STABILITY OF THE TAYLOR–HOOD ELEMENT

We prove the discrete inf-sup condition for the Taylor–Hood element following an argument of [Ver84].

Proposition 0.1 (Taylor–Hood element). *Let*

$$V_h = \mathcal{S}^2_0(\mathcal{T}_h)^d, \quad Q_h = \mathcal{S}^1(\mathcal{T}_h) \cap L^2_0(\Omega),$$

and assume that for every $T \in \mathcal{T}_h$ at most one side belongs to $\partial\Omega$. Then there exists $\beta'' > 0$ such that for all $p_h \in Q_h$ we have

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{\int_\Omega p_h \operatorname{div} v_h \,\mathrm{d}x}{\|\nabla v_h\|} \ge \beta'' \|p_h\|.$$

Proof. Let $p_h \in Q_h$. As in the proof of Lemma 7.3 we find that

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{\int_\Omega p_h \operatorname{div} v_h \,\mathrm{d}x}{\|\nabla v_h\|} \ge \beta' \|p_h\| - c_1 \|h_\mathcal{T} \nabla p_h\|.$$

We show that there exists $c_2 > 0$ such that

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{\int_\Omega p_h \operatorname{div} v_h \,\mathrm{d}x}{\|\nabla v_h\|} \ge c_2 \|h_\mathcal{T} \nabla p_h\|.$$

Multiplying this inequality by c_1/c_2 and adding it to the perturbed inf-sup inequality above then proves the statement. To derive the required inequality, we introduce for every edge $E = [z_1, z_2] \in \mathcal{E}_h$ the quantities

$$b_E = \varphi_{z_1}\varphi_{z_2}, \quad h_E = |z_1 - z_2|, \quad t_E = (z_2 - z_1)/h_E.$$

We have $b_E \in \mathcal{S}^2(\mathcal{T}_h)$ with $\operatorname{supp} b_E \subset \omega_E = \bigcup\{T \in \mathcal{T}_h : E \subset T\}$. Letting

$$\partial_E p_h = \big(p_h(z_2) - p_h(z_1)\big)/h_E,$$

we find that $\nabla p_h|_T \cdot t_E = \partial_E p_h$ for every $T \subset \omega_E$. We define

$$w_h = \sum_{E \in \mathcal{E}_h} \alpha_E t_E b_E,$$

with $\alpha_E = h_E^2 \partial_E p_h$ if $E \not\subset \partial\Omega$ and $\alpha_E = 0$ otherwise. Then, we have

$$\int_\Omega w_h \cdot \nabla p_h \,\mathrm{d}x = \sum_{E \in \mathcal{E}_h} \partial_E p_h h_E^2 \int_{\omega_E} \nabla p_h \cdot t_E b_E \,\mathrm{d}x = c_d \sum_{E \in \mathcal{E}_h \setminus \partial\Omega} |\partial_E p_h|^2 h_E^2 |\omega_E|,$$

where $c_d = d!/(d+2)!$. Since the patches $(\omega_E : E \in \mathcal{E}_h)$ have a finite overlap and $\|\nabla b_E\|_{L^\infty(\omega_E)} \le c h_E^{-1}$, we deduce that

$$\|\nabla w_h\|^2 \le c \sum_{E \in \mathcal{E}_h} \alpha_E^2 \|\nabla b_E\|^2_{L^2(\omega_E)} \le c \sum_{E \in \mathcal{E}_h \setminus \partial\Omega} |\partial_E p_h|^2 h_E^2 |\omega_E|.$$

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We thus find, after integrating by parts and using $w_h|_{\partial\Omega} = 0$, that

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{\int_\Omega p_h \operatorname{div} v_h \,\mathrm{d}x}{\|\nabla v_h\|} \geq c\Big(\sum_{E \in \mathcal{E}_h \setminus \partial\Omega} |\partial_E p_h|^2 h_E^2 |\omega_E| \Big)^{1/2}.$$

Because of the assumption on $\mathcal{T}_h$, every element $T \in \mathcal{T}_h$ has d linearly independent tangent vectors t_E that do not belong to $\partial\Omega$. Hence, the expression on the right-hand side is equivalent to the weighted norm $\|h_\mathcal{T} \nabla p_h\|$. This proves the asserted inequality. $\square$

Remarks 0.2. *(i) If $\Gamma_\mathrm{D} \neq \partial\Omega$ then we have $\mathcal{S}_0^2(\mathcal{T}_h)^d \subset \mathcal{S}_\mathrm{D}^2(\mathcal{T}_h)^d$ so that the supremum in the inf-sup condition becomes larger, the fact that p_h has vanishing integral mean has not been used in the proof.*
(ii) The assumption on the triangulation is of technical nature and can be avoided provided that $\mathcal{T}_h$ contains sufficiently many elements, cf. [BBF13].

References

[BBF13] D. Boffi, F. Brezzi, and M. Fortin, *Mixed finite element methods and applications*, Springer Series in Computational Mathematics, vol. 44, Springer, Heidelberg, 2013.

[Ver84] R. Verfürth, *Error estimates for a mixed finite element approximation of the Stokes equations*, RAIRO Anal. Numér. **18** (1984), no. 2, 175–182.