CONTROL OF QUADRATURE

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The use of the midpoint rule in the treatment of the right-hand side in the Poisson problem, i.e., the approximation

$$\ell_h(v_h) = \sum_{T\in\mathcal{T}_h} |T| f(x_T) v_h(x_T)$$

of the exact functional

$$\ell(v_h) = \int_\Omega f v_h \,\mathrm{d}x$$

leads to an error contribution that is controlled by the first Strang lemma. Defining the elementwise constant function $f_h \in L^2(\Omega)$ by requiring that

$$f_h|_T = f(x_T)$$

for all $T \in \mathcal{T}_h$ and noting that the midpoint rule is exact for affine functions we have that

$$\ell_h(v_h) = \sum_{T\in\mathcal{T}_h} f(x_T) \int_T v_h \,\mathrm{d}x = \int_\Omega f_h v_h \,\mathrm{d}x.$$

This implies that

$$\ell(v_h) - \ell_h(v_h) = \int_\Omega (f - f_h) v_h \,\mathrm{d}x.$$

With Hölder's inequality we find that

$$\ell(v_h) - \ell_h(v_h) \le \|f - f_h\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)}.$$

Assuming that $f \in W^{1,\infty}(\Omega)$ or that f is Lipschitz continuous it follows from the mean value theorem that for $T \in \mathcal{T}_h$ and $x \in T$ we have

$$|f(x) - f(x_T)| \le h_T \|\nabla f\|_{L^\infty(T)}.$$

With this estimate we deduce that

$$\|\ell(v_h) - \ell_h(v_h)\|_{V_h'} \le h \|\nabla f\|_{L^\infty(\Omega)}.$$

The argument also applies if f is only elementwise Lipschitz continuous. A quadratic error contribution can be obtained if the midpoint rule is applied with the nodal interpolant $\mathcal{I}_h f$ of f, i.e., if we use

$$\ell_h(v_h) = \sum_{T\in\mathcal{T}_h} |T| \,\mathcal{I}_h f(x_T) v_h(x_T) = \int_\Omega f_h v_h \,\mathrm{d}x,$$

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where f_h is the elementwise constant function with $f_h|_T = \mathcal{I}_h f(x_T)$ for every $T \in \mathcal{T}_h$. Since $\mathcal{I}_h f$ is elementwise affine we have that

$$\int_T \mathcal{I}_h f \,\mathrm{d}x = |T|\,\mathcal{I}_h f(x_T) = \int_T f_h \,\mathrm{d}x,$$

which implies that

$$\int_T (\mathcal{I}_h f - f_h) v_h \,\mathrm{d}x = \int_T (\mathcal{I}_h f - f_h)(v_h - \alpha_T) \,\mathrm{d}x$$

for every $\alpha_T \in \mathbb{R}$. We let $\overline{v}_h$ be the elementwise constant function with

$$\overline{v}_h|_T = \frac{1}{|T|} \int_T v_h \,\mathrm{d}x$$

for every $T \in \mathcal{T}_h$. We thus have that

$$\begin{aligned}
\ell(v_h) - \ell_h(v_h) &= \int_\Omega (f - \mathcal{I}_h f) v_h \,\mathrm{d}x + \int_\Omega (\mathcal{I}_h f - f_h) v_h \,\mathrm{d}x \\
&= \int_\Omega (f - \mathcal{I}_h f) v_h \,\mathrm{d}x + \int_\Omega (\mathcal{I}_h f - f_h)(v_h - \overline{v}_h) \,\mathrm{d}x \\
&\le \|f - \mathcal{I}_h f\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} + \|\mathcal{I}_h f - f_h\|_{L^2(\Omega)} \|v_h - \overline{v}_h\|_{L^2(\Omega)}.
\end{aligned}$$

Using nodal interpolation estimates we find that $\|f - \mathcal{I}_h f\| \le ch^2 \|D^2 f\|$. To control the second term we note that for every $v \in W^{1,2}(T)$ with vanishing integral we have that

$$\|v\|_{L^2(T)} \le c_T h_T \|\nabla v\|_{L^2(T)}$$

with a constant c_T that depends on the geometry of the element T and which is uniformly bounded if $\mathcal{T}_h$ is shape-regular. This leads to the estimate

$$\|\mathcal{I}_h f - f_h\|_{L^2(\Omega)} \le ch \|\nabla \mathcal{I}_h f\|_{L^2(\Omega)}, \quad \|v_h - \overline{v}_h\|_{L^2(\Omega)} \le ch \|\nabla v_h\|_{L^2(\Omega)},$$

and leads to

$$|\ell(v_h) - \ell_h(v_h)| \le ch^2 \|D^2 f\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} + ch^2 \|\nabla \mathcal{I}_h f\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)}.$$

Interpolation estimates and the triangle inequality show that

$$\|\nabla \mathcal{I}_h f\|_{L^2(\Omega)} \le \|\nabla f\|_{L^2(\Omega)} + ch \|D^2 f\|_{L^2(\Omega)}.$$

Combining the estimates we find that

$$\|\ell - \ell_h\|_{V_h'} \le ch^2 \|f\|_{H^2(\Omega)}.$$

The calculations show that it is useful to choose an elementwise polynomial function f_h such that the discrete functional

$$\ell_h(v_h) = \int_\Omega f_h v_h \,\mathrm{d}x$$

can be computed exactly.