EACH AVERAGING TECHNIQUE YIELDS RELIABLE A POSTERIORI ERROR CONTROL IN FEM ON UNSTRUCTURED GRIDS PART I: LOW ORDER CONFORMING, NONCONFORMING, AND MIXED FEM

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ABSTRACT. Averaging techniques are popular tools in adaptive finite element methods for the numerical treatment of second order partial differential equations since they provide efficient a posteriori error estimates by a simple postprocessing. In this paper, their reliablility is shown for conforming, nonconforming, and mixed low order finite element methods in a model situation, the Laplace equation with mixed boundary conditions. Emphasis is on possibly unstructured grids, non-smoothness of exact solutions, and a wide class of averaging techniques. Theoretical and numerical evidence supports that the reliability is up to the smoothness of given right-hand sides.

1. INTRODUCTION

Error control and efficient mesh-design in finite element simulations of computational engineering and scientific computing finite element simulations is frequently based on a posteriori error estimates. One of the more popular techniques is local or global averaging, e.g., in form of the ZZ-error indicator [ZZ]. Efficiency and reliability of this estimator were known only for very structured grids and for solutions of higher regularity and then we have even asymptotic exactness [V]. Numerical experiments in [Baetal] showed that averaging techniques were quite more reliable on irregular meshes than expected. For homogeneous Dirichlet conditions and conforming finite element methods, the reliability and efficiency of the ZZ-estimator is proven on unstructured, merely shape-regular grids [R2].

This work is devoted to give theoretical and numerical support for the robust reliability of *all* averaging techniques, robust with respect to violated (local) symmetry of meshes and super-convergence and robust with respect to *other boundary conditions* or *other finite element methods.*

For a more precise description of averaging techniques, let us discuss a discretisation of a conservation equation

$$f + \operatorname{div} p = 0 \tag{1.1}$$

with a given right-hand side $f \in L^2(\Omega)$ and a known approximation $p_h \in L^2(\Omega)^d$ to the unknown exact flux $p \in H(\operatorname{div};\Omega)$ in the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. The test function finite element space should include the continuous piecewise linears $\mathcal{S}^1_D(\mathcal{T})$ (with homogeneous Dirichlet boundary conditions) based on a regular triangulation $\mathcal{T}$ of Ω. Suppose a

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Galerkin property for $p - p_h$ with $\mathcal{S}^1_D(\mathcal{T})$, i.e.,

$$\int_\Omega p_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \qquad \text{for all } v_h \in \mathcal{S}^1_D(\mathcal{T}). \tag{1.2}$$

What can be said about the error $\|p - p_h\|_{L^2(\Omega)}$ when we regard p as an unknown and p_h as a known variable?

In averaging techniques, the error estimator is based on a smoother approximation, e.g., in $\mathcal{S}^1(\mathcal{T})^d$, the continuous $\mathcal{T}$-piecewise linears, to the (components of the) discrete solution p_h. For instance,

$$\eta_Z := \min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|p_h - q_h\|_{L^2(\Omega)} \tag{1.3}$$

may serve as a computable error estimator and the elementwise contributions as local error indicators in an adaptive mesh-refining algorithm.

The triangle inequality shows that η_Z is efficient up to higher order terms of the exact solution p, indeed,

$$\eta_Z \le \|p - p_h\|_{L^2(\Omega)} + \min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|p - q_h\|_{L^2(\Omega)} \tag{1.4}$$

and the last term converges as $O(h^2)$ (provided p is smooth enough and h denotes the maximal mesh–size in $\mathcal{T}$) and so, generically, of higher order than the error $\|p - p_h\|_{L^2(\Omega)} = O(h)$ in the lowest order finite element method.

In practise, we may apply an averaging operator $\mathcal{A} : L^2(\Omega)^d \to \mathcal{S}^1(\mathcal{T})^d$ to p_h and compute the upper bound $\|p_h - \mathcal{A}p_h\|_{L^2(\Omega)}$ of η_Z. Then, efficiency depends strongly on the approximation properties of $\mathcal{A}$ and deserved further investigations.

In this paper, the focus is on the reliability of η_Z, i.e., we investigate under which conditions an estimate

$$\|p - p_h\|_{L^2(\Omega)} \le c_1 \, \eta_Z + \text{ h.o.t.} \le c_1 \, \|p_h - \mathcal{A}p_h\|_{L^2(\Omega)} + \text{ h.o.t.} \tag{1.5}$$

holds, we study what the constant $c_1 > 0$ depends on, what affects the higher-order contributions "h.o.t.", how to modify the definition of η_Z in presence of mixed boundary conditions, and how to modify the general setting presented for nonconforming and mixed lowest order finite element methods.

Recall from (1.5) that *any* averaging technique, described by $\mathcal{A}$, then is reliable up to higher order terms. We also prove equivalence to local modifications of η_Z where the minimisation is over smaller domains, e.g., patches of nodes or edges.

The outline of the paper is as follows. Preliminaries and notation is introduced in Section 2 where we state and prove stability and first order estimates for a certain approximation operator $\mathcal{J} : H^1_D(\Omega) \to \mathcal{S}^1_D(\mathcal{T})$ essentially designed to yield further local orthogonality properties as in [CV, C2]. Basic estimates are provided in Section 3 for a local and global averaging technique and their equivalence. The subsequent Sections 4–6 display the consequences to averaging techniques in a posteriori error control for first order conforming, nonconforming and mixed finite element schemes. Numerical evidence, reported in Section 7, supports the theoretical results for adaptively refined and even perturbed meshes. Although asymptotic exactness is not claimed in this paper, our numerical experiments illustrate that η_Z is a very good approximate to $\|p - p_h\|_{L^2(\Omega)}$ even on perturbed grids.

The proofs are given for a simple elliptic model example with mixed boundary conditions for conforming, nonconforming, and mixed finite elements in two dimensions for notational simplicity. More interesting examples such as higher-order schemes, the application to the Stokes problem or the Navier-Lamé equations without incompressibility-locking will appear elsewhere [CB, CF2, CF3].

2. Approximation in finite element spaces

The Lipschitz boundary $? = \partial\Omega$ of the bounded domain Ω is split into a closed Dirichlet part $?_D$ with positive surface measure and a remaining, relatively open and possibly empty, Neumann part $?_N := ? \setminus ?_D$. Suppose $\mathcal{T}$ be a regular triangulation of the domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, in the sense of Ciarlet [BS, Ci] (no hanging node, domain is matched exactly) with piecewise affine Lipschitz boundary $? = \partial\Omega = ?_D \cup ?_N$, i.e., $\mathcal{T}$ consists of a finite number of closed subsets of $\overline{\Omega}$, that cover $\overline{\Omega} = \cup\mathcal{T}$. Each element $T \in \mathcal{T}$ is either an interval $T = \mathrm{conv}\{a, b\}$ if $d = 1$, a triangle $T = \mathrm{conv}\{a, b, c\}$ or a parallelogram $T = \mathrm{conv}\{a, b, c, d\}$. The extremal points a, b, c are called vertices, the faces $E \subseteq \partial T$, e.g. $E = \mathrm{conv}\{a, b\}$, are called edges. The set of all vertices and all edges appearing for some T in $\mathcal{T}$ are denoted as $\mathcal{N}$ and $\mathcal{E}$. Two distinct and intersecting T_1 and T_2 share either an entire edge or a vertex. Each edge $E \in \mathcal{E}$ on the boundary $?$ belongs either to $?_D$, written $E \in \mathcal{E}_D$, or to $\overline{?}_N$, written $E \in \mathcal{E}_N$. Therefore the set of edges is partitioned into $\mathcal{E}_\Omega := \{E \in \mathcal{E} : E \not\subset ?\}$, $\mathcal{E}_D$, and $\mathcal{E}_N$. We stress that $\cup\mathcal{E}$, the union of all egdes, denotes the skeleton of egdes in $\mathcal{T}$, i.e., the set of all points x that belong to some boundary $x \in \partial T$ of some element $T \in \mathcal{T}$. Finally, $\mathcal{K} := \mathcal{N} \setminus ?_D$ denotes the set of free nodes.

For $T \in \mathcal{T}$, let $P_T^k := \mathcal{P}_k(T)$ if T is a triangle or $P_T^k := \mathcal{Q}_k(T)$ if T is a parallelogram. Here, $\mathcal{P}_k(K)$ resp. $\mathcal{Q}_k(K)$ denotes the set of algebraic polynomials in d variables on K of total resp. partial degree $\leq k$. The space $\mathcal{L}^k(\mathcal{T})$ of (possibly discontinuous) $\mathcal{T}$–piecewise polynomials of degree $\leq k$ is the set of all $U \in L^\infty(\Omega)$ with $U|_T \in P_T^k$ for all T in $\mathcal{T}$. Set

$$\mathcal{S}^k(\mathcal{T}) := \mathcal{L}^k(\mathcal{T}) \cap C(\Omega) \quad \text{and} \quad \mathcal{S}^1_D(\mathcal{T}) := \{u_h \in \mathcal{S}^1(\mathcal{T}) : u|_{\Gamma_D} = 0\}.$$

Let $(\varphi_z | z \in \mathcal{N})$ denote the nodal basis of $\mathcal{S}^1(\mathcal{T})$, i.e., $\varphi_z \in \mathcal{S}^1(\mathcal{T})$ satisfies $\varphi_z(x) = 0$ if $x \in \mathcal{N} \setminus \{z\}$ and $\varphi_z(z) = 1$. Note that $(\varphi_z | z \in \mathcal{N})$ is a partition of unity and the open patches

$$\omega_z := \{x \in \Omega : 0 < \varphi_z(x)\} \tag{2.1}$$

form an open cover $(\omega_z : z \in \mathcal{N})$ of Ω with finite overlap.

In order to define a weak interpolation operator $\mathcal{J} : H^1_D(\Omega) \to \mathcal{S}^1_D(\mathcal{T})$ we modify $(\varphi_z | z \in \mathcal{K})$ to a partition of unity $(\psi_z | z \in \mathcal{K})$. For each fixed node $z \in \mathcal{N} \setminus \mathcal{K}$, we choose a node $\zeta(z) \in \mathcal{K}$ and let $\zeta(z) := z$ if $z \in \mathcal{K}$. In this way, we define a partition of $\mathcal{N}$ into card $(\mathcal{K})$ classes $I(z) := \{\tilde{z} \in \mathcal{N} : \zeta(\tilde{z}) = z\}$, $z \in \mathcal{K}$. For each $z \in \mathcal{K}$ set

$$\psi_z := \sum_{\zeta \in I(z)} \varphi_\zeta \tag{2.2}$$

and notice that $(\psi_z | z \in \mathcal{K})$ is a partition of unity. It is required that

$$\Omega_z := \{x \in \Omega : 0 < \psi_z(x)\} \tag{2.3}$$

is connected and that $\psi_z \neq \varphi_z$ implies that $?_D \cap \partial\Omega_z$ has a positive surface measure.

For $g \in L^1(\Omega)$ and $z \in \mathcal{K}$ let $g_z \in \mathbb{R}$ be

$$g_z := \frac{\int_{\Omega_z} g\psi_z \, dx}{\int_{\Omega_z} \varphi_z \, dx} \tag{2.4}$$

and then define

$$\mathcal{J}g := \sum_{z\in\mathcal{K}} g_z\varphi_z \in \mathcal{S}^1_D(\mathcal{T}). \tag{2.5}$$

The local mesh-sizes are denoted by $h_\mathcal{T}$ and $h_\mathcal{E}$ where $h_\mathcal{T} \in \mathcal{L}^0(\mathcal{T})$ denotes the element-size, $h_\mathcal{T}|_T := h_T := \operatorname{diam}(T)$ for $T \in \mathcal{T}$, and the edge-size $h_\mathcal{E} \in L^\infty(\cup\mathcal{E})$ is defined on the union or skeleton $\cup\mathcal{E}$ of all edges E in $\mathcal{E}$ by $h_\mathcal{E}|_E := h_E := \operatorname{diam}(E)$. The patch-size $h_z := \operatorname{diam}(\Omega_z)$ is defined for each node $z \in \mathcal{K}$ separately.

Theorem 2.1. *There exist $(h_\mathcal{T}, h_\mathcal{E})$-independent constants $c_2, c_3, c_4, c_5 > 0$ such that for all $g \in H^1_D(\Omega)$ and $f \in L^2(\Omega)$ there holds*

$$\|\nabla\mathcal{J}g - \nabla g\|_{L^2(\Omega)} \leq c_2\|\nabla g\|_{L^2(\Omega)}, \tag{2.6}$$

$$\int_\Omega f(g - \mathcal{J}g)\,dx \leq c_3\|\nabla g\|_{L^2(\Omega)}\Big(\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|f - f_z\|^2_{L^2(\Omega_z)}\Big)^{1/2}, \tag{2.7}$$

$$\|h_\mathcal{T}^{-1}(g - \mathcal{J}g)\|_{L^2(\Omega)} \leq c_4\|\nabla g\|_{L^2(\Omega)}, \tag{2.8}$$

$$\|h_\mathcal{E}^{-1/2}(g - \mathcal{J}g)\|_{L^2(\Gamma_N)} \leq c_5\|\nabla g\|_{L^2(\Omega)}. \tag{2.9}$$

The constants c_2, c_3, c_4, c_5 only depend on Ω, $?_D$, $?_N$ and the shape of the elements and patches (not on their sizes).

Remark 2.1. The assertion of the theorem holds verbatim for three space dimensions where $\mathcal{T}$ consists of tetrahedra or parallelepipeds with the same proof.

Proof. In this proof and at similar occasions, $\lesssim$ abbreviates an inequality $\leq$ up to a constant $(h_\mathcal{T}, h_\mathcal{E})$-independent factor. Also, $\|\cdot\|_{p,K}$ abbreviates $\|\cdot\|_{L^p(K)}$ and we neglect K if Ω is meant, i.e., $\|\cdot\|_2 := \|\cdot\|_{2,\Omega}$. Hence, e.g., (2.6) could be phrased as $\|\nabla\mathcal{J}g - \nabla g\|_2 \lesssim \|\nabla g\|_2$.

The key estimate for the stability and the approximation property of $\mathcal{J}$ will be

$$\|g_z\varphi_z - g\psi_z\|_{2,\Omega_z} \lesssim h_z\|\nabla g\|_{2,\Omega_z} \qquad (z \in \mathcal{K}). \tag{2.10}$$

For the proof of (2.10), let $\overline{g}_z$ denote the integral mean of g on Ω_z. Then, using the definition (2.4) for the coefficients g_z, Cauchy's and Young's inequality, we infer, with $c_6 := \|1\|_{2,\Omega_z}/\|\varphi_z^{1/2}\|_{2,\Omega_z}$,

$$\begin{aligned}\|\varphi_z^{1/2}(g_z - \overline{g}_z)\|^2_{2,\Omega_z} &= \int_{\Omega_z} \varphi_z(\overline{g}_z - g)(\overline{g}_z - g_z)\,dx + \int_{\Omega_z} (\psi_z - \varphi_z)g(g_z - \overline{g}_z)\,dx \\ &\leq \frac{1}{4}\|\varphi_z^{1/2}(g_z - \overline{g}_z)\|^2_{2,\Omega_z} + \|g - \overline{g}_z\|^2_{2,\Omega_z} + \frac{1}{4c_6^2}\|g_z - \overline{g}_z\|^2_{2,\Omega_z} + c_6^2\|(\psi_z - \varphi_z)g\|^2_{2,\Omega_z}.\end{aligned} \tag{2.11}$$

Absorbing $\frac{1}{4c_6^2}\|g_z - \overline{g}_z\|^2_{2,\Omega_z} \leq \frac{1}{4}\|\varphi_z^{1/2}(g_z - \overline{g}_z)\|^2_{2,\Omega_z}$ we deduce

$$\|\varphi_z^{1/2}(g_z - \overline{g}_z)\|^2_{2,\Omega_z} \lesssim \|g - \overline{g}_z\|^2_{2,\Omega_z} + \|(\psi_z - \varphi_z)g\|^2_{2,\Omega_z}. \tag{2.12}$$

A Poincaré inequality yields

$$\|g - \overline{g}_z\|_{2,\Omega_z} \lesssim h_z\|\nabla g\|_{2,\Omega_z}.$$

Note that $(\psi_z - \varphi_z)g$ is non-zero only if $?_D \cap (\partial\Omega_z)$ has positive surface measure. Since g vanishes there, Friedrichs' inequality shows

$$\|g\|_{2,\Omega_z} \lesssim h_z \|\nabla g\|_{2,\Omega_z}.$$

Therefore, (2.12) results in

$$\|\varphi_z^{1/2}(g_z - \overline{g}_z)\|_{2,\Omega_z} \lesssim h_z \|\nabla g\|_{2,\Omega_z}. \tag{2.13}$$

To prove (2.10), we use the triangle inequality, (2.13), and again Cauchy's and Friedrichs' inequality to verify

$$\begin{aligned}\|g_z\varphi_z - g\psi_z\|_{2,\Omega_z} &\le \|(g_z - \overline{g}_z)\varphi_z\|_{2,\Omega_z} + \|(g - \overline{g}_z)\varphi_z\|_{2,\Omega_z} \\ &\quad + \|(\psi_z - \varphi_z)g\|_{2,\Omega_z} \lesssim h_z\|\nabla g\|_{2,\Omega_z}.\end{aligned} \tag{2.14}$$

To prove (2.7), we use that $(\psi_z | z \in \mathcal{K})$ is a partition of unity and obtain with (2.10), (2.4) for any $f_z \in \mathbb{R}$ that

$$\begin{aligned}\int_\Omega f(g - \mathcal{J}g)\,dx &= \sum_{z\in\mathcal{K}} \int_{\Omega_z} f(g\psi_z - g_z\varphi_z)\,dx = \sum_{z\in\mathcal{K}} \int_{\Omega_z} (f - f_z)(g\psi_z - g_z\varphi_z)\,dx \\ &\lesssim \sum_{z\in\mathcal{K}} \|f - f_z\|_{2,\Omega_z} h_z \|\nabla g\|_{2,\Omega_z} \lesssim \Big(\sum_{z\in\mathcal{K}} h_z^2 \|f - f_z\|_{2,\Omega_z}^2\Big)^{1/2} \|\nabla g\|_{2,\Omega}.\end{aligned} \tag{2.15}$$

In the last step we used that $(\varphi_z | z \in \mathcal{K})$ has a finite overlap that depends on the shape of the elements only. The proof of (2.7) is thus finished.

Notice that $h_z \lesssim h_T$ for all $z \in \mathcal{K}$ and $T \in \mathcal{T}$ with $T \subset \Omega_z$. Letting $f := h_{\mathcal{T}}^{-2}(g - \mathcal{J}g)$ and $f_z = 0$, $z \in \mathcal{K}$, we deduce from (2.7) that

$$\begin{aligned}\|h_{\mathcal{T}}^{-1}(g - \mathcal{J}g)\|_2^2 &\lesssim \|\nabla g\|_2 \Big(\sum_{z\in\mathcal{K}} \|h_{\mathcal{T}}^{-1}(g - \mathcal{J}g)\|_{2,\Omega_z}^2\Big)^{1/2} \\ &\lesssim \|\nabla g\|_2 \|h_{\mathcal{T}}^{-1}(g - \mathcal{J}g)\|_2,\end{aligned} \tag{2.16}$$

which implies (2.8). To verify (2.6) we argue as above and start utilising $\sum_{z\in\mathcal{K}} \psi_z = 1$ and $\sum_{z\in\mathcal{K}} \nabla\psi_z = 0$. Repeating the triangle inequality several times we deduce

$$\|\nabla g - \nabla \mathcal{J}g\|_2^2 \lesssim \sum_{z\in\mathcal{K}} \|\nabla(\psi_z g - \varphi_z g_z)\|_2^2. \tag{2.17}$$

Employing Friedrichs' and Poincaré's inequality and utilising $\|\nabla\varphi_\zeta\|_{\infty,\Omega_z} \lesssim 1/h_z$, we infer

$$\begin{aligned}\|\nabla(g\psi_z - g_z\varphi_z)\|_{2,\Omega_z} &\le \|(\psi_z - \varphi_z)\nabla g\|_{2,\Omega_z} + \|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z} \\ &\quad + \|\nabla(\varphi_z(\overline{g}_z - g))\|_{2,\Omega_z} + \|g\nabla(\psi_z - \varphi_z)\|_{2,\Omega_z} \lesssim \|\nabla g\|_{2,\Omega_z} + \|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z}.\end{aligned} \tag{2.18}$$

To estimate $\|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z}$ we utilise (2.10), Friedrichs' inequality and argue as above to find that

$$\begin{aligned}h_z\|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z} &\lesssim \|\varphi_z(g_z - \overline{g}_z)\|_{2,\Omega_z} \\ &\le \|\varphi_z g_z - \psi_z g\|_{2,\Omega_z} + \|(\psi_z - \varphi_z)g\|_{2,\Omega_z} + \|\varphi_z(g - \overline{g}_z)\|_{2,\Omega_z} \lesssim h_z\|\nabla g\|_{2,\Omega_z}.\end{aligned}$$

Employing this estimate in (2.18) and the resulting estimate in (2.17) we eventually verify (2.6).

A trace inequality [BS, Cl, CF1] is required for the proof of (2.9). For $E \in \mathcal{E}_N$ and a neighbouring element $T \in \mathcal{T}$ with $E \subset \partial T \cap \overline{?}_N$ we have, for all $w \in H^1(T)$,

$$\|w\|_{2,E} \lesssim h_E^{-1/2}\|w\|_{2,T} + h_E^{1/2}\|\nabla w\|_{2,T}. \tag{2.19}$$

We denote $\omega_E := T$ for $E \in \mathcal{E}_N \cup \mathcal{E}_D$ and $E \subseteq \partial T$ while for $E \in \mathcal{E}_\Omega$ we let $\omega_E := T_1 \cup T_2$ for $E = T_1 \cap T_2$. Since $h_T \lesssim h_E$ we deduce from (2.19) with $w = g - \mathcal{J}g$ by summing all edges E on $?_N$ that

$$\sum_{E\in\mathcal{E}_N} h_E^{-1}\|g-\mathcal{J}g\|_{2,E}^2 \lesssim \sum_{E\in\mathcal{E}_N} h_T^{-2}\|g-\mathcal{J}g\|_{2,\omega_E}^2 + \sum_{E\in\mathcal{E}_N} \|\nabla(g-\mathcal{J}g)\|_{2,\omega_E}^2 \lesssim \|\nabla g\|_{2,\Omega}^2$$

according to (2.6) and (2.8). The proof is finished. □

3. Basic estimates

In this section we first derive with the approximation operator $\mathcal{J}$ a global error estimate for a posteriori error control by averaging processes in an abstract setting. We then show the equivalence of local and global averaging techniques. The estimates of this section are then specified, and thereby proved to be substantial, in the subsequent sections to conforming, nonconforming, and mixed finite element methods.

Theorem 3.1. *Suppose $p, q \in H(\mathrm{div}\,;\Omega)$ and $p_h \in \mathcal{L}^k(\mathcal{T})^d$ with $p\cdot n, q\cdot n \in L^2(?_N)$ and*

$$\int_\Omega (p-p_h)\cdot\nabla w_h\,dx = 0 \quad \textit{for all } w_h \in \mathcal{S}^1_D(\mathcal{T}). \tag{3.1}$$

Then there holds

$$\sup_{\substack{w\in H^1_D(\Omega)\\ \|\nabla w\|_{L^2(\Omega)}=1}} \int_\Omega (p-p_h)\cdot\nabla w\,dx \le c_2\|p_h-q\|_{L^2(\Omega)} + c_3\Big(\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}} \|\mathrm{div}\,(p-q)-f_z\|^2_{L^2(\Omega_z)}\Big)^{1/2} + c_5\|h_{\mathcal{E}}^{1/2}\,(p-q)\cdot n\|_{L^2(\Gamma_N)}. \tag{3.2}$$

Proof. According to (3.1), (2.6), Cauchy's inequality, and an integration by parts we have, for each $w \in H^1_D(\Omega)$ with $\|\nabla w\|_{L^2(\Omega)} = 1$, that

$$\begin{aligned}\int_\Omega (p-p_h)\cdot\nabla w\,dx &= \int_\Omega (p-p_h)\cdot\nabla(w-\mathcal{J}w)\,dx\\ &= \int_\Omega (p-q)\cdot\nabla(w-\mathcal{J}w)\,dx + \int_\Omega (q-p_h)\cdot\nabla(w-\mathcal{J}w)\,dx\\ &\le \int_{\Gamma_N} (w-\mathcal{J}w)\,(p-q)\cdot n\,dx - \int_\Omega (w-\mathcal{J}w)\,\mathrm{div}\,(p-q)\,dx + c_2\|p_h-q\|_{2,\Omega}\end{aligned} \tag{3.3}$$

since w and $\mathcal{J}w$ vanish on $\partial\Omega\setminus ?_N$. Owing to (2.7) and (2.9) in Theorem 2.1, we conclude (3.2) from (3.3) and Cauchy's inequality. □

The second result justifies local averaging. For each edge $E \in \mathcal{E}_\Omega$ let $\omega_E := \mathrm{int}\,(T_1 \cup T_2)$ and $\mathcal{T}_E := \{T_1, T_2\}$ for the two distinct elements $T_1, T_2 \in \mathcal{T}$ with $E = T_1 \cap T_2$ and for each edge $E \in \mathcal{E}_N$ let $\omega_E := \mathrm{int}\,(T)$ and $\mathcal{T}_E := \{T\}$ for the element $T \in \mathcal{T}$ with $E = T \cap \overline{?}_N$. Let

$\mathcal{L}^k(\mathcal{E}_N)$ denote the (possibly discontinuous) $\mathcal{E}_N$-piecewise polynomials of degree $\le k$ on $?_N$ and let $\mathcal{S}^k(\mathcal{T}_E) := \mathcal{L}^k(\mathcal{T}_E) \cap C(\omega_E)$.

Theorem 3.2. *There exists an $(h_\mathcal{T}, h_\mathcal{E})$-independent constant $c_7 > 0$ which depends on the shape of the elements in $\mathcal{T}$ and on the polynomial degree $k \ge 1$, $c_8 = \max_{T\in\mathcal{T}} \operatorname{card}\{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N : E \subseteq \partial T\}$, such that, for all $(p_h, g_h) \in \mathcal{L}^{k-1}(\mathcal{T})^d \times \mathcal{L}^k(\mathcal{E}_N)$, we have*

$$
\begin{aligned}
(3.4)\quad c_7 \min_{q_h\in\mathcal{S}^k(\mathcal{T})^d} &\Big(\|p_h - q_h\|^2_{L^2(\Omega)} + \|h_\mathcal{E}^{1/2}(g_h - q_h\cdot n)\|^2_{L^2(\Gamma_N)}\Big) \\
&\le \sum_{E\in\mathcal{E}_\Omega\cup\mathcal{E}_N} \min_{q_E\in\mathcal{S}^k(\mathcal{T}_E)^d} \Big(\|p_h - q_E\|^2_{L^2(\omega_E)} + h_E\|g_h - q_E\cdot n\|^2_{L^2(E\cap\Gamma_N)}\Big) \\
&\le c_8 \min_{q_h\in\mathcal{S}^k(\mathcal{T})^d} \Big(\|p_h - q_h\|^2_{L^2(\Omega)} + \|h_\mathcal{E}^{1/2}(g_h - q_h\cdot n)\|^2_{L^2(\Gamma_N)}\Big).
\end{aligned}
$$

Proof. The upper estimate follows from $q_E := q_h|_{\omega_E} \in \mathcal{S}^k(\mathcal{T}_E)^d$ for all $q_h \in \mathcal{S}^k(\mathcal{T})^d$ and a rearrangement of the sums over edges and elements.
To verify the lower estimate in (3.4) we consider a subspace $\tilde{\mathcal{S}}^k(\mathcal{T})$ of $\mathcal{S}^k(\mathcal{T})$,

$$
\tilde{\mathcal{S}}^k(\mathcal{T}) := \{\sum_{z\in\mathcal{N}} q_z\varphi_z : q_z \in \mathcal{S}^{k-1}(\mathcal{T}_z)\} \subseteq \mathcal{S}^k(\mathcal{T}),
$$

where $\mathcal{T}_z = \{T \in \mathcal{T} : T \subseteq \overline{\omega}_z\}$ denotes the restriction of the triangulation $\mathcal{T}$ to ω_z. Since $\{(q_h, q_h\cdot n|_{\Gamma_N}) : q_h \in \tilde{\mathcal{S}}^k(\mathcal{T})^d\}$ is a closed convex subset of $L^2(\Omega)^d \times L^2(?_N)$, the best–approximation problem

$$
(3.5)\qquad \min_{q_h\in\tilde{\mathcal{S}}^k(\mathcal{T})^d} \Big(\|p_h - q_h\|_2^2 + \|h_\mathcal{E}^{1/2}(g_h - q_h\cdot n)\|^2_{2,\Gamma_N}\Big)
$$

defines an orthogonal relation, namely, for all $q_z \in \mathcal{S}^{k-1}(\mathcal{T}_z)^d$,

$$
(3.6)\qquad \int_\Omega (p_h - \tilde{q_h})\cdot q_z\varphi_z\,dx + \int_{\Gamma_N} h_\mathcal{E}(g_h - \tilde{q}_h\cdot n)\, q_z\cdot n\varphi_z\,ds = 0,
$$

where $\tilde{q}_h = \sum_{z\in\mathcal{N}} \tilde{q}_z\varphi_z \in \tilde{\mathcal{S}}^k(\mathcal{T})^d$, $\tilde{q}_z \in \mathcal{S}^{k-1}(\mathcal{T}_z)^d$, denotes the minimiser in (3.5). From $\sum_{z\in\mathcal{N}}\varphi_z = 1$, (3.6), and Cauchy's inequality we deduce, for arbitrary $q_z \in \mathcal{S}^{k-1}(\mathcal{T}_z)^d$,

$$
\begin{aligned}
&\|p_h - \tilde{q}_h\|_2^2 + \|h_\mathcal{E}^{1/2}(g_h - \tilde{q}_h\cdot n)\|^2_{2,\Gamma_N} \\
&\quad= \sum_{z\in\mathcal{N}} \Big(\int_\Omega (p_h - \tilde{q}_h)\varphi_z(p_h - \tilde{q}_z)\,dx + \int_{\Gamma_N} h_\mathcal{E}(g_h - \tilde{q}_h\cdot n)\varphi_z(g_h - \tilde{q}_z\cdot n)ds\Big) \\
&\quad= \sum_{z\in\mathcal{N}} \Big(\int_\Omega (p_h - \tilde{q}_h)\varphi_z(p_h - q_z)dx + \int_{\Gamma_N} h_\mathcal{E}(g_h - \tilde{q}_h\cdot n)\varphi_z(g_h - q_z\cdot n)\,ds\Big) \qquad (3.7)\\
&\quad\le (\|p_h - \tilde{q}_h\|_2 + \|h_\mathcal{E}^{1/2}(g_h - \tilde{q}_h\cdot n)\|_{2,\Gamma_N}) \\
&\qquad\times \Big(\sum_{z\in\mathcal{N}} \Big(\|\varphi_z^{1/2}(p_h - q_z)\|^2_{2,\omega_z} + \|h_\mathcal{E}^{1/2}\varphi_z^{1/2}(g_h - q_z\cdot n)\|^2_{2,\Gamma_N}\Big)\Big)^{1/2}.
\end{aligned}
$$

For each $z \in \mathcal{N}$, we consider the semi-norms on a finite dimensional subspace of $L^2(\omega_z)^d \times L^2((\partial\omega_z) \cap ?_N)$

$$|||(p_h, g_h)|||_{z,1} := \min_{q_z \in \mathcal{S}^{k-1}(\mathcal{T}_z)^d} \big(\|\varphi_z^{1/2}(p_h - q_z)\|_{2,\omega_z} + \|\varphi_z^{1/2} h_{\mathcal{E}}^{1/2}(g_h - q_z \cdot n)\|_{2,\Gamma_N} \big),$$

$$|||(p_h, g_h)|||_{z,2} := \Big(\sum_{\substack{E \in \mathcal{E} \\ z \in E}} \min_{q_E \in \mathcal{S}^k(\mathcal{T}_E)^d} \Big(\|p_h - q_E\|_{2,\omega_E}^2 + h_E \|g_h - q_E \cdot n\|_{2,\Gamma_N \cap E}^2 \Big) \Big)^{1/2}.$$

Then, (3.7) and $\tilde{\mathcal{S}}^k(\mathcal{T}) \subseteq \mathcal{S}^k(\mathcal{T})$ yield

$$\min_{q_h \in \mathcal{S}^k(\mathcal{T})^d} \big(\|p_h - q_h\|_2^2 + \|h_{\mathcal{E}}^{1/2}(g_h - q_h \cdot n)\|_{2,\Gamma_N}^2 \big) \lesssim \sum_{z \in \mathcal{N}} |||(p_h, g_h)|||_{z,1}^2. \tag{3.8}$$

We claim $|||\cdot|||_{z,1} \lesssim |||\cdot|||_{z,2}$. For a proof, suppose $|||(p_h, g_h)|||_{z,2} = 0$. Then, for each E which is an inner edge of ω_z, we have $p_h = q_E$ on the open set ω_E for some $q_E \in \mathcal{S}^k(\mathcal{T}_E)^d$. Since $p_h \in \mathcal{L}^{k-1}(\mathcal{T})^d$, we find that $p_h|_{\omega_E} \in \mathcal{S}^{k-1}(\mathcal{T}_E)$. The set of all such ω_E is a cover of ω_z and there is a sequence $E_1, ..., E_m$ of such inner edges such that $\omega_{E_j} \cap \omega_{E_{j+1}} \neq \emptyset$, so that we deduce $p_h|_{\omega_z} \in \mathcal{S}^{k-1}(\mathcal{T}_z)$. Moreover, $g_h = p_h \cdot n$ on each edge $E \subseteq \overline{?}_N$ with $z \in E$; while for edges $E \subseteq \partial\omega_z \cap \overline{?}_N$ with $z \notin E$ we have $\varphi_z|_E = 0$. Altogether, we deduce $|||(p_h, g_h)|||_{z,1} = 0$. A compactness and scaling argument then shows our claim

$$|||\cdot|||_{z,1} \lesssim |||\cdot|||_{z,2} \quad \text{on } \mathcal{L}^{k-1}(\mathcal{T}_z)^d \times \mathcal{L}^k(\{E \in \mathcal{E} : E \subseteq \partial\omega_z\}). \tag{3.9}$$

Utilizing (3.9) in (3.8) we conclude

$$\begin{aligned} \min_{q_h \in \mathcal{S}^k(\mathcal{T})^d} \big(\|p_h - q_h\|_2^2 + \|h_{\mathcal{E}}^{1/2}(g_h - q_h \cdot n)\|_{2,\Gamma_N}^2 \big) &\lesssim \sum_{z \in \mathcal{N}} |||(p_h, g_h)|||_{z,1}^2 \\ \lesssim \sum_{z \in \mathcal{N}} |||(p_h, g_h)|||_{z,2}^2 &\lesssim \sum_{E \in \mathcal{E}} \min_{q_E \in \mathcal{S}^k(\mathcal{T}_E)^d} \big(\|p_h - q_E\|_{2,\omega_E}^2 + h_E \|g_h - q_E \cdot n\|_{2,\Gamma_N \cap E}^2 \big). \end{aligned} \tag{3.10}$$

□

Remark 3.1. The assertions of Theorem 3.1 and 3.2 hold verbatim for three space dimensions where $\mathcal{T}$ consists of tetrahedra or parallelepipeds with the same proofs.

4. Applications to conforming finite element schemes

Given right-hand sides $f \in L^2(\Omega)$, $g \in L^2(?_N)$, and $u_D \in H^1(?_D)$, let $u \in H^1(\Omega)$ denote the unique weak solution to

$$-\Delta u = f \quad \text{in } \Omega, \tag{4.1}$$

$$u = u_D \quad \text{on } ?_D, \tag{4.2}$$

$$\partial u/\partial n = g \quad \text{on } ?_N. \tag{4.3}$$

Suppose a finite element scheme, based on a regular triangulation $\mathcal{T}$, provided a discrete flux $p_h := \nabla u_h$ to the exact flux $p := \nabla u \in H(\mathrm{div}\,; \Omega)$ such that $u_h \in \mathcal{S}^1(\mathcal{T})$, $u_h(z) = u_D(z)$ for all $z \in \mathcal{N} \cap ?_D$ and

$$\int_\Omega \nabla u_h \cdot \nabla w_h \, dx = \int_\Omega f w_h \, dx + \int_{\Gamma_N} g w_h \, ds \quad \text{for all } w_h \in \mathcal{S}_D^1(\mathcal{T}). \tag{4.4}$$

Theorem 4.1. *There exists an $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant $c_9 > 0$ (that depends on k and the shape of the elements and patches) such that*

$$\|\nabla(u-u_h)\|_{L^2(\Omega)} \le \min_{q_h\in\mathcal{S}^k(\mathcal{T})^d}\big(c_9\|\nabla u_h - q_h\|_{L^2(\Omega)} + 2c_5\|h_{\mathcal{E}}^{1/2}(g-q_h\cdot n)\|_{L^2(\Gamma_N)}\big) + \inf_{v|_{\Gamma_D}=u_D}\|\nabla(u_h-v)\|_{L^2(\Omega)} + 2c_3\Big(\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}}\|f-f_z\|^2_{L^2(\Omega_z)}\Big)^{1/2}. \tag{4.5}$$

In the infimum, "$v|_{\Gamma_D} = u_D$" stands for all $v \in H^1(\Omega)$ with $v = u_D$ on $?_D$.

Proof. Abbreviate $e := u - u_h$ and let $q_h \in \mathcal{S}^k(\mathcal{T})^d$. Assume that $v \in H^1(\Omega)$ satisfies $v = u_D$ on $?_D$ and $\|\nabla(u_h - v)\|_2 \le \|\nabla e\|_2$. Recall $p = \nabla u$ and $p_h = \nabla u_h$. Then (4.1)-(4.4) imply (3.1). Hence, we may choose $q = q_h$ and $w = u - v$ in Theorem 3.1 to obtain with Cauchy's inequality for the second term that

$$\begin{aligned}\|\nabla e\|_2^2 &= \int_\Omega \nabla e\cdot\nabla w\,dx + \int_\Omega \nabla e\cdot\nabla(v-u_h)\,dx \\ &\le \|\nabla w\|_2\Big(c_2\|p_h-q_h\|_2 + c_5\|h_{\mathcal{E}}^{1/2}(g-q_h\cdot n)\|_{2,\Gamma_N} \\ &\quad + c_3\Big(\sum_{z\in\mathcal{K}} h_z^2\min_{f_z\in\mathbb{R}}\|f+\operatorname{div} q_h - f_z\|^2_{2,\Omega_z}\Big)^{1/2}\Big) + \|\nabla(u_h-v)\|_2\,\|\nabla e\|_2.\end{aligned} \tag{4.6}$$

Since $\|\nabla w\|_2 \le \|\nabla e\|_2 + \|\nabla(u_h - v)\|_2 \le 2\|\nabla e\|_2$, we can divide (4.6) by $\|\nabla e\|_2$ to verify

$$\begin{aligned}\|\nabla e\|_2 &\le 2c_2\|p_h-q_h\|_2 + 2c_5\|h_{\mathcal{E}}^{1/2}(g-q_h\cdot n)\|_{2,\Gamma_N} + \|\nabla(u_h-v)\|_2 \\ &\quad + 2c_3\Big(\sum_{z\in\mathcal{K}} h_z^2\min_{f_z\in\mathbb{R}}\|f+\operatorname{div} q_h - f_z\|^2_{2,\Omega_z}\Big)^{1/2}.\end{aligned} \tag{4.7}$$

Let $\operatorname{div}_{\mathcal{T}}$ denote the $\mathcal{T}$-piecewise action of the div-operator. The triangle inequality in the last summand in (4.7) and $h_z \lesssim h_T$ for $z \in T\cap\mathcal{N}$ and $T \in \mathcal{T}$ and a summation over elements show

$$\begin{aligned}\sum_{z\in\mathcal{K}} h_z^2\min_{f_z\in\mathbb{R}}\|f+\operatorname{div} q_h - f_z\|^2_{2,\Omega_z} &\lesssim \|h_{\mathcal{T}}\operatorname{div}_{\mathcal{T}}(p_h-q_h)\|_2^2 \\ &\quad + \sum_{z\in\mathcal{K}} h_z^2\min_{f_z\in\mathbb{R}}\|f+\operatorname{div}_{\mathcal{T}} p_h - f_z\|^2_{2,\Omega_z}.\end{aligned} \tag{4.8}$$

Note that $\operatorname{div}_{\mathcal{T}} p_h = \Delta_{\mathcal{T}} u_h = 0$ for our choices of $u_h \in \mathcal{S}^1(\mathcal{T})$. A $\mathcal{T}$-elementwise inverse estimate shows $\|h_{\mathcal{T}}\operatorname{div}_{\mathcal{T}}(p_h - q_h)\|_2 \lesssim \|p_h - q_h\|_2$ (with a constant that depends on the shape of the finite elements only). Utilising this in (4.7)-(4.8) we deduce (4.5). □

The subsequent lemma shows that $\inf_{v|_{\Gamma_D}=u_D}\|\nabla(u_h-v)\|_{L^2(\Omega)}$ is a higher order term.

Lemma 4.1. *Suppose that $u_h(z) = u_D(z)$ for all $z \in \mathcal{N}\cap ?_D$. Then there exists an $h_{\mathcal{E}}$-independent constant $c_{10} > 0$ (that depends on the shapes of the elements only) such that*

$$\inf_{v|_{\Gamma_D}=u_D}\|\nabla(u_h-v)\|_{L^2(\Omega)} \le c_{10}\|h_{\mathcal{E}}^{1/2}\partial(u_h-u_D)/\partial s\|_{L^2(\Gamma_D)}. \tag{4.9}$$

If $u_D \in H^2(\mathcal{E}_D) := \{v \in L^2(?_D) : \forall E \in \mathcal{E}_D,\ v|_E \in H^2(E)\}$, *we have*

$$\inf_{v|_{\Gamma_D}=u_D} \|\nabla(u_h - v)\|_{L^2(\Omega)} \leq c_{10} \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}}^2 u_D/\partial s^2\|_{L^2(\Gamma_D)}. \tag{4.10}$$

Proof. Let $E \in \mathcal{E}_D$ belong to some $T \in \mathcal{T}$ and denote $\gamma := ?_D \cap \partial T$. We determine $w \in H^1(T)$ by extending the boundary values $w|_\gamma = u_h - u_D$ and $w|_{\partial T \setminus \gamma} = 0$. Note that w is continuous on ∂T since u_h interpolates $u_D = v$ at each node on $?_D$. An harmonic extension of $w|_{\partial T}$ to $w \in H^1(\mathcal{T})$ yields

$$\|\nabla w\|_{2,T} \lesssim \|w\|_{H^{1/2}(\partial T)} \lesssim \|w\|_{2,\partial T}^{1/2} \|\partial w/\partial s\|_{2,\partial T}^{1/2}, \tag{4.11}$$

where we applied an interpolation estimate. A one-dimensional integration argument shows $\|w\|_{2,\partial T} \leq h_T \|\partial w/\partial s\|_{2,\partial T}$. Consequently,

$$\|\nabla w\|_{2,T} \lesssim h_T^{1/2} \|\partial w/\partial s\|_{2,\partial T} = h_T^{1/2} \|\partial(u_h - u_D)/\partial s\|_{2,\gamma}. \tag{4.12}$$

A scaling argument guarantees that the constant in (4.12) is h_T-independent. Defining v by $u_h - w$ on elements on $?_D$ and by zero on other elements then shows the lemma. The second estimate follows from $\|w\|_{2,\partial T} \leq h_T^2 \|\partial^2 w/\partial s^2\|_{2,\partial T}$. □

Lemma 4.2. *Suppose* $g \in H^1(\mathcal{E}_N)$ *and, for each node* $z \in \mathcal{N} \cap \overline{?}_N$ *where the outer unit normal* n *on* $?_N$ *is continuous (hence constant in a neighbourhood of* z *as* $?_N$ *is a polygon), let* g *be continuous. Then, the set*

$$\mathcal{S}^1_N(\mathcal{T}, g) := \{q_h \in \mathcal{S}^1(\mathcal{T})^d : \forall E \in \mathcal{E}_N\ \forall z \in E \cap \mathcal{N},\ q_h(z) \cdot n_E = g(z)\} \tag{4.13}$$

is non-void and, for each $q_h \in \mathcal{S}^1_N(\mathcal{T}, g)$,

$$\|h_{\mathcal{E}}^{1/2}(g - q_h \cdot n)\|_{L^2(\Gamma_N)} \leq \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}} g/\partial s\|_{L^2(\Gamma_N)}. \tag{4.14}$$

Proof. Elementary estimates on each edge on $?_N$ verify (4.14); the proof of $\mathcal{S}^1_N(\mathcal{T}, g) \neq \emptyset$ follows from an explicit construction in Example 4.1. □

Example 4.1. We define an operator $\mathcal{A} : L^2(\Omega)^2 \to \mathcal{S}^1_N(\mathcal{T}, g)$ by

$$\mathcal{A}p := \sum_{z \in \mathcal{N}} p_z \varphi_z, \tag{4.15}$$

where $p_z := \fint_{\omega_z} p\,dx := \frac{1}{|\omega_z|} \int_{\omega_z} p\,dx \in \mathbb{R}^2$ for $z \in \mathcal{N} \setminus \overline{?}_N$ while we incorporate $\mathcal{A}p(z) \cdot n_E = g(z)$ for $z \in \mathcal{N} \cap \overline{?}_N$. In case $z = E_1 \cap E_2$ for two distinct edges $E_1, E_2 \in \mathcal{E}_N$ with distinct outer unit normals n_{E_1}, n_{E_2} on E_1, E_2 at a corner z we choose $p_z \in \mathbb{R}^2$ to be the unique solution of the 2×2 linear system

$$n_{E_1} \cdot p_z = g|_{E_1}(z) \quad \text{and} \quad n_{E_2} \cdot p_z = g|_{E_2}(z). \tag{4.16a}$$

In the remaining cases $z \in E_1 \cap ?_D$ for $E_1 \in \mathcal{E}_N$ or $z = E_1 \cap E_2$ with two parallel edges $E_1, E_2 \in \mathcal{E}_N$ with the unit tangent vector t_{E_1} let $p_z \in \mathbb{R}^2$ solve

$$n_{E_1} \cdot p_z = g|_{E_1}(z) \quad \text{and} \quad t_{E_1} \cdot p_z = \fint_{\omega_z} t_{E_1} \cdot p\,dx. \tag{4.16b}$$

□

The following corollary is (1.5) with a constant $c_1 = c_9$ as in Theorem 4.1 and with specified higher order terms from Lemma 4.1 and 4.2 and a Poincaré inequality.

Corollary 4.1. *Under the conditions of Theorem 4.1, Lemma 4.1, and Lemma 4.2 we have for $f \in H^1(\Omega)$ that*

$$\|\nabla(u-u_h)\|_{L^2(\Omega)} \le c_9 \min_{q_h \in \mathcal{S}^1_N(\mathcal{T},g)} \|\nabla u_h - q_h\|_{L^2(\Omega)} + c_{11}\Big(\|h_{\mathcal{E}}^{3/2}\,\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|h_{\mathcal{E}}^{3/2}\,\partial_{\mathcal{E}} g/\partial s\|_{L^2(\Gamma_N)} + \|h_{\mathcal{T}}^2\,\nabla f\|_{L^2(\Omega)}\Big). \tag{4.17}$$

The $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant $c_{11} > 0$ depends on the shape of the elements and patches only. □

Remark 4.1. Let us emphasise that the derivatives along ? are required only $\mathcal{E}$-piecewisely while f needs to be patch-wise (not only elementwise) in H^1 and so $f \in H^1(\Omega)$. For a non-smooth right-hand side f, $\|h_{\mathcal{T}}^2\,\nabla f\|_{L^2(\Omega)}$ may be replaced by a patch-wise L^2–best-approximation error in the approximation through constants of f (cf. (2.7)).

The global averaging process might be too expensive or its approximation may be inefficient and hence a local averaging process of interest. Recall that ω_E is the (interior of the) union of all elements in $\mathcal{T}$ that share the edge $E \in \mathcal{E}$.

Corollary 4.2. *Under the conditions of Theorem 4.1, Lemma 4.1, and Lemma 4.2 we have for $f \in H^1(\Omega)$ that*

$$\begin{aligned}\|\nabla(u-u_h)\|_{L^2(\Omega)} \;\le\; & c_{12}\Big(\sum_{E\in\mathcal{E}} \min_{q_E \in \mathcal{S}^1(\mathcal{T}_E)^d}\Big(\|\nabla u_h - q_E\|^2_{L^2(\omega_E)} + h_E\|g_h - q_E\cdot n\|^2_{L^2(E\cap\Gamma_N)}\Big)\Big)^{1/2} \\ & + c_{11}\Big(\|h_{\mathcal{E}}^{3/2}\,\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|h_{\mathcal{E}}^{3/2}\,\partial_{\mathcal{E}} g/\partial s\|_{L^2(\Gamma_N)} + \|h_{\mathcal{T}}^2\,\nabla f\|_{L^2(\Omega)}\Big). \qquad (4.18)\end{aligned}$$

The $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant $c_{12} = \max\{c_9, 2c_5\}/c_7$ depends on the shape of the elements and patches only.

Proof. Theorem 4.1, Lemma 4.1, an approximation g_h of g as in Lemma 4.2 and a Poincaré inequality show

$$\|\nabla(u-u_h)\|_2 \lesssim \min_{q_h\in\mathcal{S}^1(\mathcal{T})^d}(\|\nabla u_h - q_h\|_2 + \|h_{\mathcal{E}}^{1/2}(g_h - q_h\cdot n)\|_{2,\Gamma_N}) + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}} g/\partial s\|_{2,\Gamma_N} + \|h_{\mathcal{E}}^{3/2}\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{2,\Gamma_D} + \|h_{\mathcal{T}}^2\nabla f\|_2. \tag{4.19}$$

This and the first inequality of Theorem 3.2 imply the assertion. □

Remark 4.2. The results of this section hold also in three dimensions where $\mathcal{T}$ consists of tetrahedra or parallelepipeds. The proofs of some details as Lemma 4.1 or Lemma 4.2 require much more technical preparations and so are omitted in this overview.

Remark 4.3. It is shown in [CV, C2] that the edge–contributions (jump differences in the normal fluxes components across edges) dominate in standard residual a posteriori error estimates [BaR, B, BS, CF1, EEHJ, V]. Arguing as in [R1, R2, DMR] one can hence derive alternative proofs of (4.18) and then of (4.17).

Remark 4.4. In an L^∞-estimate of [HSWW] it is suggested to average over a domain of size $O(h\log(1/h))$ instead of merely patches or the entire domain to obtain asymptotic exact results.

5. Applications to nonconforming finite element schemes

In the Laplace problem with mixed boundary conditions (4.1)-(4.3) we suppose that the discrete flux $p_h := \nabla_{\mathcal{T}} u_h \in \mathcal{L}^0(\mathcal{T})^d$, where $\nabla_{\mathcal{T}}$ denotes the $\mathcal{T}$-piecewise application of the gradient, satisfies

$$\int_\Omega \nabla_{\mathcal{T}} u_h \cdot \nabla w_h \, dx = \int_\Omega f w_h \, dx + \int_{\Gamma_N} g w_h \, ds \quad \text{for all } w_h \in \mathcal{S}^1_D(\mathcal{T}). \tag{5.1}$$

The usual conformity conditions read for all $E \in \mathcal{E}_\Omega \cup \mathcal{E}_D$,

$$\int_E [u_h] \, ds = 0, \tag{5.2}$$

where $[u_h]|_E$ denotes the jump of u_h across $E \in \mathcal{E}_\Omega$ and denotes $u_D - u_h$ on $?_D$. Those conditions are satisfied by construction for Crouzeix-Raviart finite elements of lowest order.

Remark 5.1. It is stressed that $\mathcal{S}^1_D(\mathcal{T})$ is a conforming test function space which is included in the nonconforming finite element spaces for triangles or tetrahedra. For parallelograms, (5.1) means that the polynomial degrees are at least of second order to include the conforming term $x_1 x_2$. This technical detail could actually be dropped since the contribution from an enhanced finite element space leads to a higher order term [KS]. We restrict our analysis to triangles or tetrahedra for simplicity.

Theorem 5.1. *Suppose that $?_N$ is connected and that $?_D$ belongs to only one connectivity component of $\partial\Omega$. Then, there exists an $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant $c_{13} > 0$ (that depends on $k \geq 1$ and the shape of the elements and patches) such that*

$$\begin{aligned} \|\nabla_{\mathcal{T}}(u - u_h)\|_{L^2(\Omega)} \leq \min_{q_h \in \mathcal{S}^k(\mathcal{T})^d} \big(& c_{13} \|\nabla_{\mathcal{T}} u_h - q_h\|_{L^2(\Omega)} + c_5 \|h_{\mathcal{E}}^{1/2}(g - q_h \cdot n)\|_{L^2(\Gamma_N)} \\ & + c_5 \|h_{\mathcal{E}}^{1/2}(q_h \cdot t - \partial u_D/\partial s)\|_{L^2(\Gamma_D)} \big) + c_3 \Big(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathbb{R}} \|f - f_z\|^2_{L^2(\Omega_z)} \Big)^{1/2}. \end{aligned} \tag{5.3}$$

Here, $t \in \mathcal{L}^0(\mathcal{E}_D)^d$ denotes the unit tangent vector on $?_D$.

Remark 5.2. The following lemma is based on the Helmholtz decomposition of a vector–field. The decomposition is available in three dimensions as well (cf., e.g., [GR]) but the notation is more involved so we restrict the discussion to the two-dimensional setting for brevity.

Lemma 5.1. *For all $p - p_h \in L^2(\Omega)^2$, there exist $\alpha, \beta \in H^1(\Omega)$ that satisfy boundary conditions $\alpha|_{\Gamma_D} = 0$ and $\beta|_{\Gamma_N}$ is constant such that*

$$p - p_h = \nabla\alpha + \operatorname{Curl}\beta \quad \text{and} \quad \|p - p_h\|^2_{L^2(\Omega)} = \|\nabla\alpha\|^2_{L^2(\Omega)} + \|\nabla\beta\|^2_{L^2(\Omega)}. \tag{5.4}$$

Proof. The lemma follows from the Helmholtz decomposition where $\alpha \in H^1_D(\Omega)$ solves $\Delta\alpha = \operatorname{div}(p - p_h)$ and $\Delta\beta = \operatorname{curl}(p - p_h)$ with proper boundary conditions, cf., e.g., [GR]. □

Proof of Theorem 5.1. For $p = \nabla u$ and $p_h = \nabla_{\mathcal{T}} u_h$ Lemma 5.1 yields

$$\|p - p_h\|_2^2 = \int_\Omega (p - p_h) \cdot \nabla \alpha \, dx + \int_\Omega (p - p_h) \cdot \operatorname{Curl} \beta \, dx. \tag{5.5}$$

Since $?_N$ is connected we may and will assume without loss of generality that $\beta = 0$ on $?_N$. According to (4.1)-(4.3) and (5.1) we infer (3.1) and hence may choose $q = q_h \in \mathcal{S}^k(\mathcal{T})^d$ and, in case $\alpha \not\equiv 0$, $w = \alpha / \|\alpha\|_{2,\Omega}$ in Theorem 3.1 to obtain

$$\begin{aligned} \int_\Omega (p - p_h) \cdot \nabla \alpha \, dx \le \|\nabla \alpha\|_2 \Big(& c_2 \|p_h - q_h\|_2 + c_5 \|h_{\mathcal{E}}^{1/2} (g - q_h \cdot n)\|_{2,\Gamma_N} \\ & + c_3 \Big(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathbb{R}} \|f + \operatorname{div} q_h - f_z\|_{2,\Omega_z}^2 \Big)^{1/2} \Big). \end{aligned} \tag{5.6}$$

The estimate of the last term in (5.5) will follow from Theorem 3.1 as well once we established an analogy to (3.1), namely

$$\int_\Omega (p - p_h) \cdot \operatorname{Curl} \; w_h \, dx = 0 \quad \text{for all } w_h \in \mathcal{S}_N^1(\mathcal{T}), \tag{5.7}$$

where $\mathcal{S}_N^1(\mathcal{T}) := \{v_h \in \mathcal{S}^1(\mathcal{T}) : v_h = 0 \text{ on } ?_N\}$. It is essential to notice that $\partial w_h / \partial s$ is constant and $[u_h]$ has vanishing integral on any edge. An elementwise integration by parts on the left-hand side of (5.7) yields volume terms $(u - u_h) \operatorname{div}_{\mathcal{T}} \operatorname{Curl} w_h = 0$ and edge terms $[(u - u_h) \partial w_h / \partial s] = [u_h] \partial w_h / \partial s$ whose integral vanishes on any E (the case $E \in \mathcal{E}_\Omega$ is indicated and the assertion is true for $E \in \mathcal{E}_D$ as well; $w_h = 0$ on $?_N$ shows it for $E \in \mathcal{E}_N$). In this way we establish (5.7).

To employ Theorem 3.1, we interchange components, writing in this proof $Q(a_1, a_2) := (-a_2, a_1)$ for vectors, and we interchange the role of the boundaries and adopt Theorem 2.1 and (3.1) where $\tilde{?}_D = ?_N$ acts as the Dirichlet boundary and $\tilde{?}_N = ?_D$ acts as the Neumann boundary. Writing $\tilde{p} = Qp$ and $\tilde{p}_h = Qp_h$, (5.7) reads $\int_\Omega (\tilde{p} - \tilde{p}_h) \cdot \nabla w_h \, dx = 0$ for all $w_h \in \tilde{\mathcal{S}}_D^1(\mathcal{T}) = \mathcal{S}_N^1(\mathcal{T})$ and this is (3.1). Reading Theorem 3.1 in the present notation we obtain

$$\begin{aligned} \int_\Omega (p - p_h) \cdot \operatorname{Curl} \beta \, dx \le \|\nabla \beta\|_2 \Big(& c_2 \|p_h - q_h\|_2 + c_5 \|h_{\mathcal{E}}^{1/2} (\partial u_D / \partial s - q_h \cdot t)\|_{2,\Gamma_D} \\ & + c_3 \Big(\sum_{z \in \mathcal{K}} h_z^2 \|\operatorname{curl}_{\mathcal{T}} (q_h - p_h)\|_{2,\Omega_z}^2 \Big)^{1/2} \Big) \end{aligned} \tag{5.8}$$

with $\operatorname{curl}_{\mathcal{T}} (q_h - p_h) := \operatorname{div}_{\mathcal{T}} Q(q_h - p_h)$. In the second last term, $t = Qn$ denotes the unit tangent vector and in the last term we used that $\operatorname{curl}_{\mathcal{T}} p = 0 = \operatorname{curl}_{\mathcal{T}} p_h$.

The remaining arguments are similar to those in the proof of Theorem 4.1 and hence omitted. □

In contrast to the conforming situation, Theorem 5.1 demands averaging functions to satisfy some conditions on the Dirichlet boundary.

Lemma 5.2. *Suppose $u_D \in H^2(\mathcal{E}_D)$ and, for each node $z \in \mathcal{N} \cap ?_D$ where the outer unit normal n on $?$ is continuous let $\partial u_D / \partial s$ be continuous. Then, the set*

$$\mathcal{S}_D^1(\mathcal{T}, u_D) := \{q_h \in \mathcal{S}^1(\mathcal{T})^d : \forall E \in \mathcal{E}_D \, \forall z \in E \cap \mathcal{N}, \, q_h(z) \cdot t_E = \partial u_D / \partial s(z)\} \tag{5.9}$$

is non–void and, for each $q_h \in \mathcal{S}^1_D(\mathcal{T}, u_D)$,

$$\|h_{\mathcal{E}}^{1/2}(q_h \cdot t_{\mathcal{E}} - \partial u_D/\partial s)\|_{L^2(\Gamma_D)} \le \|h_{\mathcal{E}}^{3/2}\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\Gamma_D)}. \tag{5.10}$$

Proof. Similar to (4.10) in Lemma 4.1 or 4.2. □

Example 5.1. Assume the conditions of Lemma 4.2 and Lemma 5.2 on the data g and u_D. We define an operator $\mathcal{A} : L^2(\Omega)^2 \to \mathcal{S}^1_N(\mathcal{T}, g) \cap \mathcal{S}^1_D(\mathcal{T}, u_D)$ by (4.15) and $p_z := \fint_{\omega_z} p\,dx$ for $z \in \mathcal{N} \setminus ?$. In case $z \in (\mathcal{N} \cap \overline{?}_N) \setminus ?_D$ we preceed as in (4.16a) resp. (4.16b). In case $z \in (\mathcal{N} \cap ?_D) \setminus \overline{?}_N$ we consider the analogous 2×2 systems

$$t_{E_1} \cdot p_z = \partial u_D|_{E_1}/\partial s(z) \quad \text{and} \quad t_{E_2} \cdot p_z = \partial u_D|_{E_2}/\partial s(z) \tag{5.11a}$$

(cf. (4.16a) and notation from Example 4.1), resp., as an analog to (4.16b),

$$t_{E_1} \cdot p_z = \partial u_D|_{E_1}/\partial s(z) \quad \text{and} \quad n_{E_1} \cdot p_z = \fint_{\omega_z} n_{E_1} \cdot p\,dx. \tag{5.11b}$$

For $z \in ?_D \cap \overline{?}_N \subseteq \mathcal{N}$ we require a compatibility condition if $n_{E_1} = t_{E_2}$, namely $g(z) = \partial u_D/\partial s(z)$.
Then, we define $p_z \in \mathbb{R}^2$ as in (5.11b) when $E_1 \subseteq ?_D$ (the case $E_2 \subseteq ?_D$ is analogous). For $n_{E_1} \neq \pm t_{E_2}$ we need no further compatibility of the data and solve the 2×2 linear system

$$t_{E_1} \cdot p_z = \partial u_D|_{E_1}/\partial s(z) \quad \text{and} \quad n_{E_2}(z) \cdot p_z = g(z). \tag{5.12}$$

(Here $E_1 \subseteq ?_D$; the case $E_2 \subseteq ?_D$ is analogous.) □

The modification of (1.5) in the nonconforming setting is a direct consequence of Theorem 5.1, Lemma 5.2 and Example 5.1. Note that Corollary 4.1 is a special case apart from the different treatment of the Dirichlet boundary conditions.

Corollary 5.1. *Under the conditions of Theorem 5.1, Lemma 4.2, and Lemma 5.2 we have for* $f \in H^1(\Omega)$ *that*

$$\begin{aligned}\|\nabla_{\mathcal{T}}(u - u_h)\|_{L^2(\Omega)} &\le c_{13} \min_{q_h \in \mathcal{S}^1_N(\mathcal{T},g)\cap\mathcal{S}^1_D(\mathcal{T},u_D)} \|\nabla_{\mathcal{T}}(u_h - q_h)\|_{L^2(\Omega)} \\ &\quad + c_{14}\big(\|h_{\mathcal{E}}^{3/2}\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}} g/\partial s\|_{L^2(\Gamma_N)} + \|h^2_{\mathcal{T}}\nabla f\|_{L^2(\Omega)}\big).\end{aligned} \tag{5.13}$$

□

The analog to Corollary 4.2 concludes this section on lowest order Raviart–Crouzeix finite elements.

Corollary 5.2. *Under the conditions of Theorem 5.1, Lemma 4.2, and Lemma 5.2 there exists a constant* $c_{15} > 0$ *such that we have for* $f \in H^1(\Omega)$

$$\begin{aligned}\|\nabla_{\mathcal{T}}(u - u_h\|_{L^2(\Omega)} &\le c_{15}\Big(\sum_{E\in\mathcal{E}} \min_{q_E \in \mathcal{S}^1(\mathcal{T}_E)^d}\big(\|\nabla_{\mathcal{T}} u_h - q_E\|^2_{L^2(\omega_E)} \\ &\quad + h_E\|g_h - q_E \cdot n\|^2_{L^2(E\cap\Gamma_N)} + h_E\|q_E \cdot t - u'_{D,h}\|^2_{L^2(E\cap\Gamma_D)}\big)\Big)^{1/2} \\ &\quad + c_{14}\big(\|h_{\mathcal{E}}^{3/2}\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}} g/\partial s\|_{L^2(\Gamma_N)} + \|h^2_{\mathcal{T}}\nabla f\|_{L^2(\Omega)}\big).\end{aligned} \tag{5.14}$$

Here, $u'_{D,h}$ *denotes an approximation of* $\partial u_D/\partial s$ *as in Lemma 5.2, i.e.,* $u'_{D,h} = q_h \cdot t_{\mathcal{E}}$ *on* $?_D$ *for some* $q_h \in \mathcal{S}^1_D(\mathcal{T}, u_D)$. □

Remark 5.3. The results of this section can be generalised to three space dimensions as all the required tools such as a Helmholtz decomposition are available then as well. Details on the three–dimensional case are omitted for notational simplicity.

Remark 5.4. Arguing as in [CV, C2] one can prove that edge contributions (jumps in the fluxes across edges) dominate the residual based error estimates from [DDPV, C2, KS]. Arguing in the spirit of [R1, R2, DMR] one can hence derive alternative proofs of (5.14) and then of (5.13).

6. Applications to mixed finite element schemes

In the Laplace problem with mixed boundary conditions (4.1)-(4.3) we suppose that the discrete flux $p_h \in H(\mathrm{div},\Omega) \cap \mathcal{L}^k(\mathcal{T})^2$ and the displacement approximation $u_h \in \mathcal{L}^k(\mathcal{T})$ satisfy, for all $q_h \in \mathrm{Curl}\,\mathcal{S}^1(\mathcal{T})$ with $q_h \cdot n = 0$ on $?_N$, and for all $T \in \mathcal{T}$ and $E \in \mathcal{E}_N$ that

$$\int_\Omega (p_h \cdot q_h + u_h \,\mathrm{div}\, q_h)dx = \int_{\Gamma_D} u_D\, q_h \cdot n\, ds, \tag{6.1}$$

$$\int_T (f + \mathrm{div}\, p_h)dx = 0, \tag{6.2}$$

$$\int_E (g - p_h \cdot n)ds = 0. \tag{6.3}$$

Remark 6.1. Standard mixed finite element methods of any order such as Raviart-Thomas (RT), Brezzi-Douglas-Marini (BDM), or Brezzi-Douglas-Fortin-Marini (BDFM) elements (cf. [BF] for details) provide (6.1)-(6.3) [C1].

Theorem 6.1. *Suppose that $?_N$ is connected and that $?_D$ belongs to only one connectivity component of $\partial\Omega$ and let $f \in H^1(\mathcal{T})$, i.e. $f|_T \in H^1(T)$ for all $T \in \mathcal{T}$. Then, there exists an $(h_\mathcal{T}, h_\mathcal{E})$-independent constant $c_{16} > 0$ (that depends on $k \geq 1$ and the shape of the elements and patches) such that*

$$\begin{aligned}\|p - p_h\|_{L^2(\Omega)} \leq \min_{q_h \in \mathcal{S}^k(\mathcal{T})^2} \Big(& c_2 \|p_h - q_h\|_{L^2(\Omega)} + c_5 \|h_\mathcal{E}^{1/2} (\partial u_D/\partial s - q_h \cdot t)\|_{L^2(\Gamma_D)} \\ & + c_3 \Big(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathbb{R}} \|\mathrm{curl}_\mathcal{T} q_h - f_z\|^2_{L^2(\Omega_z)} \Big)^{1/2} \Big) \\ & + c_{16} \|h_\mathcal{T}^2 \nabla_\mathcal{T} (f + \mathrm{div}\, p_h)\|_{2,\Omega} + c_{16} \|h_\mathcal{E}^{1/2} (g - p_h \cdot n)\|_{2,\Gamma_N}.\end{aligned} \tag{6.4}$$

Proof. Lemma 5.1 provides (5.5) and we may and will assume without loss of generality that $\beta = 0$ on $?_N$. An integration by parts and (6.2)-(6.3) show for the $\mathcal{T}$-piecewise integral mean $\alpha_\mathcal{T} \in \mathcal{L}^0(\mathcal{T})$ of $\alpha \in H^1_D(\Omega)$

$$\begin{aligned}\int_\Omega (p - p_h) \cdot \nabla \alpha \, dx &= \int_\Omega (f + \mathrm{div}\, p_h)\, \alpha \, dx + \int_{\Gamma_N} (g - p_h \cdot n)\, \alpha \, dx \\ &= \int_\Omega (f + \mathrm{div}\, p_h)\, (\alpha - \alpha_\mathcal{T})\, dx + \int_{\Gamma_N} (g - p_h \cdot n)\, (\alpha - \alpha_\mathcal{T})\, dx.\end{aligned} \tag{6.5}$$

The second last term is estimated with an elementwise Poincaré inequality while the last term in (6.5) involves a trace theorem [BS, CF1, Cl], namely

$$\|a\|_{2,E} \lesssim h_E^{-1/2} \|a\|_{2,T_E} + h_E^{1/2} \|\nabla a\|_{2,T_E} \tag{6.6}$$

for $a = \alpha - \alpha_{\mathcal{T}} \in H^1(T_E)$ on the triangle $T_E \in \mathcal{T}$ and the edge $E \in \mathcal{E}_N$, $E \subset \partial T_E$. With a second application of Poincaré's inequality, (6.6), and Cauchy's inequality show

$$\int_\Omega (p - p_h)\cdot\nabla\alpha\,dx \lesssim \|h_{\mathcal{T}}(f+\operatorname{div} p_h)\|_2\,\|\nabla\alpha\|_2 + \|h_{\mathcal{E}}^{1/2}(g - p_h\cdot n)\|_{2,\Gamma_N}\|h_{\mathcal{E}}^{-1/2}(\alpha-\alpha_{\mathcal{T}})\|_{2,\Gamma_N}$$

$$\lesssim \|\nabla\alpha\|_2\Big(\|h_{\mathcal{T}}^2\nabla_{\mathcal{T}}(f+\operatorname{div} p_h)\|_2 + \|h_{\mathcal{E}}^{1/2}(g-p_h\cdot n)\|_{2,\Gamma_N}\Big). \tag{6.7}$$

The second contribution on the right-hand side of (5.5) is analysed with Theorem 3.1, where, as in the proof of Theorem 5.1, we interchange components and the role of the boundary conditions. As already employed in [C1, C2], $\operatorname{curl} w_h \in H(\operatorname{div},\Omega)$ for all $w_h \in \mathcal{S}^1_N(\mathcal{T})$. Moreover, $\operatorname{curl} w_h\cdot n = \partial w_h/\partial s = 0$ on $?_N$. Hence, (6.1) and an integration by parts for p yield (5.7) because of (4.2). Arguing as in the proof of Theorem 5.1, we deduce for arbitrary $q_h \in \mathcal{S}^k(\mathcal{T})^2$ that

$$\int_\Omega (p-p_h)\cdot\operatorname{Curl}\beta\,dx \le \|\nabla\beta\|_2\Big(c_2\|p_h-q_h\|_2 + c_5\|h_{\mathcal{E}}^{1/2}(\partial u_D/\partial s - q_h\cdot t)\|_{2,\Gamma_D} + c_3\Big(\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}}\|\operatorname{curl}_{\mathcal{T}} q_h - f_z\|^2_{2,\Omega_z}\Big)^{1/2}\Big). \tag{6.8}$$

The remaining details are analogous to the proof of Theorem 5.1 and hence omitted. □

The precise version of (1.5) for lowest order mixed finite element methods is summarised as follows.

Corollary 6.1. *Suppose that the discrete flux p_h satisfies $\operatorname{curl}_{\mathcal{T}} p_h = 0$, $\operatorname{div}_{\mathcal{T}} p_h \in \mathcal{L}^0(\mathcal{T})$ and $p_h\cdot n \in \mathcal{L}^0(\mathcal{E}_N)$. Then,*

$$\|p-p_h\|_{L^2(\Omega)} \le \min_{q_h\in\mathcal{S}^1_D(\mathcal{T},u_D)} c_{17}\|p_h-q_h\|_{L^2(\Omega)} + c_{18}\big(\|h_{\mathcal{E}}^{3/2}\partial^2_{\mathcal{E}} u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}} g/\partial s\|_{L^2(\Gamma_N)} + \|h_{\mathcal{T}}^2\nabla_{\mathcal{T}} f\|_{L^2(\Omega)}\big). \tag{6.9}$$

Proof. Combine Theorem 6.1, Lemma 4.1 and 5.2 and use an inverse estimate to prove

$$\sum_{z\in\mathcal{K}} h_z^2\|\operatorname{curl}_{\mathcal{T}} q_h\|^2_{2,\Omega_z} = \sum_{z\in\mathcal{K}} h_z^2\|\operatorname{curl}_{\mathcal{T}}(q_h-p_h)\|^2_{2,\Omega_z} \lesssim \sum_{z\in\mathcal{K}}\|q_h-p_h\|^2_{2,\Omega_z} \lesssim \|q_h-p_h\|_2. \quad \square$$

Remark 6.2. The assumptions in Corollary 6.1 are satisfied for lowest order Raviart-Thomas and Brezzi-Douglas-Fortin-Marini finite elements.

Example 6.1. Assume the conditions of Lemma 5.2 on the data u_D. We define an operator $\mathcal{A} : L^2(\Omega)^2 \to \mathcal{S}^1_D(\mathcal{T},u_D)$ by (4.15) and $p_z := \fint_{\omega_z} p\,dx$ for $z \in \mathcal{N}\setminus ?_D$. In case $z\in\mathcal{N}\cap ?_D$ we consider 2×2 systems

$$t_{E_1}\cdot p_z = \partial u_D|_{E_1}/\partial s(z) \quad\text{and}\quad t_{E_2}\cdot p_z = \partial u_D|_{E_2}/\partial s(z) \tag{6.10a}$$

(cf. (4.16a) and notation from Example 4.1), resp., as an analog to (4.16b),

$$t_{E_1}\cdot p_z = \partial u_D|_{E_1}/\partial s(z) \quad\text{and}\quad n_{E_1}\cdot p_z = \fint_{\omega_z} n_{E_1}\cdot p\,dx. \quad \square \tag{6.10b}$$

A local version follows from Theorem 3.2 and concludes this section on mixed finite element methods.

Corollary 6.2. *Under the conditions of Theorem 6.1 and Corollary 6.1 we have*

$$\|p-p_h\|_{L^2(\Omega)} \le c_{19}\Big(\sum_{E\in\mathcal{E}} \min_{q_E\in\mathcal{S}^1(\mathcal{T}_E)^2} \|p_h-q_E\|^2_{L^2(\omega_E)} + h_E\|u'_{D,h}-q_E\cdot t\|^2_{L^2(\Gamma_D)}\Big)^{1/2}$$
$$+ c_{18}\big(\|h_{\mathcal{E}}^{3/2}\partial^2_{\mathcal{E}}u_D/\partial s^2\|_{L^2(\Gamma_D)} + \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}}g/\partial s\|_{L^2(\Gamma_N)} + \|h_{\mathcal{T}}^2\nabla_{\mathcal{T}}f\|_{L^2(\Omega)}\big). \quad \square$$

Remark 6.3. The results of this section could be generalised to three space dimensions. Details are omitted for brevity.

Remark 6.4. For related residual based a posteriori error estimates we refer to [A, BV, C1, C2, HW].

7. Numerical Experiments

The theoretical results of this paper are supported by numerical experiments. In this section, we report on two examples of the problem (4.1)-(4.3) on uniform, adapted, and perturbed meshes for conforming, nonconforming, and mixed finite element methods.

Example 7.1. Let $f := 0$ on the L–shaped domain $\Omega := (-1,1)^2 \setminus [0,1]\times[-1,0]$, $u_D := 0$ on the Dirichlet boundary $?_D := \{0\}\times[-1,0]\cup[0,1]\times\{0\}$, and on the Neumann boundary $?_N := \partial\Omega \setminus ?_D$,

$$g(r,\varphi) := 2/3\, r^{-1/3}(-\sin(\varphi/3),\cos(\varphi/3))\cdot n$$

using polar coordinates (r,φ). The exact solution $u(r,\varphi) := r^{2/3}\sin(2\varphi/3)$ of (4.1)-(4.3) has a typical corner singularity at the origin. In this example, the right-hand sides are smooth, but the solution is not. The coarsest triangulation $\mathcal{T}_0$ consists of three squares halved by diagonals parallel to the vector $(1,1)$, cf. Fig. 1.

Example 7.2. Let $f := -\Delta u$ for the function $u(x,y) := x(1-x)y(1-y)\arctan(60(r-1))$, $r^2 := (x-1.25)^2+(y+0.25)^2$ on the unit square $\Omega := (0,1)^2$ and set $u_D := 0$ on the entire boundary $?_D := \partial\Omega$ ($?_N = \emptyset$). The solution u to (4.1)-(4.3) is H^2-regular but f (although theoretically smooth) has huge gradients on the circle with radius 1 around $(1.25,-0.25)$. The coarsest triangulation $\mathcal{T}_0$ consists of four squares halved by diagonals parallel to the vector $(1,1)$, cf. Fig. 3.

The following adaptive algorithm generates all the sequences of meshes $\mathcal{T}_0,\mathcal{T}_1,\mathcal{T}_2,\ldots$ in this paper which are uniform for $\Theta = 0$ or adapted for $\Theta = 1/2$ in (7.2). Since the resulting meshes might show local symmetries, we considered meshes that are either unperturbed (relative to $\mathcal{T}_0$) for $\vartheta = 0$ and randomly perturbed for $\vartheta = 1$ in step (e). The implementation was performed in Matlab in the spirit of [ACF] with a direct solution of linear systems of equations. For details on the red-blue-green-refinements we refer to [V].

Algorithm (A^{ϑ}_{Θ}). (a) Start with a coarse mesh $\mathcal{T}_0$, $k = 0$.
(b) Compute the discrete solution p_h on the actual mesh $\mathcal{T}_k$.
(c) Compute error indicators

$$\eta_{Z,T} := \|p_h - \mathcal{A}p_h\|_{L^2(T)} \tag{7.1}$$

for all $T\in\mathcal{T}_k$ and plot energy error $e_N := \|p-p_h\|_{L^2(\Omega)}$ and its estimator $\eta_N^2 := \sum_{T\in\mathcal{T}}\eta^2_{Z,T}$ versus the degree of freedom N of the triangulation $\mathcal{T}_k$.

(d) Mark the element T for *red*-refinement provided

$$\eta_{Z,T} \geq \Theta \max_{T' \in \mathcal{T}_k} \eta_{Z,T'}. \tag{7.2}$$

(e) Mark further elements (*red–blue–green*-refinement) to avoid hanging nodes. Generate a new triangulation $\tilde{\mathcal{T}}_{k+1}$ using edge–midpoints if $\vartheta = 0$ and points on the edges at a random distance at most $0.3\,h_E$ from the edge–midpoints if $\vartheta = 1$. Perturbe the nodes $z \in \mathcal{N}_{k+1}$ of the mesh $\tilde{\mathcal{T}}_{k+1}$ at random with values taken uniformly from a ball around z of radius $\vartheta\, 2^{-k}/15$. Correct boundary nodes by orthogonal projection onto that boundary piece they are expected such that $\Omega, ?_D, ?_N$ are matched by the resulting mesh $\mathcal{T}_{k+1}$ exactly. Update k and go to (b).

7.1. **Results for conforming finite element methods.** In the conforming finite element scheme, we use operator $\mathcal{A}$ from Example 4.1 in (7.1) of Algorithm (A^ϑ_Θ) and report on results obtained for $(\Theta, \vartheta) = (0,0)$ (uniform), $(1/2,0)$ (adaptive), and $(1/2,1)$ (adaptive, perturbed).

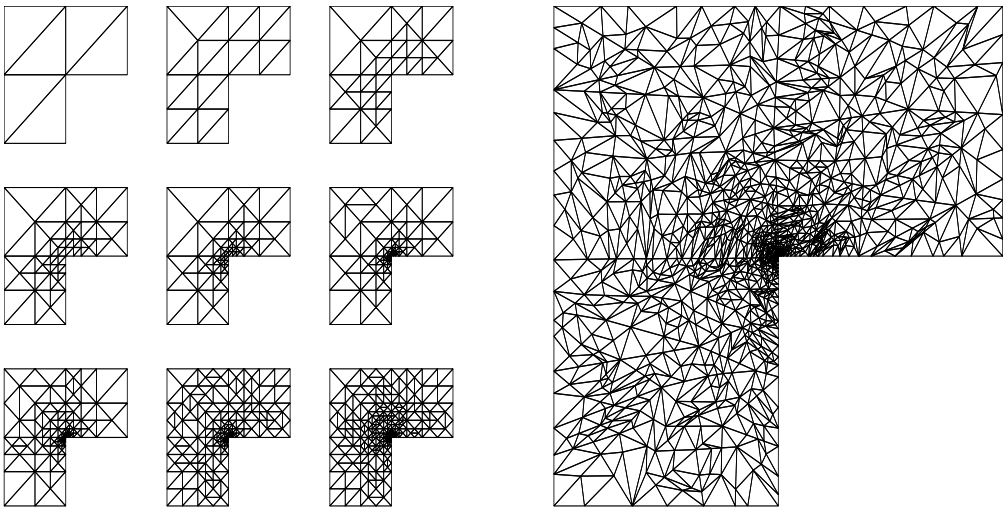

FIGURE 1. Adaptively refined meshes $\mathcal{T}_0$ (left upper) to $\mathcal{T}_8$ (right lower) (left) and perturbed triangulation $\mathcal{T}_{16}$ with 1157 free nodes (right) in Example 7.1 for the conforming finite element scheme

Some obtained meshes for Example 7.1 are shown in Fig. 1 and illustrate a high automatic mesh-refinement of the adapted meshes towards the origin, which is expected to improve the convergence rate of 2/3 possibly to the optimal value 1. The result of the perturbation in step (e) of Algorithm $(A^1_{1/2})$ is seen in the right half of Fig. 1. We believe that the meshes generated by Algorithm $(A^1_{1/2})$ have less local symmetry than that by $(A^0_{1/2})$. According to local extrapolation, symmetry could cause superconvergence phenomena. To check the practical convergence behaviour, we plotted in Fig. 2 for each mesh $\mathcal{T}_k$ an entry (N, e_N) and (N, η_N). A log-scaling on both axes allows a slope $-\alpha$ of a straight line in the plot, that connects two subsequent entries for a series of meshes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ generated by Algorithm (A^ϑ_Θ), to be interpreted as an experimental convergence rate 2α (owing to $N \propto h^{-2}$ in two dimensions). We observe experimental convergence rates 2/3 resp. 1 for uniform resp. adapted meshes (generated by Algorithm (A^ϑ_Θ) for $\Theta = 0$ resp. $\Theta = 1/2$). Furthermore,

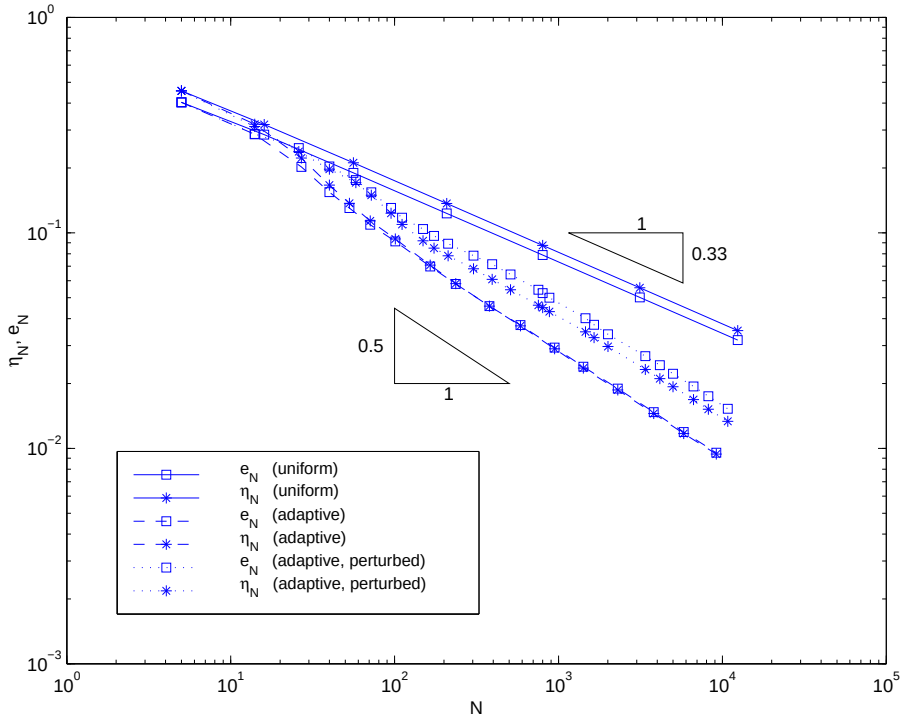

FIGURE 2. Error and error estimator for uniform, adaptive, and perturbed adaptive mesh–refinement in the conforming finite element scheme in Example 7.1

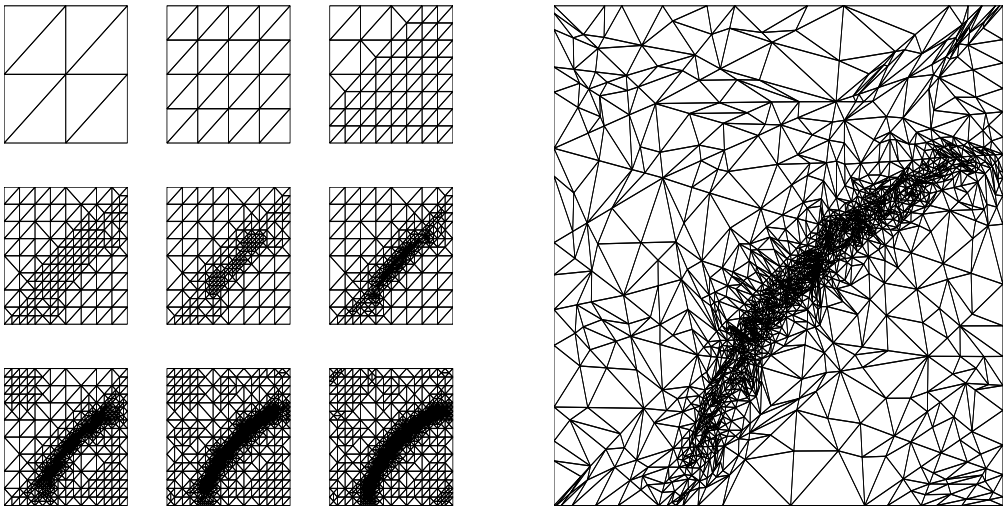

FIGURE 3. Adaptively refined meshes $\mathcal{T}_0$ (left upper) to $\mathcal{T}_8$ (right lower) (left) and perturbed triangulation $\mathcal{T}_{12}$ with 1909 free nodes (right) for the conforming finite element scheme in Example 7.2

even for coarse meshes, η_N appears to be a very good approximation to e_N; corresponding entries almost coincide for $(\Theta, \vartheta) = (1/2, 0)$. If these meshes are perturbed, cf. Fig. 1, the quotient η_N/e_N is almost a constant very close to 1. Numerical checks with different

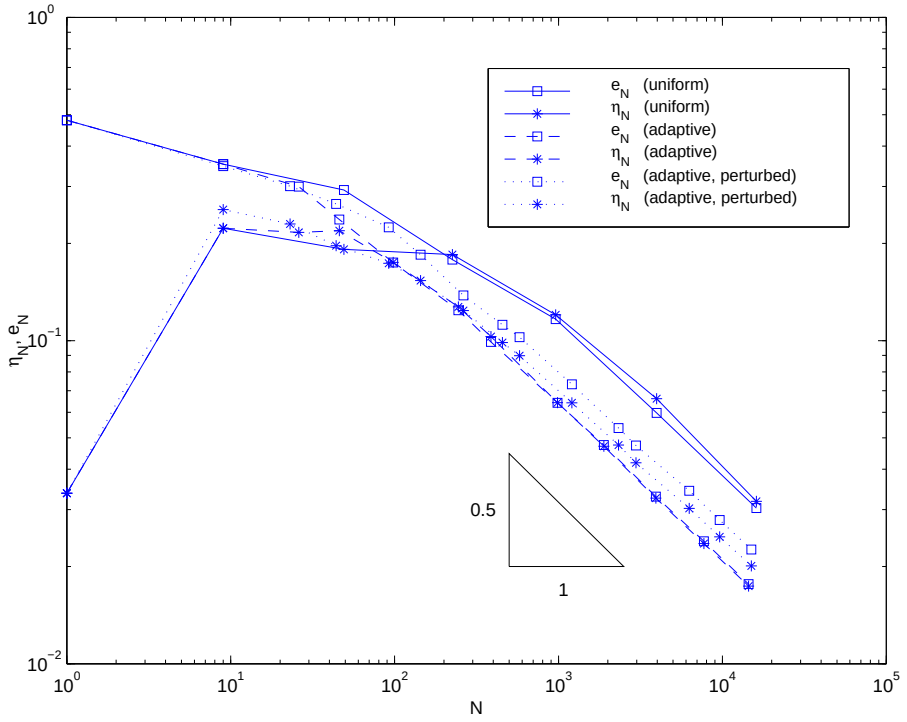

FIGURE 4. Error and error estimator for uniform, adaptive, and perturbed adaptive mesh–refinement in the conforming finite element scheme in Example 7.2

numerical quadrature rules (used to evaluate e_N) convinced us that in general, η_N behaves not asymptotically exact in practise but is very accurate.

In Example 7.2 we obtained meshes and experimental convergence rates displayed in Fig. 3 and 4. Although u belongs to $H^2(\Omega)$ and we expect linear convergence, u has huge second order derivatives along a circular arc where f is steep. We observe high refinements in the adapted meshes towards this arc. In this example the preasymptotic range is very large, an experimental convergence rate 1 can be observed only for $N \geq 300$ for all refinement strategies. In this regime the estimator η_Z appears as a good approximate for e_N and the entries (N, e_N) and (N, η_N) almost conincide for $(\Theta, \vartheta) = (0, 0), (1/2, 0)$. This is not the case for $(\Theta, \vartheta) = (1/2, 1)$ but the quotient $\eta_N / e_N \approx 0.9$ is still close to 1.

7.2. **Results for nonconforming finite element methods.** The operator $\mathcal{A}$ from Example 5.1 serves in (7.1) to define $\eta_{Z,T}$ for first order Crouzeix-Raviart finite elements (cf., e.g., [BS, Ci]) in Algorithm (A^{ϑ}_{Θ}). The generated meshes look similar to those shown in Fig. 1 resp. Fig. 3 and therefore are not displayed in this paper. The experimental convergence rates for Example 7.1 resp. 7.2 are illustrated in Fig. 5 resp. Fig. 6. The overall picture appears similarly to the above discussions and we draw the same conclusions. For uniform meshes, the quotient η_N / e_N is nearly constant 1.2 but significantly larger than 1.1 in Fig. 2.

7.3. **Results for mixed finite element methods.** For the Raviart–Thomas finite element method (cf. [B, BF, BS]) we use the operator $\mathcal{A}$ from Example 6.1 to define $\eta_{Z,T}$. The adapted meshes look similar to those shown in Fig. 1 and Fig. 3 and therefore are not displayed in this paper. Fig. 7 and Fig. 8 display the error and the estimator for the mixed

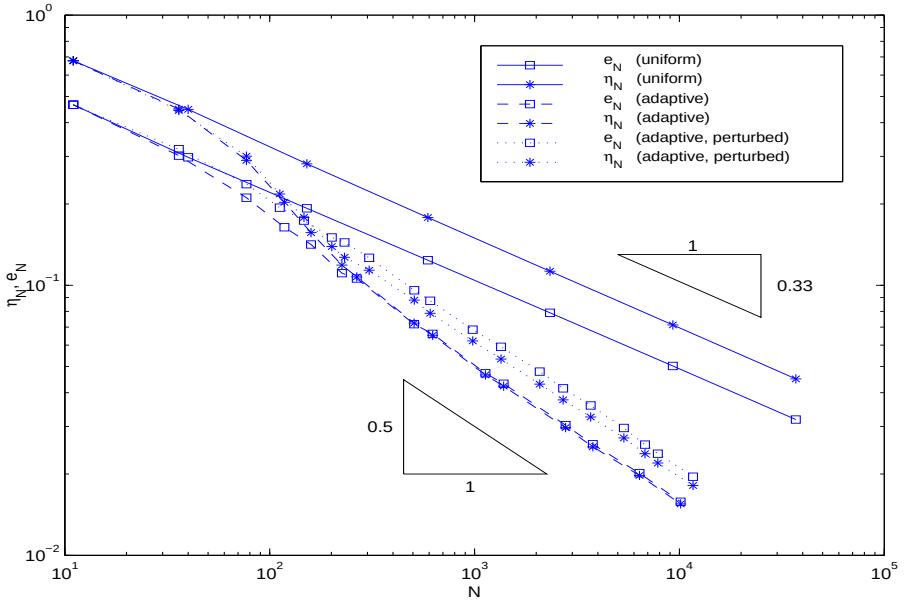

FIGURE 5. Error and error estimator for uniform, adaptive, and perturbed adaptive mesh–refinement in the nonconforming finite element scheme in Example 7.1

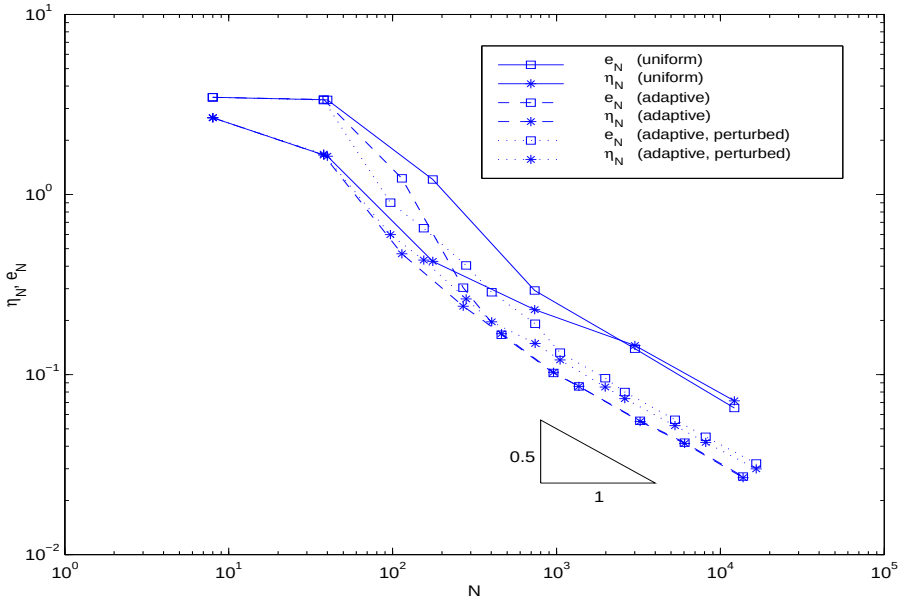

FIGURE 6. Error and error estimator for uniform, adaptive, and perturbed adaptive mesh–refinement in the nonconforming finite element scheme in Example 7.2

finite element scheme in Examples 7.1 and 7.2 obtained from Algorithm (A^{ϑ}_{Θ}) and $(\Theta, \vartheta) = (0,0), (1/2,0), (1/2,1)$. We obtain the same experimental convergence rates as in the previous methods.

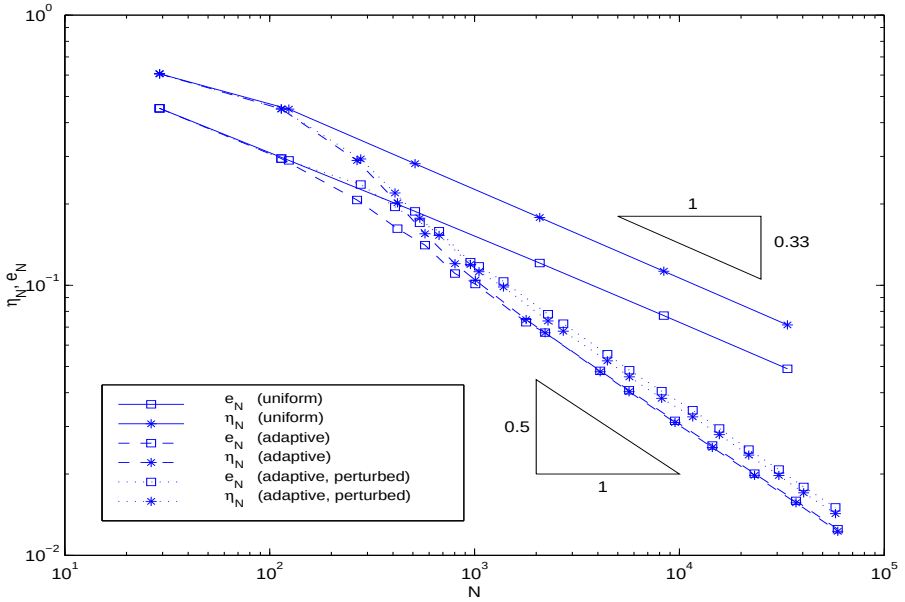

FIGURE 7. Error and error estimator for uniform, adaptive, and perturbed adaptive mesh–refinement in the mixed finite element scheme in Example 7.1

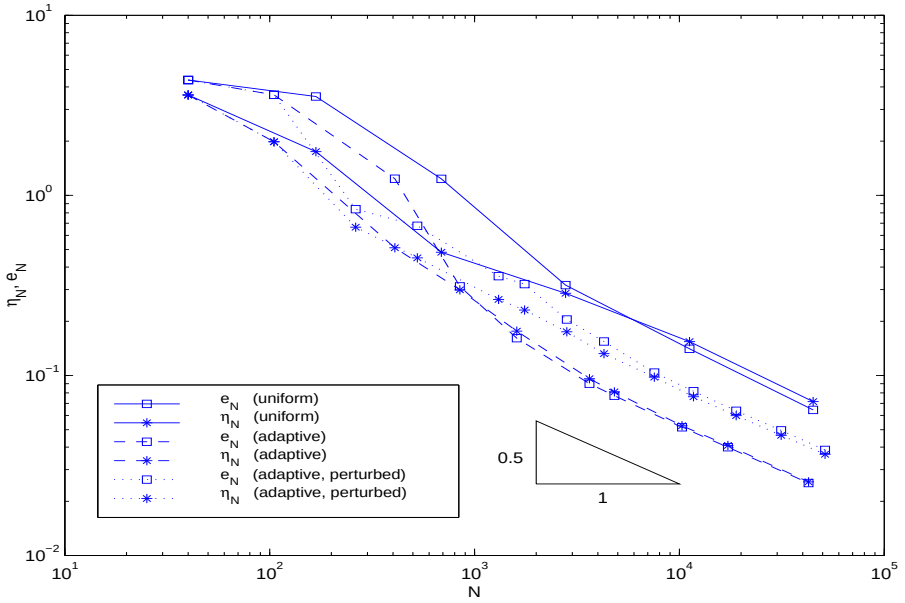

FIGURE 8. Error and error estimator for uniform, adaptive, and perturbed adaptive mesh–refinement in the mixed finite element scheme in Example 7.2

7.4. **Remarks.** (i) Our overall experience with Algorithm (A_Θ^ϑ) and other (e.g. residual–based) adaptive algorithms supports that all such adaptive algorithms yield a considerable convergence improvement.
(ii) Although asymptotic exactness of η_N is not observed, the reliability-constant c_1 in (1.5) and the efficiency constant are experimentally very close to 1 since η_N is a very good approximation to e_N for very fine meshes (i.e. when h.o.t. is neglegible, say, for $N \geq 100$).
(iii) Note that the efficiency constant is not known to be one as η_N (based on the averaging

operator $\mathcal{A}$) is different from

$$\eta_Z := \min_{q_h \in \mathcal{S}^1_N(\mathcal{T},g)} \|p_h - q_h\|_{L^2(\Omega)}. \tag{7.3}$$

For conforming linear triangular finite elements, the efficiency of η_N with an averaging operator follows from [R1, R2].
(iv) Instead of the averaging operator $\mathcal{A}$, we tested the error estimator $\eta_Z \leq \eta_N$ from (7.3) and found that sometimes the performance is poorer than that of η_N: In Fig. 4, for instance, the results of η_Z are much smaller than those of $\eta_N \approx e_N$. The averaging technique suggested in [HSWW], average over a domain of size $O(h\log(1/h))$, is expected to give values between η_N and e_N.
(v) The error estimation in Example 7.1 is very accurate even for very coarse meshes. Hence the higher order terms do not seem to be important here although u is nonsmooth. This agrees with our theoretical prediction in Equation (1.5) since f and u_D are zero and g is piecewise analytic. For a generic corner singularity of u, we expect

$$\min_{q_h \in \mathcal{S}^1_N(\mathcal{T},g)} \|p - q_h\|_{L^2(\Omega)} \ll \min_{q_h \in \mathcal{L}^0(\mathcal{T})} \|p - q_h\|_{L^2(\Omega)} \leq \|p - p_h\|_{L^2(\Omega)}$$

and so the h.o.t. in Equation (1.4) are not expected to be dominant even for coarse meshes.
(vi) For coarse meshes in Example 7.2, higher order terms may cause the overall observation that η_N is much smaller than e_N. Assuming $c_1 \approx 1$ (which is seen for fine meshes, whence for neglegible h.o.t.) non-smooth data (∇f is large) indicate that $e_N \leq c_1\eta_N$ + h.o.t cannot be improved to $e_N \leq c_1\eta_N$.

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References

[ACF] J. Alberty, C. Carstensen, S.A. Funken: Remarks around 50 lines of Matlab: short finite element implementation. Berichtsreihe des Mathematischen Seminars Kiel, Technical report **98-11** Universität Kiel (1998). Num. Alg. accepted for publication. (http://www.numerik.uni-kiel.de/reports/1997/98-11.html).

[A] A. Alonso: Error estimators for a mixed method. Numer. Math. **74** (1996), 385–395.

[BaR] I. Babuška, W.C. Rheinboldt: Error estimates for adaptive finite element computations. SIAM J. Numer. Anal. **15** (1978) 736–754.

[Baetal] I. Babuška, T. Strouboulis, C.S. Upadhyay, S.K. Gangaraj, K. Copps: Validation of a posteriori error estimators by numerical approach. Int. J. Numer. Meth. Engrg. **37** (1994) 1073–1123.

[BeR] R. Becker, R. Rannacher: A feed-back approach to error control in finite element methods: basic analysis and examples. East-West J. Numer. Math., **4**, No. 4 (1996) 237–264.

[B] D. Braess: *Finite Elements.* Cambridge University Press (1997).

[BV] D. Braess, R. Verfürth: A posteriori error estimators for the Raviart-Thomas element. SIAM J. Numer. Anal. **33** (1996) 2431–2444.

[BS] S.C. Brenner, L.R. Scott: *The Mathematical Theory of Finite Element Methods.* Texts Appl. Math. 15, Springer, New-York (1994).

[BF] F. Brezzi, M. Fortin: *Mixed and hybrid finite element methods.* Springer-Verlag (1991).

[C1] C. Carstensen: A posteriori error estimate for the mixed finite element method. Math. Comp. **66** (1997), 465–476.

[C2] C. Carstensen: Weighted Clement-type interpolation and a posteriori analysis for FEM. Berichtsreihe des Mathematischen Seminars Kiel, Technical report **97-19**, Christian–Albrechts–Universität zu

Kiel, Kiel (1997), M^2AN accepted for publication. (http://www.numerik.uni-kiel.de/reports/1997/97-19.html).

[CB] C. Carstensen, S. Bartels: Each Averaging Technique Yields Reliable A Posteriori Error Control in FEM on Unstructured Grids Part II: Higher Order FEM. In preparation (1999).

[CF1] C. Carstensen, S.A. Funken: Constants in Clément-interpolation error and residual based a posteriori error estimates in Finite Element Methods. Berichtsreihe des Mathematischen Seminars Kiel, Technical report **97-11**, Christian–Albrechts–Universität zu Kiel, Kiel (1997). East-West Numer. Anal. (to appear). (http://www.numerik.uni-kiel.de/reports/1997/97-11.html).

[CF2] C. Carstensen, S.A. Funken: A posteriori error control in low-order finite element discretisations of incompressible stationary flow problems. Berichtsreihe des Mathematischen Seminars Kiel, Technical report **99-5**, Christian–Albrechts–Universität zu Kiel, Kiel (1999). (http://www.numerik.uni-kiel.de/reports/1999/99-5.html).

[CF3] C. Carstensen, S.A. Funken: Averaging Techniques for FE-A Posteriori Error Control in Elasticity. Part I: Conforming FEM. In preparation (1999).

[CV] C. Carstensen, R. Verfürth: Edge residuals dominate a posteriori error estimates for low order finite element methods. Berichtsreihe des Mathematischen Seminars Kiel, Technical report **97-6**, Christian–Albrechts–Universität zu Kiel, SIAM J. Numer. Anal. (1998) accepted for publication. (http://www.numerik.uni-kiel.de/reports/1997/97-6.html).

[Cl] P. Clément: Approximation by finite element functions using local regularization. RAIRO Sér. Rouge Anal. Numér. **R-2** (1975) 77–84.

[Ci] P.G. Ciarlet: *The finite element method for elliptic problems.* North-Holland, Amsterdam (1978).

[DDPV] E. Dari, R. Duran, C. Padra, and V. Vampa: A posteriori error estimators for nonconforming finite element methods. Math. Modelling Numer. Anal. **30** (1996) 385–400.

[DMR] R. Duran, M.A. Muschietti, R. Rodriguez: On the asymptotic exactness of error estimators for linear triangular elements. Numer. Math. **59** (1991), 107–127.

[EEHJ] K. Eriksson, D. Estep, P. Hansbo, C. Johnson: Introduction to adaptive methods for differential equations. Acta Numerica (1995) 105–158.

[GR] V. Girault, P.A. Raviart: *Finite Element Methods for Navier-Stokes Equations.* Springer, Berlin (1986).

[HSWW] W. Hoffmann, A.H. Schatz, L.B. Whalbin, G. Wittum: Asymptotically exact a posteriori error estimators for the pointwise gradient error on each element in irregular meshes. Part 1: A smooth problem and globally quasi–uniform meshes. Preprint (1999).

[HW] R.H.W. Hoppe, B. Wohlmuth: Element-orientated and edge-orientated local error estimates for nonconforming finite element methods. Math. Modelling Numer. Anal. **30** (1996) 237–263.

[KS] G. Kanschat, F.-T. Suttmeier: A posteriori error estimates for nonconforming finite element schemes. Preprint 98-56 (SFB 359) (1998) Universität Heidelberg.

[R1] R. Rodriguez: Some remarks on Zienkiewicz–Zhu estimator. Int. J. Numer. Meth. in PDE **10** (1994) 625–635.

[R2] R. Rodriguez: A posteriori error analysis in the finite element method. *Finite element methods. 50 years of the Courant element. Conference held at the University of Jyvaeskylae, Finland, 1993.* Inc. Lect. Notes Pure Appl. Math. 164, 389-397 (1994).

[V] R. Verfürth: *A review of a posteriori error estimation and adaptive mesh-refinement techniques.* Wiley-Teubner (1996).

[ZZ] O.C. Zienkiewicz, J.Z. Zhu: A simple error estimator and adaptive procedure for practical engineering analysis. *Int. J. Numer. Meth. Engrg.* **24** (1987) 337—357.

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