A POSTERIORI ERROR ESTIMATES FOR NONCONFORMING FINITE ELEMENT METHODS

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ABSTRACT. Computable a posteriori error bounds for a large class of nonconforming finite element methods are provided for a model Poisson-problem in two and three space dimensions. Besides a refined residual-based a posteriori error estimate an averaging estimator is established and an L^2-estimate is included. The a posteriori error estimates are reliable and efficient; the proof of reliability relies on a Helmholtz decomposition.

1. INTRODUCTION

Nonconforming finite element methods play an important practical role in partial differential equations when conforming methods seem too expensive or unstable within low order mixed methods. In this paper, we establish tools for error control and adaptive mesh-refinement for a simple model problem for nonconforming elements, i.e., we prove sharp a posteriori error bounds for a class of nonconforming finite element methods for the Poisson problem with mixed boundary conditions: Given $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, $u_D \in H^{1/2}(\Gamma_D)$, find $u \in H^1(\Omega)$ which satisfies

$$\operatorname{div}(A\nabla u) + f = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$(A\nabla u)\cdot n = g \qquad \text{on } \Gamma_N, \tag{1.2}$$

$$u = u_D \qquad \text{on } \Gamma_D. \tag{1.3}$$

Here, Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, with boundary Γ, which is split into a closed Dirichlet boundary $\Gamma_D \subseteq \Gamma$ with positive surface measure and the remaining Neumann boundary $\Gamma_N := \Gamma \backslash \Gamma_D$. The coefficients form a pointwise symmetric uniformly positive definite $(d\times d)$-matrix $A \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{sym})$ so there exist $0 < \mu < M < \infty$ such that for all $y \in \mathbb{R}^d$ and almost all $x \in \Omega$ there holds

$$\mu|y|^2 \leq y \cdot A(x) y \leq M|y|^2. \tag{1.4}$$

In this paper we focus on a posteriori error estimates which allow error control for a computed approximation U to the unknown u in terms of U. We generalise the residual based estimates of [C1, DDPV] to the case of three space dimensions in Section 3 and prove that edge contributions dominate provided the given data functions are smooth, namely

$$\|\nabla_\mathcal{T}(u-U)\|^2_{L^2(\Omega)} \leq c_1\Big(\sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} h_E\|[(A\nabla_\mathcal{T} U)\cdot n_E]\|^2_{L^2(E)} + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D} h_E\|[\gamma_{t_E}(\nabla_\mathcal{T} U)]\|^2_{L^2(E)}\Big) + \text{h.o.t.},$$

where higher order terms h.o.t. depend on given data. The sums are over edges in a triangulation of Ω and $h_E^{1/2}[(A\nabla_\mathcal{T} U)\cdot n_E]$, respectively $h_E^{1/2}[\gamma_{t_E}(\nabla_\mathcal{T} U)]$ are weighted jumps of the normal, respectively tangential components of the elementwise gradient of U. We suppose that the $\mathcal{T}$-piecewise divergence $\operatorname{div}_\mathcal{T} A\nabla_\mathcal{T} U$ exists as an L^2-function so that the jumps $[A\nabla_\mathcal{T} U]\cdot n_E$ are well defined.

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The reliable error estimate is efficient in the sense that its converse estimate holds up to different higher order terms and a different multiplicative constant.

In Section 4 we investigate error estimates based on averaging techniques as in [BC, CB] and prove their reliability, i.e., estimates of the form

$$\|\nabla_{\mathcal{T}}(u-U)\|^2_{L^2(\Omega)} \le c_2 \min_{q_h \in \mathcal{S}(\mathcal{T},g,u_D)} \|\nabla_{\mathcal{T}} U - q_h\|^2_{L^2(\Omega)} + \text{h.o.t.}$$

The minimisation is over a space of smoother functions (than $\nabla_{\mathcal{T}} U$). Note that any choice of $q_h^* \in \mathcal{S}(\mathcal{T},g,u_D)$, possibly calculated from $\nabla_{\mathcal{T}} U$ in a post-processing step, yields reliable error control. Our proofs of the reliability estimates also show the connection to the residual based error estimates. The efficiency of averaging estimates will be proved up to higher order terms depending on the smoothness of the exact solution. In Section 5 we briefly establish an L^2-a posteriori estimate for $\|u-U\|_{L^2(\Omega)}$.

Our reliability arguments employ a Helmholtz decomposition as in [A, C1, DDPV]. Related adaptive mesh-refining algorithms can be found in [EEHJ, V2, HW, W].

2. Notation and Preliminaries

The nonconforming finite elements are described by a regular triangulation $\mathcal{T}$ of Ω in the sense of Ciarlet [Ci] which is a finite partition of Ω into closed polyhedral domains, namely into triangles or parallelograms if $d=2$, and in tetrahedra or parallelepipeds if $d=3$, respectively, such that two distinct T and T' in $\mathcal{T}$ are either disjoint, or $T \cap T'$ is a complete face, a complete edge, or a common node of both T and T'. By $\mathcal{N}$ we denote the set of nodes in $\mathcal{T}$ and by $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$ the subset of free nodes. With $\mathcal{T}$ let $\mathcal{E}$ denote the set of all faces (or edges), and we assume that $E \in \mathcal{E}$ either belongs to Γ_D or $E \cap \Gamma_D$ has vanishing surface measure, so there is no change of boundary conditions within one face $E \subseteq \Gamma$. We define a partition of $\mathcal{E}$ into $\mathcal{E}_\Omega$, $\mathcal{E}_D$, and $\mathcal{E}_N$ consisting of inner faces, those on Γ_D and Γ_N, respectively. By h_T and h_E we denote the diameter of an element $T \in \mathcal{T}$ and an edge $E \in \mathcal{E}$ and introduce functions $h_{\mathcal{T}}$ and $h_{\mathcal{E}}$ on Ω and $\cup\mathcal{E}$, respectively, which satisfy $h_{\mathcal{T}}|_T = h_T$ and $h_{\mathcal{E}}|_E = h_E$;

$$\omega_E := \cup\{T \in \mathcal{T} : E \subseteq \partial T\}$$

denotes the neighbourhood of E.

We assume that the conforming lowest order method described by

$$\mathcal{P}_1(T) := \begin{cases} P_1(T) & \text{if } T \text{ is a triangle or thetrahedron,} \\ Q_1(T) & \text{if } T \text{ is a parallelogram or parallelepiped,} \end{cases} \tag{2.1}$$

is included in our scheme; $P_1(T)$ and $Q_1(T)$ denote the set of those algebraic polynomials of total and partial degree ≤ 1, respectively. The Lebesgue and Sobolev spaces $L^2(\Omega)$, $H^1(\Omega)$, $H(\text{div},\Omega)$ etc. are defined as usual (e.g., as in [H, LM, GR]), with corresponding norms $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_{H(\text{div},\Omega)}$ etc., we set $H^1_D(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$. The class of nonconforming finite elements under consideration is defined by a finite dimensional space $\mathcal{S} \subseteq H^1(\mathcal{T})$, $H^\ell(\mathcal{T}) = H^\ell(\cup_{T\in\mathcal{T}} \operatorname{int} T)$, which satisfies

$$\mathcal{S}_D \subseteq \mathcal{S} \subseteq H^2(\mathcal{T}), \tag{2.2}$$

where

$$\mathcal{S}_D := \{v_h \in C(\overline{\Omega}) : v_h = 0 \text{ on } \Gamma_D \text{ and } \forall T \in \mathcal{T}, v_h|_T \in \mathcal{P}_1(T)\}.$$

Then, the discrete solution $U \in H^2(\mathcal{T})$ satisfies, for all $V \in \mathcal{S}$,

$$\int_\Omega \nabla_\mathcal{T} V \cdot A \nabla_\mathcal{T} U \, dx = \int_\Omega fV \, dx + \int_{\Gamma_N} gV \, ds. \tag{2.3}$$

Here $\nabla_\mathcal{T} U(x)$ denotes $\nabla U|_T$ for $x \in T$ in $\mathcal{T}$, which may be different from the distributional gradient $\nabla U \in \mathcal{D}'(\Omega)$. Similarly, $\operatorname{div}_\mathcal{T}$ denotes the $\mathcal{T}$-elementwise action of the divergence operator. The conditions imposed on U are Dirichlet conditions $\int_E (U - u_D)\, ds = 0$ for all $E \in \mathcal{E}_D$ and weak continuity conditions $\int_E [U]\, ds = 0$ for all $E \in \mathcal{E}_\Omega$; $[U]$ denotes the jump of U across E.

Remark 2.1. Via integration by parts we infer from (1.1)–(1.3) and (2.3) for $v_h \in \mathcal{S}_D$, that

$$\int_\Omega \nabla v_h \cdot A \nabla_\mathcal{T} U \, dx = \int_\Omega \nabla v_h \cdot A \nabla u \, dx. \tag{2.4}$$

This leads to the Galerkin orthogonality, i.e., for all $v_h \in \mathcal{S}_D$ we have

$$\int_\Omega \nabla v_h \cdot A (\nabla u - \nabla_\mathcal{T} U)\, dx = 0. \tag{2.5}$$

Example 2.1. The Crouzeix-Raviart elements are described by $\mathcal{S}$ as follows. $\mathcal{S}$ consists of all elementwise affine functions V which are continuous in all midpoints z_E of faces $E \in \mathcal{E}_\Omega$ and $V(z_E) = 0$ if $E \in \mathcal{E}_D$. Moreover, the approximate solution U of (2.3) is defined analogously except that $U(z_E) = \int_E u_D \, ds / |E|$ of a face $E \in \mathcal{E}_D$ with length or area $|E|$.

Example 2.2. For parallelograms we refer to the rotated bilinear element by Rannacher and Turek [RT] which is *not* included in our analysis as then, in general, $\mathcal{S}_D \not\subseteq \mathcal{S}$.

Example 2.3. It is stressed that $\mathcal{S}_D$ is a conforming test function space which is included in the nonconforming finite element spaces for triangles or tetrahedra. For parallelograms or parallelepipeds, (2.2) means that the polynomial degrees are at least of second order to include the conforming Q_1-finite elements as suggested in [KS].

We consider $d = 2, 3$ simultaneously and let $k := 1$ if $d = 2$ and $k := 3$ if $d = 3$. The Curl of a function $\psi \in H^1(\Omega)^k$ is defined by

$$\operatorname{Curl} \psi := Q \nabla \psi \quad \text{if } d = 2 \quad \text{and} \quad \operatorname{Curl} \psi := \nabla \times \psi \quad \text{if } d = 3,$$

where Q is such that $Q(a_1, a_2) = (-a_2, a_1)$ for any $(a_1, a_2) \in \mathbb{R}^2$ and $v \times w$ denotes the usual vector product of two vectors $v, w \in \mathbb{R}^3$. Given a unit normal n_E we define the tangential component of a vector $v \in \mathbb{R}^d$ with respect to n_E by

$$\gamma_{t_E}(v) := \begin{cases} v \cdot Q n_E & \text{if } d = 2, \\ v \times n_E & \text{if } d = 3. \end{cases} \tag{2.6}$$

Note that we have, for $d = 2, 3$, $\phi \in H^1(T)$, $\psi \in H^1(T)^k$, by an integration by parts

$$\sum_{E \subset \partial T} \int_E \phi \operatorname{Curl} \psi \cdot n \, ds = \int_T \nabla \phi \cdot \operatorname{Curl} \psi \, dx = \sum_{E \subset \partial T} \int_E \psi \cdot \gamma_{t_E}(\nabla \phi)\, ds.$$

Let $\Gamma_0, ..., \Gamma_p$ denote the connectivity components of Γ. The subsequent characterisation of divergence free vector fields is a refined version of Theorem 3.5 in [GR] for Lipschitz boundaries.

Theorem 2.1. *A function $v \in L^2(\Omega)^d$ satisfies*

$$\operatorname{div} v = 0 \quad \text{and} \quad \int_{\Gamma_j} v \cdot n \, ds = 0 \quad \text{for all } j = 0, ..., p$$

if and only if there exists a stream function $\phi \in H^1(\Omega)^k$ such that $v = \operatorname{Curl}\phi$. Moreover, in this case ϕ can be chosen such that

$$\|\nabla\phi\|_{L^2(\Omega)} \le c_3\|v\|_{L^2(\Omega)}$$

with a constant $c_3 > 0$ that only depends on Ω.

Proof. In this proof and at similar occasions, $\lesssim$ abbreviates an inequality $\le$ up to a constant $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent factor. Also, $\|\cdot\|_{p,K}$ abbreviates $\|\cdot\|_{L^p(K)}$ and $\|\cdot\|_2 := \|\cdot\|_{2,\Omega}$.

We refer to [GR] for the proof of the if-and-only-if part of the theorem and focus on the estimate $\|\nabla\phi\|_2 \lesssim \|v\|_2$ for $d = 3$ (the estimate is obvious for $d = 2$ since then $\|v\|_2 = \|\operatorname{Curl}\phi\|_2 = \|Q\nabla\phi\|_2 = \|\nabla\phi\|_2$). The function ϕ is constructed as follows. Let B be a large ball containing $\overline{\Omega}$ and let θ be the solution to

$$\Delta\theta = 0 \quad \text{in } B\setminus\overline{\Omega}, \quad \partial\theta/\partial n = v\cdot n \quad \text{on } \Gamma_j,\ j = 0, ..., p, \quad \partial\theta/\partial n = 0 \quad \text{on } \partial B.$$

The compatibility condition $\int_{\Gamma_j} v\cdot n\, ds = 0$, $j = 0, ..., p$ ensures the existence of a solution which is unique up to an additive constant on each connectivity component of $B\setminus\overline{\Omega}$. We choose this constant in $\theta \in H^1(B\setminus\overline{\Omega})$ such that θ has vanishing integral mean on each component. Then we extend v to $\mathbb{R}^3$ by

$$\tilde{v} = v \quad \text{in } \Omega, \quad \tilde{v} = \nabla\theta \quad \text{in } B\setminus\overline{\Omega}, \quad \tilde{v} = 0 \quad \text{in } \mathbb{R}^3\setminus\overline{B}.$$

The function $\tilde{v} \in H(\operatorname{div};\mathbb{R}^3)$ satisfies $\operatorname{div}\tilde{v} = 0$ and has compact support. To estimate $\|\tilde{v}\|^2_{H(\operatorname{div};\mathbb{R}^3)} = \|\tilde{v}\|^2_{2,\mathbb{R}^3} = \|v\|_2^2 + \|\nabla\theta\|^2_{2,B\setminus\overline{\Omega}}$, we perform an integration by parts and use $\Delta\theta = 0$ in $B\setminus\overline{\Omega}$ and $\nabla\theta\cdot n = 0$ on ∂B. This leads to

$$\|\nabla\theta\|^2_{2,B\setminus\overline{\Omega}} = \int_{\partial(B\setminus\overline{\Omega})}(\nabla\theta\cdot n)\theta\, ds = \int_{\partial(\mathbb{R}^3\setminus\overline{\Omega})}(\nabla\theta\cdot n)\theta\, ds \le \|\nabla\theta\cdot n\|_{H^{-1/2}(\partial(\mathbb{R}^3\setminus\overline{\Omega}))}\|\theta\|_{H^{1/2}(\partial(\mathbb{R}^3\setminus\overline{\Omega}))}.$$

The existence of a bounded extension operator $H^1(B\setminus\overline{\Omega}) \to H^1(\mathbb{R}^3\setminus\overline{\Omega})$ and a Poincaré inequality show

$$\|\theta\|_{H^{1/2}(\partial(\mathbb{R}^3\setminus\overline{\Omega}))} = \inf_{\substack{w\in H^1(\mathbb{R}^3\setminus\overline{\Omega})\\ w|_\Gamma=\theta}} \|w\|_{H^1(\mathbb{R}^3\setminus\overline{\Omega})} \lesssim \|\theta\|_{H^1(B\setminus\overline{\Omega})} \lesssim \|\nabla\theta\|_{2,B\setminus\overline{\Omega}}.$$

Since the norms of $\nabla\theta\cdot n$ and $v\cdot n$ in $H^{-1/2}(\Gamma)$ are equivalent (though defined from different sides of Γ), the continuity of the mapping $H(\operatorname{div};\Omega) \to H^{-1/2}(\Gamma)$, $v \mapsto v\cdot n$, yields

$$\|\nabla\theta\cdot n\|_{H^{-1/2}(\partial(\mathbb{R}^3\setminus\overline{\Omega}))} \lesssim \|v\cdot n\|_{H^{-1/2}(\Gamma)} \lesssim \|v\|_{H(\operatorname{div};\Omega)} = \|v\|_2.$$

The last three estimates show $\|\nabla\theta\|_{2,B\setminus\overline{\Omega}} \lesssim \|v\|_2$ which implies $\|\tilde{v}\|_{2,\mathbb{R}^3} \lesssim \|v\|_2$. The function $\tilde{v}$ satisfies the conditions of Theorem 3.4 in [GR] on the smooth domain B, namely

$$\operatorname{div}\tilde{v} = 0 \quad \text{in } B \text{ and } \int_{\partial B}\tilde{v}\cdot n\, ds = 0.$$

Hence, there exists $\phi \in H^1(\Omega)^3$ with $\operatorname{Curl}\phi = \tilde{v}$ and $\operatorname{div}\phi = 0$ in B. By Theorem 3.5 in [GR], ϕ can be chosen such that

$$\phi\cdot n = 0 \quad \text{on } \partial B$$

and solves $-\Delta\phi = \operatorname{Curl}\tilde{v}$ in B and $\partial\phi/\partial s = \tilde{v}\cdot n = 0$ on ∂B. We then have

$$\|\nabla\phi\|^2_{2,B} \le \|\tilde{v}\|_{2,B}\|\operatorname{Curl}\phi\|_{2,B} \lesssim \|v\|_2\|\nabla\phi\|_{2,B}. \quad \square$$

Lemma 2.1. *Assume that $\Gamma_D \subseteq \Gamma_0$. Then, there exist $\alpha \in H^1(\Omega)$ and $\beta \in H^1(\Omega)^k$ such that*

$$A\nabla_{\mathcal{T}} U = A\nabla\alpha - \operatorname{Curl}\beta. \tag{2.7}$$

The functions α and β can be chosen such that

$$\alpha|_{\Gamma_D} = u_D \quad \text{on } \Gamma_D \quad \text{and} \quad \operatorname{Curl}\beta \cdot n = 0 \quad \text{on } \Gamma_N. \tag{2.8}$$

Proof. Choose $\alpha \in H^1(\Omega)$ such that $\alpha|_{\Gamma_D} = u_D$ and for all $v \in H^1_D(\Omega)$ we have

$$\int_\Omega A(\nabla\alpha - \nabla_{\mathcal{T}} U)\cdot\nabla v\,dx = 0. \tag{2.9}$$

Then, $A(\nabla\alpha - \nabla_{\mathcal{T}} U)$ is divergence-free, and by (2.9) we have, for all $v \in H^1_D(\Omega)$,

$$\int_{\Gamma_N} A(\nabla\alpha - \nabla_{\mathcal{T}} U)\cdot n\,v\,ds = 0,$$

which implies $A(\nabla\alpha - \nabla_{\mathcal{T}} U)\cdot n = 0$ on $\Gamma_1 \cup \ldots \cup \Gamma_p$. Moreover, we deduce from integration by parts

$$\int_{\Gamma_0} A(\nabla\alpha - \nabla_{\mathcal{T}} U)\cdot n\,ds = \int_{\partial\Omega} A(\nabla\alpha - \nabla_{\mathcal{T}} U)\cdot n\,ds = 0.$$

According to Theorem 2.1 we can find $\beta \in H^1(\Omega)^k$ such that $A(\nabla\alpha - \nabla_{\mathcal{T}} U) = \operatorname{Curl}\beta$. ☐

The following approximation operator is one key ingredient for the reliability proofs. For each $z \in \mathcal{K}$ the set Ω_z is a (possibly enlarged) patch (i.e., union of neighbouring elements) of diameter h_z which satisfies $h_z \le c_4 h_T$ if $T \subseteq \Omega_z$ with an $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant $c_4 > 0$. We refer to [C2, CB] for definitions and proofs.

Theorem 2.2 ([C2, CB])**.** *There exists a linear mapping $\mathcal{J} : H^1_D(\Omega) \to \mathcal{S}_D$ which satisfies*

$$\|\nabla\,\mathcal{J}\varphi\|_{L^2(\Omega)} + \|h_{\mathcal{T}}^{-1}(\varphi - \mathcal{J}\varphi)\|_{L^2(\Omega)} + \|h_{\mathcal{E}}^{-1/2}(\varphi - \mathcal{J}\varphi)\|_{L^2(\cup\mathcal{E})} \le c_5\,\|\nabla\varphi\|_{L^2(\Omega)} \tag{2.10}$$

for all $\varphi \in H^1_D(\Omega)$. Moreover, for all $f \in L^2(\Omega)$, we have

$$\int_\Omega f(\varphi - \mathcal{J}\varphi)\,dx \le c_6\,\|\nabla\varphi\|_{L^2(\Omega)}\Big(\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f - f_z\|^2_{L^2(\Omega_z)}\Big)^{1/2}. \tag{2.11}$$

The $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constants $c_5, c_6 > 0$ depend on the shape of the elements only. ☐

Definition 2.1. For $E \in \mathcal{E}_\Omega$ and $T_1, T_2 \in \mathcal{T}$ such that $E = T_1 \cap T_2$ let n_E be the unit vector perpendicular to E pointing from T_1 to T_2 and define

$$[A\nabla_{\mathcal{T}} U \cdot n_E] := (A\nabla U|_{T_2} - A\nabla U|_{T_1})\cdot n_E.$$

For $E \in \mathcal{E}_N$ and $T \in \mathcal{T}$ with $E \subset \partial T$ and the outer unit normal n to $E \cap \Gamma_N$ let

$$[A\nabla_{\mathcal{T}} U \cdot n_E] := g - A\nabla U|_T \cdot n.$$

We assume that $u_D \in H^1(\Gamma_D) \cap C(\Gamma_D)$ and $u_D|_E \in H^2(E)$ for all $E \in \mathcal{E}_D$ and denote by $u_{h,D}$ the nodal $\mathcal{E}_D$-piecewise bilinear interpolant of u_D on Γ_D which satisfies $u_{h,D}(z) = u_D(z)$ for all $z \in \mathcal{N} \cap \Gamma_D$.

Definition 2.2. For $E \in \mathcal{E}_\Omega$ with $E = T_1 \cap T_2$, $T_1, T_2 \in \mathcal{T}$, define

$$[\gamma_{t_E}(\nabla_{\mathcal{T}} U)] := \gamma_{t_E}(\nabla U|_{T_2}) - \gamma_{t_E}(\nabla U|_{T_1}) \quad \text{and} \quad [U] := (U|_{T_2} - U|_{T_1})|_E \quad \text{on } E,$$

where γ_{t_E} is defined in (2.6) through n_E which points from T_1 to T_2. For $E \in \mathcal{E}_D$ and $T \in \mathcal{T}$ with $E \subset \partial T$ we set

$$[\gamma_{t_E}(\nabla_{\mathcal{T}} U)] := \partial u_D/\partial s - \gamma_{t_E}(\nabla U|_T) \quad \text{and} \quad [U] := u_D - U|_E,$$

Here, $\partial u_D/\partial s$ denotes the surface gradient of u_D along E and γ_{t_E} is defined in (2.6) via the outer unit normal $n_E = n$ on $E \cap \Gamma_D$.

3. A Residual-Based Reliable a Posteriori Error Estimate

The error $u - U \in H^1(\mathcal{T})$ of the approximate solution U will be measured in the discrete H^1-seminorm $\|\nabla_{\mathcal{T}}(u-U)\|_{L^2(\Omega)}$ by employing a Helmholtz decomposition of $A\nabla_{\mathcal{T}} U \in L^2(\Omega)^d$. The proof of reliability in Theorem 3.1 will follow the proof of efficiency in Theorem 3.2.

Theorem 3.1. *Let $u \in H^1(\Omega)$ and $U \in H^2(\mathcal{T})$ satisfy (1.1)–(1.3) and (2.3), respectively. Suppose $\Gamma_D \subseteq \Gamma_0$ and that $\int_E [U]\, ds = 0$ for all $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$. Then, there exists an $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant $c_1 > 0$ such that*

$$\|\nabla_{\mathcal{T}}(u-U)\|^2_{L^2(\Omega)} \le c_1 \Big(\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z \in \mathbb{R}} \|f + \operatorname{div}_{\mathcal{T}} A \nabla_{\mathcal{T}} U - f_z\|^2_{L^2(\Omega_z)}$$
$$+ \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} h_E \|[(A\nabla_{\mathcal{T}} U)\cdot n_E]\|^2_{L^2(E)} + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D} h_E \|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|^2_{L^2(E)} + \|h_{\mathcal{E}}^{3/2} \partial^2_{\mathcal{E}} u_D/\partial s^2\|^2_{L^2(\Gamma_D)} \Big).$$

Remark 3.1. The term that includes u_D is of higher order for the lowest order schemes.

Remark 3.2. If $\operatorname{div}_{\mathcal{T}} A\nabla_{\mathcal{T}} U \equiv 0$ and $f \in H^1(\Omega)$ we can choose f_z as the integral mean of f over Ω_z to verify $\min_{f_z\in\mathbb{R}} \|f + \operatorname{div}_{\mathcal{T}} A\nabla_{\mathcal{T}} U - f_z\|_{L^2(\Omega_z)} \le c_7 h_z \|\nabla f\|_{L^2(\Omega_z)}$ which leads to a higher order term.

The reliability estimate of Theorem 3.1 is sharp according to the converse, efficiency, inequality.

Theorem 3.2. *Let $u \in H^1(\Omega)$ and $U \in H^2(\mathcal{T})$ satisfy (1.1)–(1.3) and (2.3), respectively. Then, there exists an $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant $c_8 > 0$ such that, for all $T \in \mathcal{T}$ and $\mathcal{T}_T := \{T' \in \mathcal{T} : T' \cap T \in \mathcal{E}_\Omega\} \cup \{T\}$,*

$$h_T^2 \|f + \operatorname{div} A\nabla U\|^2_{L^2(T)} + \sum_{\substack{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N,\\ E\subset\partial T}} h_E \|[A\nabla_{\mathcal{T}} U \cdot n_E]\|^2_{L^2(E)} + \sum_{\substack{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D,\\ E\subset\partial T}} h_E \|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|^2_{L^2(E)}$$
$$\le c_8 \Big(\sum_{T'\in\mathcal{T}_T} \big(\|\nabla u - \nabla_{\mathcal{T}} U\|^2_{L^2(T')} + h_{T'}^2 \inf_{f_{T'}\in\mathcal{P}_1(T')} \|f - f_{T'}\|^2_{L^2(T')} \big)$$
$$+ \|h_{\mathcal{E}}^{3/2} \partial^2_{\mathcal{E}} u_D/\partial s^2\|^2_{L^2(\partial T\cap\Gamma_D)} + \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}} g/\partial s\|^2_{L^2(\partial T\cap\Gamma_N)} \Big).$$

Proof. Estimates regarding the volume terms and the jumps of the normal components of $\nabla_{\mathcal{T}} U$ can be proved as in the conforming situation [V1, V2]. Concerning the jumps of the tangential derivatives of U along $E \in \mathcal{E}_\Omega$, we let b_E be the bubble function on ω_E vanishing on $\partial\omega_E$ and normed by $\max_{\omega_E} b_E = 1$. Using the extension operator $P\colon C(E) \to C(\omega_E)$ of [V1] we find

$$0 = \int_{\omega_E} \operatorname{Curl}(b_E P([\gamma_{t_E}(\nabla_{\mathcal{T}} U)])) \cdot \nabla u \, dx. \tag{3.1}$$

Integrating by parts, we obtain (using equivalence of $\|\cdot\|_{2,E}$ and $\|b_E^{1/2}\cdot\|_{2,E}$ on $[\gamma_{t_E}(\mathcal{S})]|_E$ and applying P to each component of $[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]$)

$$\|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|^2_{2,E} \lesssim \|b_E^{1/2}[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|^2_{2,E} = \int_{\omega_E} \operatorname{Curl}(b_E P([\gamma_{t_E}(\nabla_{\mathcal{T}} U)])) \cdot \nabla_{\mathcal{T}} U \, dx \tag{3.2}$$

$$= \int_{\omega_E} \operatorname{Curl}(b_E P([\gamma_{t_E}(\nabla_{\mathcal{T}} U)])) \cdot \nabla_{\mathcal{T}}(U-u)\, dx \le \|\operatorname{Curl}(b_E P([\gamma_{t_E}(\nabla_{\mathcal{T}} U)]))\|_{2,\omega_E} \|\nabla_{\mathcal{T}}(u-U)\|_{2,\omega_E}.$$

An inverse estimate, $|b_E| \le 1$, and the properties of P show

$$\|\operatorname{Curl}(b_E P([\gamma_{t_E}(\nabla_{\mathcal{T}} U)]))\|_{2,\omega_E} \lesssim h_E^{-1}\|b_E P([\gamma_{t_E}(\nabla_{\mathcal{T}} U)])\|_{2,\omega_E} \lesssim h_E^{-1/2}\|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|_{2,E}$$

so that

$$h_E^{1/2}\|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|_{2,E} \lesssim \|\nabla_{\mathcal{T}}(u-U)\|_{2,\omega_E}. \tag{3.3}$$

For $E \in \mathcal{E}_D$ and $\omega_E = T$ we insert $\partial u_{h,D}/\partial s$, perform an integration by parts on T, utilise an interpolation estimate [BS], and argue as above to verify the equivalent of (3.3). □

The remaining part of this section is devoted to the proof of Theorem 3.1 which is split with Lemma 2.1 into several terms and corresponding estimates. As an immediate consequence (which uses the fact that $A^{1/2}\nabla(u-\alpha)$ is L^2-orthogonal to $\operatorname{Curl}\beta$) of Lemma 2.1 we have

$$\begin{aligned}\|A^{1/2}\nabla_{\mathcal{T}}(u-U)\|^2_{L^2(\Omega)} &= \|A^{1/2}\nabla(u-\alpha)\|^2_{L^2(\Omega)} + \|A^{-1/2}\operatorname{Curl}\beta\|^2_{L^2(\Omega)}\\ &= \int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot A\nabla(u-\alpha)\,dx + \int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot \operatorname{Curl}\beta\,dx.\end{aligned} \tag{3.4}$$

Lemma 3.1. *There exists an $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant $c_9 > 0$ such that*

$$\begin{aligned}\int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot A\nabla(u-\alpha)\,dx \le c_9\Big(&\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f + \operatorname{div}_{\mathcal{T}} A\nabla_{\mathcal{T}} U - f_z\|^2_{L^2(\Omega_z)}\\ &+ \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} h_E\|[(A\nabla_{\mathcal{T}} U)\cdot n_E]\|^2_{L^2(E)}\Big)^{1/2}\|\nabla(u-\alpha)\|_{L^2(\Omega)}.\end{aligned}$$

Proof. Let $w := u-\alpha \in H^1_D(\Omega)$. Galerkin's orthogonality (2.5) and an elementwise integration by parts yield

$$\begin{aligned}&\int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot A\nabla w\,dx = \int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot A\nabla(w-\mathcal{J}w)\,dx\\ &= \sum_{T\in\mathcal{T}}\int_T (f+\operatorname{div} A\nabla U)(w-\mathcal{J}w)\,dx + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N}\int_E [A\nabla_{\mathcal{T}} U\cdot n_E](w-\mathcal{J}w)\,dx\\ &\lesssim \Big(\sum_{z\in\mathcal{K}} h_z^2\min_{f_z\in\mathbb{R}}\|f+\operatorname{div}_{\mathcal{T}} A\nabla_{\mathcal{T}} U - f_z\|^2_{2,\Omega_z}\Big)^{1/2}\|\nabla w\|_2 + \Big(\sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} h_E\|[A\nabla_{\mathcal{T}} U\cdot n_E]\|^2_{2,E}\Big)^{1/2}\|\nabla w\|_2.\end{aligned}$$

A (discrete) Cauchy inequality proves the assertion. □

Lemma 3.2. *There exists $B \in C(\overline{\Omega})^k$ such that $B|_T \in \mathcal{P}_1(T)^k$ for all $T \in \mathcal{T}$ and, for each $E \in \mathcal{E}$,*

$$\|\beta - B\|_{L^2(E)} \le c_{10} h_E^{1/2}\|\nabla\beta\|_{L^2(\omega_E)} \quad \text{and} \quad \|\nabla B\|_{L^2(\Omega)} \le c_{11}\|\nabla\beta\|_{L^2(\Omega)} \tag{3.5}$$

with $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constants $c_{10}, c_{11} > 0$.

Proof. Apply the operator $\mathcal{J}$ of Theorem 2.2 (with $\Gamma_D = \emptyset$) to each component of β. □

The non-conformity error in (3.4) will be estimated twice. A first result for the estimation of the second term in (3.4) involves an auxiliary function v. This is the straightforward generalisation of the two-dimensional result [C1, DDPV].

Lemma 3.3. *Suppose $\int_E [U]\,ds = 0$ for all $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$ and that $\mathcal{T}$ consists of triangles or tetrahedra only. Then, there exists an $(h_\mathcal{T}, h_\mathcal{E})$-independent constant $c_{12} > 0$ such that, for all $v \in H^1(\Omega)$ which satisfy $v = u_D$ on Γ_D and $\int_E (v-U)\,ds = 0$ for all $E \in \mathcal{E}_N$,*

$$\int_\Omega \nabla_\mathcal{T}(u-U)\cdot \operatorname{Curl}\beta\,dx \le c_{12}\Big(\sum_{E\in\mathcal{E}_\Omega} h_E \|[\gamma_{t_E}(\nabla_\mathcal{T}(v-U))]\|^2_{L^2(E)} + \sum_{E\in\mathcal{E}_D} h_E\|[\gamma_{t_E}(\nabla_\mathcal{T} U)]\|^2_{L^2(E)}$$
$$+ \sum_{E\in\mathcal{E}_N} h_E \|\gamma_{t_E}(\nabla_\mathcal{T}(v-U))\|^2_{L^2(E)}\Big)^{1/2} \|A^{1/2}\nabla_\mathcal{T}(u-U)\|_{L^2(\Omega)}.$$

Proof. With B from Lemma 3.2, we have, since $\operatorname{Curl} B \in H(\operatorname{div};\Omega)$ and $\operatorname{Curl} B|_T$ is constant, that $[\operatorname{Curl} B \cdot n_E] = 0$ across all interior edges. Using $\int_E [v-U]\operatorname{Curl} B\cdot n_E\,ds = 0$ for $E \in \mathcal{E}_\Omega$ and $\int_E (v-U)\operatorname{Curl} B \cdot n_E\,ds = 0$ for $E \in \mathcal{E}_N \cup \mathcal{E}_D$, we infer $\int_\Omega \nabla_\mathcal{T}(v-U)\cdot \operatorname{Curl} B\,dx = 0$. Using $\operatorname{Curl}\beta\cdot n = 0$ on Γ_N, we also have $\int_\Omega \nabla(v-u)\cdot\operatorname{Curl}\beta\,dx = 0$. Employing these two observations and Lemma 3.2 we find

$$\int_\Omega \nabla_\mathcal{T}(u-U)\cdot\operatorname{Curl}\beta\,dx = \int_\Omega \nabla_\mathcal{T}(v-U)\cdot\operatorname{Curl}\beta\,dx = \int_\Omega \nabla_\mathcal{T}(v-U)\cdot\operatorname{Curl}(\beta-B)\,dx$$
$$= \sum_{E\in\mathcal{E}_\Omega}\int_E [\gamma_{t_E}(\nabla_\mathcal{T}(v-U))](\beta-B)\,ds + \sum_{E\in\mathcal{E}_N\cup\mathcal{E}_D}\int_E \gamma_{t_E}(\nabla_\mathcal{T}(v-U))(\beta-B)\,ds$$
$$\lesssim \Big(\sum_{E\in\mathcal{E}_\Omega} h_E\|[\gamma_{t_E}(\nabla_\mathcal{T}(v-U))]\|^2_{2,E} + \sum_{E\in\mathcal{E}_D} h_E\|[\gamma_{t_E}(\nabla_\mathcal{T}U)]\|^2_{2,E}$$
$$+ \sum_{E\in\mathcal{E}_N} h_E\|\gamma_{t_E}(\nabla_\mathcal{T}(v-U))\|^2_{2,E}\Big)^{1/2}\|\nabla\beta\|_2.$$

The estimate $\|\nabla\beta\|_2 \lesssim \|A\nabla_\mathcal{T}(U-\alpha)\|_2 \lesssim \|A^{1/2}\nabla_\mathcal{T}(u-U)\|_2$ concludes the proof. □

Remark 3.3. In two space dimensions, β is constant on each component of Γ_N (since $0 = \operatorname{Curl}\beta\cdot n = \partial\beta/\partial t$ is the tangential derivative of β). Then, $\|[\gamma_{t_E}(\nabla_\mathcal{T}(v-U))]\|_{L^2(E)}$ can be replaced by $\|[\gamma_{t_E}(\nabla_\mathcal{T}U)]\|_{L^2(E)}$ for $E\in\mathcal{E}_\Omega$ and $\|\gamma_{t_E}(\nabla_\mathcal{T}(v-U))\|_{L^2(E)}$ can be neglected for $E \in \mathcal{E}_N$ [C1, DDPV].

We do not discuss the construction of a good function v in Lemma 3.3. Instead, we focus on a second estimation which avoids v but whose proof is rather more complicated.

Lemma 3.4. *Assume that $\int_E [U]\,ds = 0$ for all $E\in\mathcal{E}_\Omega\cup\mathcal{E}_D$. Then, there exists an $(h_\mathcal{T}, h_\mathcal{E})$-independent positive constant c_{13} such that*

$$\int_\Omega \nabla_\mathcal{T}(u-U)\cdot\operatorname{Curl}\beta\,dx$$
$$\le c_{13}\Big(\sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D} h_E\|[\gamma_{t_E}(\nabla_\mathcal{T}U)]\|^2_{L^2(E)} + \|h_\mathcal{E}^{3/2}\partial^2_\mathcal{E} u_D/\partial s^2\|^2_{L^2(\Gamma_D)}\Big)^{1/2}\|\operatorname{Curl}\beta\|_{L^2(\Omega)}.$$

Proof. Recalling that $\int_\Omega \operatorname{Curl}\beta\cdot\nabla v\,dx = 0$ for $v\in H^1_D(\Omega)$, we have for $v_h \in H^1(\Omega)$,

$$\int_\Omega \nabla_\mathcal{T}(u-U)\cdot\operatorname{Curl}\beta\,dx = \int_\Omega \nabla(u-v_h-v)\cdot\operatorname{Curl}\beta\,dx + \int_\Omega \nabla_\mathcal{T}(v_h-U)\cdot\operatorname{Curl}\beta\,dx$$
$$\le \|\nabla(u-v_h-v)\|_2\|\operatorname{Curl}\beta\|_2 + \|\nabla_\mathcal{T}(v_h-U)\|_2\|\operatorname{Curl}\beta\|_2. \tag{3.6}$$

Let $(\varphi_z : z\in\mathcal{N})$ be the nodal basis of the lowest order finite element space associated to $\mathcal{T}$, i.e., $\varphi_z \in C(\overline{\Omega})$, $\varphi_z|_T \in \mathcal{P}_1(T)$ for all $T\in\mathcal{T}$, $\varphi_z(x) = 0$ for $x\in\mathcal{N}\setminus\{z\}$, and $\varphi_z(z) = 1$. Set $\omega_z := \operatorname{int}(\operatorname{supp}\varphi_z)$ and, for $z\in\mathcal{N}\cap\Gamma_D$, let $h_z := \operatorname{diam}(\omega_z)$. For each $z\in\mathcal{N}$ define

$$\mathcal{S}(z,u_D) := \{v_z \in C(\omega_z) : v_z|_{\omega_z}\in\mathcal{S}|_{\omega_z} \text{ and } v_z = u_{h,D} \text{ on } \Gamma_D\cap\partial\omega_z\}.$$

We then have, for $v_h \in \tilde{\mathcal{S}}$,

$$\tilde{\mathcal{S}} := \{\sum_{z\in\mathcal{N}} \varphi_z v_z : \forall z \in \mathcal{N}, v_z \in \mathcal{S}(z,u_D)\} \subset H^1(\Omega),$$

that

$$\inf_{v\in H^1_D(\Omega)} \|\nabla(u - v_h - v)\|_2^2 = \inf_{\substack{w\in H^1(\Omega)\\ w|_{\Gamma_D}=u_D-u_{h,D}}} \|\nabla w\|_2^2 \lesssim \|h_{\mathcal{E}}^{3/2}\partial^2_{\mathcal{E}} u_D/\partial s^2\|^2_{2,\Gamma_D}$$

(cf. [BC] for details in three dimensions) and, by an elementwise inverse estimate,

$$\min_{v_h\in\tilde{\mathcal{S}}} \|\nabla_{\mathcal{T}}(v_h - U)\|_2 \lesssim \min_{v_h\in\tilde{\mathcal{S}}} \|h_{\mathcal{T}}^{-1}(v_h - U)\|_2. \tag{3.7}$$

(The constant in (3.7) depends on the polynomial degrees of functions in $\tilde{\mathcal{S}}$.) The minimiser $w_h = \sum_{z\in\mathcal{N}} w_z\varphi_z \in \tilde{\mathcal{S}}$ of the right-hand side in (3.7) obeys the orthogonal relation

$$\int_\Omega h_{\mathcal{T}}^{-2}(w_h - U)\,\varphi_z v_z\,dx = 0 \tag{3.8}$$

for all $z \in \mathcal{N}$ and $v_z \in \mathcal{S}(z,u_D)$. Using (3.8) and noting that $\sum_{z\in\mathcal{N}}\varphi_z = 1$ we have, for $v_z \in \mathcal{S}(z,u_D)$,

$$\begin{aligned}\|h_{\mathcal{T}}^{-1}(w_h - U)\|_2^2 &= \sum_{z\in\mathcal{N}}\int_\Omega h_{\mathcal{T}}^{-2}(w_h - U)\varphi_z(w_z - U)\,dx\\ &= \sum_{z\in\mathcal{N}}\int_\Omega h_{\mathcal{T}}^{-2}(w_h - U)\varphi_z(v_z - U)\,dx \lesssim \|h_{\mathcal{T}}^{-1}(w_h - U)\|_2\big(\sum_{z\in\mathcal{N}} h_z^{-2}\|\varphi_z^{1/2}(v_z - U)\|^2_{2,\omega_z}\big)^{1/2}.\end{aligned} \tag{3.9}$$

Similar to $\mathcal{S}(z,u_D)$ we set, for $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$, $\mathcal{T}_E := \{T \in \mathcal{T} : E \subseteq \partial T\}$ and

$$\mathcal{S}(E,u_D) := \{v_E \in C(\omega_E) : \forall T \in \mathcal{T}_E, v_E|_T \in \mathcal{S}|_T \text{ and } v_E = u_{h,D} \text{ on } \Gamma_D \cap E\}.$$

For a fixed $z \in \mathcal{N}$ we consider the semi-norms on a finite dimensional subspace of $L^2(\hat\omega_z)$, $\hat\omega_z := \cup_{E\subseteq\overline{\omega}_z}\omega_E$,

$$\begin{aligned}|||V|||_{1,z} &:= \min_{v_z\in\mathcal{S}(z,u_D)} h_z^{-1}\|\varphi_z^{1/2}(V - v_z)\|_{2,\omega_z},\\ |||V|||^2_{2,z} &:= \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D, E\subseteq\overline{\omega}_z} h_E^{-2} \min_{v_E\in\mathcal{S}(E,u_D)} \|V - v_E\|^2_{2,\omega_E}.\end{aligned}$$

We claim $|||V|||_{1,z} \lesssim |||V|||_{2,z}$. Indeed, if $|||V|||_{2,z} = 0$ we have for each $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$ with $E \subseteq \overline{\omega}_z$ that $V = v_E$ on the open set ω_E for some $v_E \in \mathcal{S}(E,u_D)$. The set of all such ω_E is a cover of ω_z and there is a sequence $E_1, ..., E_m$ of inner edges such that $\omega_{E_j} \cap \omega_{E_{j+1}} \neq \emptyset$, so we deduce $V \in \mathcal{S}(z,u_D)$ and thus $|||V|||_{1,z} = 0$. A compactness and a scaling argument (in the sense of equivalence of semi-norms) show

$$|||\cdot|||_{1,z} \lesssim |||\cdot|||_{2,z} \quad \text{on } \mathcal{S}|_{\hat\omega_z}. \tag{3.10}$$

Combining (3.7)-(3.10) we find

$$\min_{v_h\in\tilde{\mathcal{S}}} \|\nabla_{\mathcal{T}}(v_h - U)\|_2^2 \lesssim \sum_{z\in\mathcal{K}} |||U|||^2_{1,z} \lesssim \sum_{z\in\mathcal{K}} |||U|||^2_{2,z} \lesssim \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D} h_E^{-2} \min_{v_E\in\mathcal{S}(E,u_D)} \|U - v_E\|^2_{2,\omega_E}. \tag{3.11}$$

Since $\|[U]\|_{2,E} = 0$ means that U is continuous across E if $E \in \mathcal{E}_\Omega$ it implies $U|_{\omega_E} \in \mathcal{S}(E,u_D)$. If $U|_E = u_{h,D}$ for $E \in \mathcal{E}_D$ we also have $U|_{\omega_E} \in \mathcal{S}(E,u_D)$. A scaling and a compactness argument thus show, for all $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$,

$$\min_{v_E \in \mathcal{S}(E,u_D)} \|U - v_E\|_{2,\omega_E} \lesssim \begin{cases} h_E^{1/2}\|[U]\|_{2,E} & \text{if } E \in \mathcal{E}_\Omega, \\ h_E^{1/2}\|U - u_{h,D}\|_{2,E} & \text{if } E \in \mathcal{E}_D. \end{cases} \tag{3.12}$$

An interpolation estimate [BS] on $E \in \mathcal{E}_D$ shows $\|U - u_{h,D}\|_{2,E} \lesssim \|[U]\|_{2,E} + h_E^2\|\partial^2 u_D/\partial s^2\|_{2,E}$. Since $\int_E [U]\,ds = 0$ an edgewise Poincaré inequality yields $\|[U]\|_{2,E} \lesssim h_E\|[\gamma_{t_E}(\nabla_\mathcal{T} U)]\|_{2,E}$. Hence,

$$\min_{v_E \in \mathcal{S}(E,u_D)} h_E^{-1}\|U - v_E\|_{2,\omega_E} \lesssim h_E^{1/2}\|[\gamma_{t_E}(\nabla_\mathcal{T} U)]\|_{2,E} + h_E^{3/2}\|\partial^2 u_D/\partial s^2\|_{2,E\cap\Gamma_D}.$$

Using this in (3.11) and the resulting estimate in (3.6) we eventually verify the assertion of the lemma. □

Proof of Theorem 3.1. The combination of Lemma 3.1 and 3.4, (3.4), and the definitions

$$\eta_1^2 := \sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f + \operatorname{div}_\mathcal{T} A\nabla_\mathcal{T} U - f_z\|_{2,\Omega_z}^2 + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} h_E\|[(A\nabla_\mathcal{T} U)\cdot n_E]\|_{2,E}^2,$$

$$\eta_2^2 := \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D} h_E\|[\gamma_{t_E}(\nabla_\mathcal{T} U)]\|_{2,E}^2 + \|h_\mathcal{E}^{3/2}\partial_\mathcal{E}^2 u_D/\partial s^2\|_{2,\Gamma_D}^2, \quad \text{yields}$$

$$\begin{aligned}\|\nabla_\mathcal{T}(u-U)\|_2^2 &\lesssim \|A^{1/2}\nabla_\mathcal{T}(u-U)\|_2^2 = \int_\Omega \nabla_\mathcal{T}(u-U)\cdot A\nabla(u-\alpha)\,dx + \int_\Omega \nabla_\mathcal{T}(u-U)\cdot \operatorname{Curl}\beta\,dx \\ &\lesssim \eta_1\|\nabla(u-\alpha)\|_2 + \eta_2\|\operatorname{Curl}\beta\|_2 \le (\eta_1^2+\eta_2^2)^{1/2}(\|\nabla(u-\alpha)\|_2^2 + \|\operatorname{Curl}\beta\|_2^2)^{1/2} \\ &\lesssim (\eta_1^2+\eta_2^2)^{1/2}(\|A^{1/2}\nabla(u-\alpha)\|_2^2 + \|A^{-1/2}\operatorname{Curl}\beta\|_2^2)^{1/2} = (\eta_1^2+\eta_2^2)^{1/2}\|A^{1/2}\nabla_\mathcal{T}(u-U)\|_2 \\ &\lesssim (\eta_1^2+\eta_2^2)^{1/2}\|\nabla_\mathcal{T}(u-U)\|_2. \quad \square\end{aligned}$$

Remark 3.4. An alternative proof of Theorem 3.1 under more restrictive conditions if $d = 3$ follows with Lemma 3.3.

4. Reliable Averaging A Posteriori Error Estimates

In this section we prove modifications of Lemma 3.1 and 3.4 and then derive a posteriori error estimates based on averaging techniques for lowest order nonconforming finite elements, i.e., for the Crouzeix-Raviart element. For higher order methods we refer to the ideas of [BC].

We suppose that to each node $z \in \mathcal{N}\cap\overline{\Gamma}_N$ at most d distinct outer unit normals can be associated. Moreover, if $g|_E \in H^1(E)$ for all $E \in \mathcal{E}_N$ and if for each node $z \in \mathcal{N}\cap\overline{\Gamma}_N$ where the outer unit normal n is continuous g is continuous then

$$\begin{aligned}\mathcal{S}(\mathcal{T},g) := \{q_h \in C(\overline{\Omega})^d : \forall T\in\mathcal{T}, q_h|_T \in \mathcal{P}_1(T)^d \text{ and} \\ \forall z\in\mathcal{N}\cap\overline{\Gamma}_N, \forall E\in\mathcal{E}_N, z\in E, Aq_h\cdot n|_E(z) = g|_E(z)\}\end{aligned}$$

is non-void and well-defined. By the assumptions on u_D we may define

$$\mathcal{S}(\mathcal{T},u_D) := \{v_h \in C(\overline{\Omega}) : \forall T\in\mathcal{T}, v_h|_T\in\mathcal{P}_1(T) \text{ and } \forall z\in\mathcal{N}\cap\Gamma_D, v_h(z) = u_D(z)\}.$$

Theorem 4.1. *Let $u \in H^1(\Omega)$ and $U \in H^2(\mathcal{T})$ satisfy* (1.1)–(1.3) *and* (2.3), *respectively, and suppose $\Gamma_D \subseteq \Gamma_0$ and $\operatorname{div}_\mathcal{T} A\nabla_\mathcal{T} U \equiv 0$. Then, there exists an $(h_\mathcal{T}, h_\mathcal{E})$-independent positive constant c_{14} such that*

$$\begin{aligned}\|\nabla_\mathcal{T}(u-U)\|_{L^2(\Omega)}^2 \le c_{14}\Big(&\min_{v_h\in\mathcal{S}(\mathcal{T},u_D)} \|h_\mathcal{T}^{-1}(U-v_h)\|_{L^2(\Omega)}^2 + \min_{q_h\in\mathcal{S}(\mathcal{T},g)}\|\nabla_\mathcal{T} U - q_h\|_{L^2(\Omega)}^2 \\ &+ \|h_\mathcal{E}^{3/2}\partial_\mathcal{E}^2 u_D/\partial s^2\|_{L^2(\Gamma_D)}^2 + \|h_\mathcal{E}^{3/2}\partial_\mathcal{E} g/\partial s\|_{L^2(\Gamma_N)}^2 + \sum_{z\in\mathcal{K}} h_z^2\min_{f_z\in\mathbb{R}}\|f - f_z\|_{L^2(\Omega_z)}^2\Big).\end{aligned}$$

Remark 4.1. The terms including u_D and g are of higher order for the lowest order schemes.

Remark 4.2. If $f \in H^1(\Omega)$ we can choose f_z as the integral mean of f over Ω_z to verify $\min_{f_z\in\mathbb{R}} \|f - f_z\|_{L^2(\Omega_z)} \le c_7 h_z \|\nabla f\|_{L^2(\Omega_z)}$ which leads to a higher order term.

Remark 4.3. The above assumptions on Γ_N and g appear restrictive but can be weakened: If $\mathcal{S}'(\mathcal{T},g) := \{q_h \in C(\overline{\Omega})^d : \forall T \in \mathcal{T}, q_h|_T \in \mathcal{P}_1(T)^d \text{ and } \forall E \in \mathcal{E}_N, \int_E (Aq_h \cdot n - g)\, ds = 0\}$ and $g|_E \in H^1(E)$ for all $E \in \mathcal{E}_N$ then the estimate of Theorem 4.1 reads

$$\|\nabla_\mathcal{T}(u-U)\|^2_{L^2(\Omega)} \le c_{15}\Big(\min_{v_h\in\mathcal{S}(\mathcal{T},u_D)} \|h_\mathcal{T}^{-1}(U-v_h)\|^2_{L^2(\Omega)} + \|h_\mathcal{E}^{3/2}\partial^2_\mathcal{E} u_D/\partial s^2\|^2_{L^2(\Gamma_D)}$$
$$+ \min_{q_h\in\mathcal{S}'(\mathcal{T},g)} \big(\|\nabla_\mathcal{T} U - q_h\|^2_{L^2(\Omega)} + \|h_\mathcal{E}^{3/2}\partial_\mathcal{E}(g - Aq_h\cdot n)/\partial s\|^2_{L^2(\Gamma_N)}\big) + \sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f-f_z\|^2_{L^2(\Omega_z)}\Big).$$

Two efficiency estimates are presented in the next theorem. In the first, the multiplicative constant is 1 while higher order terms depend on regularity of the exact solution. In the second, higher order terms are given with the smoothness of the data while the multiplicative constant c_{16} is unknown.

Theorem 4.2. *Let $u \in H^1(\Omega)$ and $U \in H^2(\mathcal{T})$ satisfy (1.1)–(1.3) and (2.3), respectively. Then, there holds*

$$\min_{v_h\in\mathcal{S}(\mathcal{T},u_D)} \|h_\mathcal{T}^{-1}(U-v_h)\|_{L^2(\Omega)} + \min_{q_h\in\mathcal{S}(\mathcal{T},g)} \|\nabla_\mathcal{T} U - q_h\|_{L^2(\Omega)} \le \|h_\mathcal{T}^{-1}(u-U)\|_{L^2(\Omega)}$$
$$+ \|\nabla_\mathcal{T}(u-U)\|_{L^2(\Omega)} + \min_{v_h\in\mathcal{S}(\mathcal{T},u_D)} \|h_\mathcal{T}^{-1}(u-v_h)\|_{L^2(\Omega)} + \min_{q_h\in\mathcal{S}(\mathcal{T},g)} \|\nabla u - q_h\|_{L^2(\Omega)}.$$

Moreover, there exists a constant $c_{16} > 0$ such that

$$\min_{v_h\in\mathcal{S}(\mathcal{T},u_D)} \|h_\mathcal{T}^{-1}(U-v_h)\|_{L^2(\Omega)} + \min_{q_h\in\mathcal{S}(\mathcal{T},g)} \|\nabla_\mathcal{T} U - q_h\|_{L^2(\Omega)} \le c_{16}\Big(\|\nabla u - \nabla_\mathcal{T} U\|_{L^2(\Omega)}$$
$$+ \big(\sum_{T\in\mathcal{T}_T} h_T^2 \inf_{f_T\in\mathcal{P}_1(T)} \|f-f_T\|^2_{L^2(T)}\big)^{1/2} + \|h_\mathcal{E}^{3/2}\partial^2_\mathcal{E} u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|h_\mathcal{E}^{3/2}\partial_\mathcal{E} g/\partial s\|_{L^2(\Gamma_N)}\Big).$$

Proof. The first estimate follows from two applications of the triangle inequality. Using that global averaging is equivalent to local averaging [CB] and that local averaging is equivalent to weighted jumps of $\nabla_\mathcal{T} U$ the second estimate follows with Theorem 3.2. ☐

The proof of Theorem 4.1 is based on the following two lemmas.

Lemma 4.1. *Assume that $\operatorname{div}_\mathcal{T} A\nabla_\mathcal{T} U \equiv 0$. Then, there exists an $(h_\mathcal{T}, h_\mathcal{E})$-independent positive constant c_{17} such that*

$$\int_\Omega \nabla_\mathcal{T}(u-U)\cdot A\nabla(u-\alpha)\,dx \le c_{17}\Big(\min_{q_h\in\mathcal{S}(\mathcal{T},g)} \|\nabla_\mathcal{T} U - q_h\|^2_{L^2(\Omega)}$$
$$+ \|h_\mathcal{E}^{3/2}\partial_\mathcal{E} g/\partial s\|^2_{L^2(\Gamma_N)} + \sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f-f_z\|^2_{L^2(\Omega_z)}\Big)^{1/2} \|\nabla(u-\alpha)\|_{L^2(\Omega)}.$$

Proof. Let $w := u - \alpha \in H^1_D(\Omega)$. We obtain with Galerkin orthogonality (2.5), Cauchy's inequality, the properties of $\mathcal{J}$, and an elementwise inverse estimate after inserting $\operatorname{div}_\mathcal{T} A\nabla_\mathcal{T} U$, for arbitrary $q_h \in \mathcal{S}(\mathcal{T},g)$,

$$\int_\Omega \nabla_\mathcal{T}(u-U)\cdot A\nabla w\,dx = \int_\Omega (\nabla u - q_h)\cdot A\nabla(w - \mathcal{J}w)\,dx + \int_\Omega (q_h - \nabla_\mathcal{T} U)\cdot A\nabla(w-\mathcal{J}w)\,dx$$
$$\lesssim \Big(\big(\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f + \operatorname{div} Aq_h - f_z\|^2_{2,\Omega_z}\big)^{1/2} + \|\nabla_\mathcal{T} U - q_h\|_2 + \|h_\mathcal{E}^{1/2}(g - Aq_h\cdot n)\|_{2,\Gamma_N}\Big)\|\nabla w\|_2$$
$$\lesssim \Big(\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f - f_z\|^2_{2,\Omega_z} + \|\nabla_\mathcal{T} U - q_h\|_2^2 + \|h_\mathcal{E}^{1/2}(g - Aq_h\cdot n)\|^2_{2,\Gamma_N}\Big)\|\nabla w\|_2.$$

It follows from an edgewise interpolation estimate (see, e.g., [BS]) that

$$\|h_{\mathcal{E}}^{1/2}(g - Aq_h \cdot n)\|_{2,\Gamma_N} \lesssim \Big(\sum_{E\in\mathcal{E}_N} h_E^3 \|\partial g/\partial s\|_{2,E}^2\Big)^{1/2} =: \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}} g/\partial s\|_{2,\Gamma_N}. \quad \square$$

Lemma 4.2. *There exists an* $(h_{\mathcal{T}}, h_{\mathcal{E}})$*-independent positive constant* c_{18} *such that*

$$\int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot \operatorname{Curl}\beta\, dx$$
$$\le c_{18}\Big(\min_{v_h\in\mathcal{S}(\mathcal{T},u_D)} \|h_{\mathcal{T}}^{-1}(v_h - U)\|_{L^2(\Omega)}^2 + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}}^2 u_D/\partial s^2\|_{L^2(\Gamma_D)}^2\Big)^{1/2} \|\operatorname{Curl}\beta\|_{L^2(\Omega)}.$$

Proof. As in (3.6) we have for $v_h \in H^1(\Omega)$

$$\int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot\operatorname{Curl}\beta\, dx$$
$$\lesssim \inf_{v\in H_D^1(\Omega)} \|\nabla(u - v_h - v)\|_2 \|\operatorname{Curl}\beta\|_2 + \|\nabla_{\mathcal{T}}(v_h - U)\|_2 \|\operatorname{Curl}\beta\|_2.$$

Each $v_h \in \mathcal{S}(\mathcal{T}, u_D)$ interpolates u_D in nodes on Γ_D (so that we can estimate the infimum as in the proof of Lemma 3.4). An elementwise inverse estimate proves the lemma. $\square$

Proof of Theorem 4.1. Using Lemma 4.1 and 4.2 and (3.4) for

$$\eta_1^2 := \min_{q_h\in\mathcal{S}(\mathcal{T},g)} \|\nabla_{\mathcal{T}}U - q_h\|_2^2 + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}} g/\partial s\|_{2,\Gamma_N}^2 + \sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f - f_z\|_{2,\Omega_z}^2,$$
$$\eta_2^2 := \min_{v_h\in\mathcal{S}(\mathcal{T},u_D)} \|h_{\mathcal{T}}^{-1}(U - v_h)\|_2^2 + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}}^2 u_D/\partial s^2\|_{2,\Gamma_D}^2,$$

we proceed as in the proof of Theorem 3.1. $\square$

We now show that the second term in the right-hand side of the inequality in Theorem 4.1 dominates the first one if $\mathcal{S}(\mathcal{T}, g)$ is modified appropriately, $\int_E [U]\, ds = 0$ is satisfied for all $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$, and some conditions on Γ, u_D, and g are satisfied.

Assume that $g|_E \in H^1(E)$ for all $E \in \mathcal{E}_N$ and that, for each node $z \in \mathcal{N}\cap\overline{\Gamma}_N$ where the outer unit normal n is continuous g is continuous. We also suppose that $u_D \in H^1(\Gamma_D)\cap C(\Gamma_D)$, $\partial u_D|_E/\partial s \in H^1(E)$ for all $E \in \mathcal{E}_D$ and that for each node $z \in \mathcal{N}\cap\Gamma_D$ where the outer unit normal n is continuous $\partial u_D/\partial s$ is continuous. Moreover, we assume that for each $z \in \mathcal{N}\cap\Gamma$ the system of linear equations

$$\text{(4.1)}\qquad \begin{cases} \forall E\in\mathcal{E}_N \text{ with } z\in E \text{ there holds } Ax\cdot n = g|_E(z), \\ \forall E\in\mathcal{E}_D \text{ with } z\in E \text{ there holds } \gamma_{t_E}(x) = \partial u_D|_E/\partial s(z), \end{cases}$$

admits at least one solution x. Then, the space

$$\mathcal{S}(\mathcal{T}, g, u_D) := \{q_h \in C(\overline{\Omega})^d : \forall T\in\mathcal{T}, q_h|_T \in \mathcal{P}_1(T)^d$$
$$\text{and } \forall z\in\mathcal{N}\cap\overline{\Gamma}_N, \forall E\in\mathcal{E}_N, z\in E, Aq_h\cdot n|_E(z) = g|_E(z),$$
$$\text{and } \forall z\in\mathcal{N}\cap\Gamma_D, \forall E\in\mathcal{E}_D, z\in E, \gamma_{t_E}(q_h)|_E(z) = \partial u_D|_E/\partial s(z)\}$$

is well-defined and non-void.

Remark 4.4. The above assumption is fulfilled if for each $z \in \mathcal{N}\cap\Gamma$ at most d distinct conditions are imposed in (4.1). Note that g and u_D might have to satisfy certain compatibility conditions in nodes $z \in \Gamma_D \cap \overline{\Gamma}_N$ to ensure that (4.1) is well-posed [CB]. This is necessary when, e.g., for all $x \in \mathbb{R}^d$ we have $Ax\cdot n_{E_1} = \gamma_{t_{E_2}}(x)$ for two edges $E_1 \in \mathcal{E}_N$ and $E_2 \in \mathcal{E}_D$ that share a node.

Theorem 4.3. *Let $u \in H^1(\Omega)$ and $U \in H^2(\mathcal{T})$ satisfy (1.1)–(1.3) and (2.3), respectively, and suppose $\Gamma_D \subseteq \Gamma_0$, $\int_E [U]\, ds = 0$ for $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$, and $\operatorname{div}_\mathcal{T} A \nabla_\mathcal{T} U \equiv 0$. Then, there exists an $(h_\mathcal{T}, h_\mathcal{E})$-independent positive constant c_2 such that*

$$\|\nabla_\mathcal{T}(u-U)\|^2_{L^2(\Omega)} \le c_2 \Big(\min_{q_h \in \mathcal{S}(\mathcal{T},g,u_D)} \|\nabla_\mathcal{T} U - q_h\|^2_{L^2(\Omega)} + \sum_{z\in\mathcal{K}} h_z^2 \min_{f_z \in \mathbb{R}} \|f - f_z\|^2_{L^2(\Omega_z)} + \|h_\mathcal{E}^{3/2} \partial^2_\mathcal{E} u_D/\partial s^2\|^2_{L^2(\Gamma_D)} + \|h_\mathcal{E}^{3/2} \partial_\mathcal{E} g/\partial s\|^2_{L^2(\Gamma_N)} \Big).$$

Remark 4.5. Note that any choice of $q_h^* \in \mathcal{S}(\mathcal{T}, g, u_D)$ yields a reliable error estimate. An operator $\mathcal{A} : \nabla_\mathcal{T}\mathcal{S} \to \mathcal{S}(\mathcal{T}, g, u_D)$ specified for $d = 2$ in [CB] yielded an error estimate $\|\nabla_\mathcal{T} U - \mathcal{A}\nabla_\mathcal{T} U\|_{L^2(\Omega)}$ which performed well in numerical experiments reported therein.

Remark 4.6. As in the previous estimates, the term including $f \in H^1(\Omega)$ is of higher order.

Two efficiency estimates with complementary properties are presented in the next theorem on the analogy of Theorem 4.2.

Theorem 4.4. *Let $u \in H^1(\Omega)$ and $U \in H^2(\mathcal{T})$ satisfy (1.1)–(1.3) and (2.3), respectively. Then, there holds*

$$\min_{q_h \in \mathcal{S}(\mathcal{T},g,u_D)} \|\nabla_\mathcal{T} U - q_h\|_{L^2(\Omega)} \le \|\nabla_\mathcal{T}(u-U)\|_{L^2(\Omega)} + \min_{q_h \in \mathcal{S}(\mathcal{T},g,u_D)} \|\nabla u - q_h\|_{L^2(\Omega)}.$$

Moreover, there exists a constant $c_{19} > 0$ such that

$$\min_{q_h \in \mathcal{S}(\mathcal{T},g,u_D)} \|\nabla_\mathcal{T} U - q_h\|_{L^2(\Omega)} \le c_{19} \Big(\|\nabla u - \nabla_\mathcal{T} U\|_{L^2(\Omega)} + \big(\sum_{T\in\mathcal{T}_T} h_T^2 \inf_{f_T \in \mathcal{P}_1(T)} \|f - f_T\|^2_{L^2(T)} \big)^{1/2} + \|h_\mathcal{E}^{3/2} \partial^2_\mathcal{E} u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|h_\mathcal{E}^{3/2} \partial_\mathcal{E} g/\partial s\|_{L^2(\Gamma_N)} \Big). \quad \square$$

Proof of Theorem 4.3. For each $E \in \mathcal{E}$ let $\mathcal{S}_E := \mathcal{S}(\mathcal{T}, g, u_D)|_{\omega_E}$ and define

$$[\nabla_\mathcal{T} U]|_E := \begin{cases} (\nabla U|_{T_1} - \nabla U|_{T_2})|_E & \text{if } E = T_1 \cap T_2 \in \mathcal{E}_\Omega, \\ [\gamma_{t_E}(\nabla_\mathcal{T} U)] & \text{if } E \in \mathcal{E}_D, \\ [A\nabla_\mathcal{T} U \cdot n_E] & \text{if } E \in \mathcal{E}_N. \end{cases}$$

Note that $[\nabla_\mathcal{T} U] = 0$ if $\nabla_\mathcal{T} U|_{\omega_E} \in \mathcal{S}_E$ so that a compactness and a scaling argument show, for each $E \in \mathcal{E}_\Omega$,

$$h_E^{1/2} \|[\nabla_\mathcal{T} U]\|_{2,E} \lesssim \min_{q_E \in \mathcal{S}_E} \|\nabla_\mathcal{T} U - q_E\|_{2,\omega_E}.$$

For $E \in \mathcal{E}_N$ and $E \subseteq \partial T \cap \overline{\Gamma}_N$ we insert the nodal interpolant g_h of g, note that for each $q_E \in \mathcal{S}_E$ we have $q_E \cdot n = g_h$ on E, and obtain by equivalence of semi-norms on a finite dimensional space, a scaling argument, and an interpolation estimate

$$\begin{aligned} h_E^{1/2} \|[\nabla_\mathcal{T} U]\|_{2,E} &\le h_E^{1/2} \big(\|A\nabla U|_T \cdot n - g_h\|_{2,E} + \|g_h - g\|_{2,E} \big) \\ &\lesssim \min_{q_E \in \mathcal{S}_E} \|\nabla_\mathcal{T} U - q_E\|_{2,\omega_E} + h_E^{3/2} \|\partial g/\partial s\|_{2,E}. \end{aligned}$$

For $E \in \mathcal{E}_D$ and $E \subseteq \partial T \cap \Gamma_D$ we denote by $u'_{h,D}$ the nodal interpolant of $\partial u_D|_E/\partial s$. For each $q_E \in \mathcal{S}_E$, we have $\gamma_{t_E}(q_E) = u'_{h,D}$ on E. By equivalence of semi-norms on a finite dimensional space, a scaling argument, and an interpolation estimate we obtain

$$\begin{aligned} h_E^{1/2} \|[\nabla_\mathcal{T} U]\|_{2,E} &\le h_E^{1/2} \big(\|\gamma_{t_E}(\nabla U|_T) - u'_{h,D}\|_{2,E} + \|u'_{h,D} - \partial u_D/\partial s\|_{2,E} \big) \\ &\lesssim \min_{q_E \in \mathcal{S}_E} \|\nabla_\mathcal{T} U - q_E\|_{2,\omega_E} + h_E^{3/2} \|\partial^2 u_D/\partial s^2\|_{2,E}. \end{aligned}$$

Local averaging is bounded by global averaging, i.e.,

$$\sum_{E\in\mathcal{E}} \min_{q_E\in\mathcal{S}_E} \|\nabla_{\mathcal{T}} U - q_E\|_{2,\omega_E}^2 \lesssim \min_{q_h\in\mathcal{S}(\mathcal{T},g,u_D)} \|\nabla_{\mathcal{T}} U - q_h\|_2^2,$$

and for each $E \in \mathcal{E}_\Omega$ we have by orthogonality of the decomposition into tangential and normal components

$$\|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|_{2,E}^2 + \|[A\nabla_{\mathcal{T}} U\cdot n_E]\|_{2,E}^2 \lesssim \|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|_{2,E}^2 + \|[\nabla_{\mathcal{T}} U\cdot n_E]\|_{2,E}^2 = \|[\nabla_{\mathcal{T}} U]\|_{2,E}^2.$$

A combination of the above estimates yields

$$\sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} h_E\|[A\nabla_{\mathcal{T}} U\cdot n_E]\|_{2,E}^2 + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D} h_E\|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|_{2,E}^2$$
$$\lesssim \min_{q_h\in\mathcal{S}(\mathcal{T},g,u_D)} \|\nabla_{\mathcal{T}} U - q_h\|_2^2 + \sum_{E\in\mathcal{E}_N} h_E^{3/2}\|\partial g/\partial s\|_{2,E}^2 + \sum_{E\in\mathcal{E}_D} h_E^{3/2}\|\partial^2 u_D/\partial s^2\|_{2,E}^2.$$

Using this to bound the right-hand side of the inequality in Theorem 3.1 proves the assertion. □

5. A Reliable L^2-A Posteriori Error Estimate

For the L^2-estimates we assume H^2-regularity of problem (1.1)–(1.3) with $u_D = 0$ and $g = 0$, i.e., we assume the existence of a constant $c_{20} > 0$ such that, for all $f \in L^2(\Omega)$ and corresponding solution $u \in H_D^1(\Omega)$, we have $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \le c_{20}\|f\|_{L^2(\Omega)}. \tag{5.1}$$

Sufficient for this is, e.g., $\Gamma_N = \emptyset$, A is Lipschitz, and Ω is convex [G].

Theorem 5.1. *Let $u \in H^1(\Omega)$ and $U \in H^2(\mathcal{T})$ satisfy* (1.1)–(1.3) *and* (2.3), *respectively. Suppose* (1.1)–(1.3) *with $u_D = 0$ and $g = 0$ is H^2-regular, $\int_E [U]\,ds = 0$ for all $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$, and $A \in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d}_{sym})$. Then, there exists an $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent positive constant c_{21} such that*

$$\|u-U\|_{L^2(\Omega)} \le c_{21}\Big(\sum_{T\in\mathcal{T}} h_T^4\|f + \mathrm{div}_{\mathcal{T}} A\nabla_{\mathcal{T}} U\|_{L^2(T)}^2$$
$$+ \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} h_E^3\|[A\nabla_{\mathcal{T}} U\cdot n_E]\|_{L^2(E)}^2 + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D} h_E^3\|[\gamma_{t_E}(\nabla_{\mathcal{T}} U)]\|_{L^2(E)}^2\Big)^{1/2}.$$

Proof. Let $\eta \in H^2(\Omega)\cap H_D^1(\Omega)$ satisfy $\mathrm{div}\, A\nabla\eta = -(u-U)$ in Ω and $(A\nabla\eta)\cdot n = 0$ on Γ_N. Let $\eta_h \in \mathcal{S}_D$ be the nodal interpolant of η. We deduce from integration by parts, Galerkin orthogonality (2.5), and the assumption on $[U]$, for $c_E \in \mathbb{R}^d$ and $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$,

$$\int_\Omega (u-U)^2\,dx = -\int_\Omega (u-U)\,\mathrm{div}\,A\nabla\eta\,dx = \int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot A\nabla\eta\,dx - \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D}\int_E [U]A\nabla\eta\cdot n_E\,ds$$
$$= \int_\Omega \nabla_{\mathcal{T}}(u-U)\cdot A\nabla(\eta-\eta_h)\,dx - \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D}\int_E [U]A\nabla\eta\cdot n_E\,ds$$
$$= \sum_{T\in\mathcal{T}}\int_T (f + \mathrm{div}_{\mathcal{T}} A\nabla_{\mathcal{T}} U)(\eta-\eta_h)\,dx + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N}\int_E [A\nabla_{\mathcal{T}} U\cdot n_E](\eta-\eta_h)\,dx$$
$$- \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D}\int_E [U](A\nabla\eta - c_E)\cdot n_E\,ds.$$

For each $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$ take $T_E \in \mathcal{T}$ with $E \subset \partial T$ and verify with a trace inequality [BS, CF]

$$\|(A\nabla\eta - c_E)\cdot n_E\|_{2,E}^2 = \|A\nabla\eta - c_E\|_{2,E}^2 \lesssim h_E^{-1}\|A\nabla\eta - c_E\|_{2,T_E}^2 + h_E\|D(A\nabla\eta)\|_{2,T_E}^2.$$

Choosing c_E as the integral mean of $A\nabla\eta$ over T_E we have $\|A\nabla\eta - c_E\|_{2,T_E} \lesssim h_E\|D(A\nabla\eta)\|_{2,T_E}$ and thus

$$\int_E [U](A\nabla\eta - c_E)\cdot n_E\,ds \lesssim h_E^{1/2}\|[U]\|_{2,E}\|D(A\nabla\eta)\|_{2,T_E} \lesssim h_E^{1/2}\|[U]\|_{2,E}\Big(\|\nabla\eta\|_{2,T_E} + \|D^2\eta\|_{2,T_E}\Big).$$

With Cauchy's and trace inequalities as well as interpolation and the above estimates,

$$\|u-U\|_2^2 \lesssim c_{20}\,||u-U||_2\Big(\sum_{T\in\mathcal{T}} h_T^4\|f + \mathrm{div}_{\mathcal{T}}A\nabla_{\mathcal{T}}U\|_{2,T}^2 + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} h_E^3\|[A\nabla_{\mathcal{T}}U\cdot n_E]\|_{2,E}^2 + \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_D} h_E\|[U]\|_{2,E}^2\Big)^{1/2}. \tag{5.2}$$

An edgewise Poincaré inequality shows, for each $E\in\mathcal{E}_\Omega\cup\mathcal{E}_D$,

$$\|[U]\|_{2,E} \lesssim h_E\|[\gamma_{t_E}(\nabla_{\mathcal{T}}U)]\|_{2,E}. \quad \square \tag{5.3}$$

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