq \Phi$. The same conclusion holds if $\Psi(b) = \Phi(b)$ and $\Psi'(a) = \Phi'(a) = 0$.
b) Let $a < b$ set $I_+ := (a, b)$ and $I_- := (-b, -a)$. For $\psi \in C(\mathbb{R})$ and $c \geq 0$ the functions $\Phi_\pm : I_\pm \to \mathbb{R}$, defined by

$$\Phi_\pm : z \mapsto \Psi(\pm a)\frac{\cosh(c(b \mp z))}{\cosh(c(b - a))}$$

satisfy $\Phi_\pm(\pm a) = \Psi(\pm a)$, $\Phi'_\pm(\pm b) = 0$, and $-\Phi''_\pm + c^2\Phi_\pm = 0$ in $I_\pm$. For $z \in I_\pm$ we have

$$|\Phi_\pm(z)| \leq 2|\Psi(\pm a)|e^{-c|z|}e^{ca}. \tag{B.1}$$

Proof. a) The function $E := \Psi - \Phi$ satisfies $E(a) = 0$, $E'(b) = 0$, and $-E'' + qE \leq 0$ in I. Suppose there exist $a \leq \alpha < \beta \leq b$ such that $E|_{(\alpha,\beta)} > 0$ and $E(\alpha) = E(\beta) = 0$. Then $E'(\alpha) > 0$ and $E'(\beta) \leq 0$ contradict $E'' \geq qE \geq 0$ in (α, β), i.e., the fact that E' is monotonically increasing in (α, β). Hence, $E \leq 0$, i.e., $\Psi \leq \Phi$. The second case is analogous.
b) The identities follow from $\cosh'' = \cosh$ and $\sinh(0) = 0$. The estimates are consequences of the bounds

$$\frac{\cosh(c(b \mp z))}{\cosh(c(b - a))} = \frac{e^{c(b\mp z)} + e^{-c(b\mp z)}}{e^{c(b-a)} + e^{-c(b-a)}} = \frac{e^{\mp cz}e^{cb}}{e^{-ca}e^{cb}}\left(\frac{1 + e^{-2c(b\mp z)}}{1 + e^{-2c(b-a)}}\right) \leq 2e^{\mp cz}e^{ca}, \tag{B.2}$$

where we used $e^{-2c(b\mp z)} \leq 1$ for $z < b$ and $-b < z$, respectively. $\square$

Lemma B.2. *Let $a < b$ such that $I = (a, b) \subseteq I_{\varepsilon,t} := \big(-|t|^{1/2}/(\sqrt{2}\varepsilon), 1/\varepsilon\big)$, $p \in C(I)$, and $c \geq 0$ such that $p \geq c^2$. Let $\Psi \in C^2(I)$ be non-negative with $\Psi'(b) = 0$ and*

$$-\tilde{J}^{-1}\frac{d}{dz}\Big(\tilde{J}\frac{d}{dz}\Big)\Psi + p\Psi = 0$$

in I, where $\tilde{J}^{-1} = 1/\tilde{J}$ with $\tilde{J}(z) = \varepsilon z + \sqrt{2}|t|^{1/2}$. Then, $\Psi(z) \leq 2\Psi(a)e^{-cz}e^{ca}$.

Proof. Defining

$$\Phi : I \to \mathbb{R}, \quad z \mapsto \Psi(a)\frac{\cosh(c(b-z))}{\cosh(c(b-a))}.$$

we have $-\Phi'' + c^2\Phi = 0$, $\Phi(a) = \Psi(a)$, and $\Phi'(b) = 0$. With $\tilde{J}_z = \varepsilon$ and $\Phi' \leq 0$, $\tilde{J} > 0$ in I we deduce

$$-\tilde{J}^{-1}\frac{d}{dz}\Big(\tilde{J}\frac{d}{dz}\Big)\Phi + c^2\Phi = -\varepsilon\tilde{J}^{-1}\Phi' - \tilde{J}^{-1}\tilde{J}\Phi'' + c^2\Phi = -\varepsilon\tilde{J}^{-1}\Phi' \geq 0.$$

Since $p \geq c^2$ and $\Psi \geq 0$ the function $E := \Psi - \Phi$ satisfies

$$-\tilde{J}^{-1}\frac{d}{dz}\Big(\tilde{J}\frac{d}{dz}\Big)E + c^2E = \varepsilon\tilde{J}^{-1}\Phi' \leq 0$$

and $E(a) = 0$, $E'(b) = 0$. Suppose that $(\alpha, \beta) \subseteq I$ is maximal with $E|_{(\alpha,\beta)} > 0$. Then, since $\tilde{J} > 0$ we have $\tilde{J}(\alpha)E'(\alpha) > 0$ and $\tilde{J}(\beta)E'(\beta) \leq 0$. This contradicts

$$\frac{d}{dz}(\tilde{J}E') = \frac{d}{dz}\Big(\tilde{J}\frac{d}{dz}\Big)E \geq \tilde{J}c^2E \geq 0$$

and shows that $E \leq 0$, i.e., $\Psi \leq \Phi$. The proof of the estimate follows from (B.2). □

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