NUMERICAL SOLUTION OF NONSMOOTH PROBLEMS

SÖREN BARTELS

Abstract. Nonsmooth minimization problems arise in problems involving constraints or models describing certain discontinuities. The existence of solutions is established with classical arguments from the calculus of variations, iterative schemes for the practical computation of discretizations have to be adjusted to the particular features of the problem. Three abstract iterative methods are devised and there application to a model obstacle problem and a problem arising in image processing are discussed.

1. Minimization problems

I.A. **Constraints.** Discretizing partial differential equations or evolution problems by numerical methods typically leads to solving finite-dimensional problems. Often, these are minimization problems or can be interpreted as optimality conditions for minimizing a given functional. We will thus consider the problem of finding a minimum for a function

$$F : A \to \mathbb{R}$$

with a subset $A \subset \mathbb{R}^n$. We will assume that the function is defined on all of $\mathbb{R}^n$ which can be done by formally extending F by the value $+\infty$, i.e., we identify F with its extension

$$F : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}, \quad F|_{\mathbb{R}^n \setminus A} = +\infty.$$

The subset $A \subset \mathbb{R}^n$ may represent the set of those elements in $\mathbb{R}^n$ that satisfy a constraint, i.e., the admissible elements. Via extending the function by the value $+\infty$, the constraint is enforced in the minimization problem, cf. Fig. 1 for an illustration.

I.B. **Existence.** Necessary for the well-posedness of the minimization problem is that F is *proper* in the sense that there exist $w_0 \in \mathbb{R}^n$ and $c_{F,1} \in \mathbb{R}$ such that

$$F(w_0) \le c_{F,1},$$

i.e., requiring that $A \neq \emptyset$. Typically, the function values $F(w)$ become large or unbounded when w becomes large. We describe this by the *coercivity* condition

$$|w_j| \to \infty \implies F(w_j) \to +\infty$$

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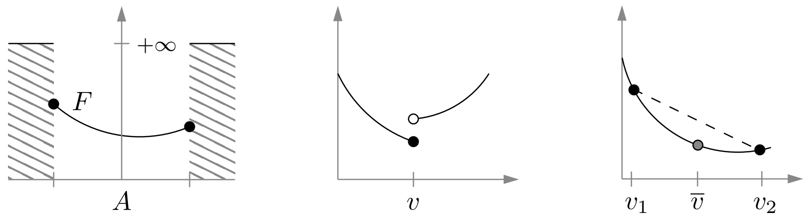

FIGURE 1. Extending a function by the value $+\infty$ outside a closed admissible set (left); examples of lower semicontinuous (middle) and strictly convex functions (right).

for every sequence $(w_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$. This condition implies that if we have a sequence for which the function values are uniformly bounded, i.e., $F(w_j) \le c_0$ for all $j \in \mathbb{N}$, then the sequence itself is bounded. We establish the existence of a sequence $(w_j)_{j\in\mathbb{N}}$ with finite function values by assuming that F is *bounded from below*, i.e., there exists $c_{F,2} \in \mathbb{R}$ such

$$F(w) \ge c_{F,2}$$

for all $w \in \mathbb{R}^n$. This implies that

$$c_{F,1} \ge \inf_{w\in\mathbb{R}^n} F(w) \ge c_{F,2}$$

and hence by definition of the infimum there exists an *infimizing sequence* $(w_j)_{j\in\mathbb{N}}$ with

$$\lim_{j\to\infty} F(w_j) = \inf_{v\in\mathbb{R}^n} F(v).$$

By the coercivity condition the sequence $(w_j)_{j\in\mathbb{N}}$ is bounded in $\mathbb{R}^n$. It hence has accumulation points. To show that these are minimizers for F, we assume that F is *lower semicontinuous* on a sufficiently large set, e.g., on the closure $\overline{V}$ of the sublevel set

$$V = \{w \in \mathbb{R}^n : F(w) \le F(w_0) \le c_{F,1}\}.$$

Lower semicontinuity of F restricted to $\overline{V}$ means that whenever we are given a sequence $(v_k)_{k\in\mathbb{N}} \subset \overline{V}$ that converges to $v \in \overline{V}$ we have that

$$F(v) \le \liminf_{k\to\infty} F(v_k),$$

cf. Fig. 1. If F is continuous on $\overline{V}$ then this holds with equality and the limit inferior can be replaced by the usual limit. Applying the lower semicontinuity to a convergent subsequence $(w_{j_k})_{k\in\mathbb{N}}$ with limit $w \in \mathbb{R}^n$ we have that

$$F(w) \le \liminf_{k\to\infty} F(w_{j_k}) = \inf_{v\in\mathbb{R}^n} F(v).$$

Here we used that except for finitely many members, the infimizing sequence belongs to $\overline{V}$ and hence also w. We have thus established in a nonconstructive way the *existence* of a minimizer for F.

I.C. **Uniqueness.** Uniqueness of minimizers can be shown if $\overline{V}$ is convex and F is *strictly convex* in $\overline{V}$, i.e., for all distinct $v_1, v_2 \in \overline{V}$ we have that $\overline{v} = (v_1 + v_2)/2 \in \overline{V}$ and

$$F(\overline{v}) < \frac{1}{2}F(v_1) + \frac{1}{2}F(v_2),$$

i.e., the graph of F lies below the secant defined by v_1 and v_2, cf. Fig. 1. The existence of two minimizers $w, w' \in \overline{V}$ leads to a contradiction since we then could construct an element with strictly lower function value. We have thus proved the following proposition.

Proposition I.1 (Existence and uniqueness)**.** *Let $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, coercive, bounded from below, and lower semicontinuous on the closure $\overline{V}$ of the sublevel set of an admissible element. Then there exists a minimizer on $\mathbb{R}^n$ for F which belongs to $\overline{V}$. There exists a unique minimizer if F is in addition strictly convex on $\overline{V}$.*

The same concepts apply to minimization problems on reflexive Banach spaces, i.e., for minimizing $F : X \to \overline{\mathbb{R}}$, cf. , e.g., [ABM06]. In this case coercivity implies existence of weakly convergent subsequences of infimizing sequences and lower semicontiuity has to be formulated with respect to weak convergence, which is a stronger condition in general.

2. Elementary examples

II.A. **Nondifferentiability.** We discuss a simple but prototypical nondifferentiable minimization problem that can be solved explicitly. For this we let $\widetilde{w} \in \mathbb{R}^n$, $\gamma > 0$, and define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(w) = \frac{1}{2}|w - \widetilde{w}|^2 + \gamma|w|.$$

The function F is proper, bounded from below by zero, coercive since $\gamma|w| \le F(w)$, and continuous on $\mathbb{R}^n$. Moreover, it is strictly convex since for $v_1, v_2 \in \mathbb{R}^n$ and $\overline{v} = (v_1 + v_2)/2$ we have by an application of the triangle inequality that

$$\gamma|\overline{v}| \le \frac{\gamma}{2}|v_1| + \frac{\gamma}{2}|v_2|$$

and, using $(a + b)^2 \le a^2 + b^2$ for $a, b \ge 0$,

$$\frac{1}{2}|\overline{v} - \widetilde{w}|^2 \le \frac{1}{2}\big(\frac{1}{2}|v_1 - \widetilde{w}| + \frac{1}{2}|v_2 - \widetilde{w}|\big)^2 \le \frac{1}{8}|v_1 - \widetilde{w}|^2 + \frac{1}{8}|v_2 - \widetilde{w}|^2.$$

These estimates imply that

$$F(\overline{v}) \le \frac{1}{2}\big(F(v_1) + F(v_2)\big)$$

with equality if and only if $v_1 = v_2$. To specify the unique minimizer w we distinguish two cases. Assume first that it satisfies $w \ne 0$. Noting that F is differentiable in $\mathbb{R}^n \setminus \{0\}$, the optimality condition reads

$$0 = DF(w) = w - \widetilde{w} + \gamma\frac{w}{|w|}.$$

The equivalent identity

$$\widetilde{w} = \big(|w| + \gamma\big)\frac{w}{|w|}$$

implies that w is parallel to the nonvanishing vector $\widetilde{w}$. Taking the modulus of the identity shows that

$$|\widetilde{w}| = |w| + \gamma \quad \Longleftrightarrow \quad |w| = |\widetilde{w}| - \gamma.$$

It thus follows that

$$w = \big(|\widetilde{w}| - \gamma\big)\frac{\widetilde{w}}{|\widetilde{w}|}.$$

Assume next that $w = 0$. To verify for which vectors $\widetilde{w}$ this is the case, let $v \in \mathbb{R}^n$ be arbitrary and consider for $t \geq 0$ the function

$$\varphi(t) = F(tv) = \frac{1}{2}|tv - \widetilde{w}|^2 + \gamma t|v|.$$

As φ is minimal for $t = 0$ we have for the right-sided derivative of φ at $t = 0$ that

$$0 \leq \varphi'(0) = -\widetilde{w} \cdot v + \gamma|v|.$$

This is the case for all $v \in \mathbb{R}^n$ if and only if $|\widetilde{w}| \leq \gamma$. We thus have that the solution of the minimization of F is given by

$$\begin{aligned} w &= \begin{cases} \big(|\widetilde{w}| - \gamma\big)\frac{\widetilde{w}}{|\widetilde{w}|} & \text{if } |\widetilde{w}| \geq \gamma \\ 0 & \text{if } |\widetilde{w}| \leq \gamma \end{cases} \\ &= \big(|\widetilde{w}| - \gamma\big)_+ \frac{\widetilde{w}}{|\widetilde{w}|}, \end{aligned}$$

where $(x)_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$.

II.B. **Nonlinear projection.** The second example concerns a function that is finite only on a closed ball. To define this in a concise way, we use the *indicator functional* $I_A : \mathbb{R}^n \to \overline{\mathbb{R}}$ associated with a set $A \subset \mathbb{R}^n$ via

$$I_A(w) = \begin{cases} 0 & \text{if } w \in A, \\ +\infty & \text{if } w \notin A. \end{cases}$$

By $K_r(0)$ we denote the closed ball of radius r in $\mathbb{R}^n$. Let $\widetilde{w} \in \mathbb{R}^n$, $r > 0$, and define $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$F(w) = \frac{1}{2}|w - \widetilde{w}|^2 + I_{K_r(0)}(w).$$

The function F is proper, bounded from below by zero, coercive, and continuous on the set $K_r(0)$. Moreover, it is strictly convex on the set $K_r(0)$. The unique minimizer is the best-approximation of $\widetilde{w}$ in $K_r(0)$, i.e.,

$$\begin{aligned} w &= \begin{cases} \widetilde{w} & \text{if } |\widetilde{w}| \leq r \\ r\frac{\widetilde{w}}{|\widetilde{w}|} & \text{if } |\widetilde{w}| \geq r \end{cases} \\ &= r\frac{\widetilde{w}}{\max\{r, |\widetilde{w}|\}}. \end{aligned}$$

3. Convex analysis

III.A. **Subgradients.** The examples from the previous section can be understood with techniques from convex analysis. We let $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, convex, and lower semicontinuous function. A generalization of derivatives is obtained via the *subdifferential* of F defined for $w \in \mathbb{R}^n$ as the set $\partial F(w) \subset \mathbb{R}^n$ consisting of all $s \in \mathbb{R}^n$ such that

$$s \cdot (v - w) + F(w) \leq F(v)$$

for all $v \in \mathbb{R}^n$, i.e., the set of all slopes $s \in \mathbb{R}^n$ that define a tangent plane to the graph of F in $(w, F(w))$. The elements in $\partial F(w)$ are called *subgradients* of F at w. A necessary condition for a minimizer w of F is that

$$0 \in \partial F(w).$$

Note that $\partial F(w) = \emptyset$ if $F(w) = +\infty$ and that $\partial F(w) = \{DF(w)\}$ if F is differentiable at w. Moreover, if G is convex and differentiable then we have that, cf. [ET76],

$$\partial(F + G)(w) = \partial F(w) + DG(w).$$

The following examples are illustrated in Fig. 2.

Examples III.1. *(i) For the indicator functional $I_{K_r(0)}$ we have*

$$\partial I_{K_r(0)}(w) = \begin{cases} \{0\} & \text{if } |w| < r, \\ \{sw : s \geq 0\} & \text{if } |w| = r, \\ \emptyset & \text{if } |w| > r. \end{cases}$$

(ii) For the function $w \mapsto \gamma|w|$ we have

$$\partial\gamma|w| = \begin{cases} \gamma\frac{w}{|w|} & \text{if } w \neq 0, \\ K_\gamma(0) & \text{if } w = 0. \end{cases}$$

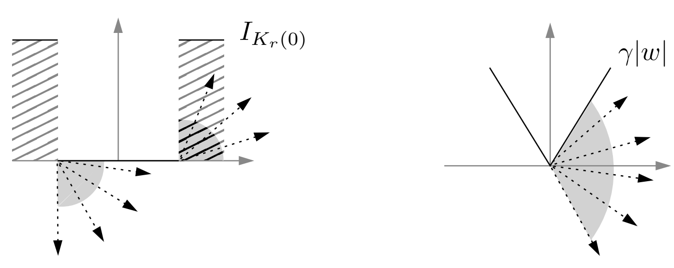

Figure 2. Typical subgradients of an indicator functional (left) and a degree-one homogeneous function (right).

III.B. **Convex conjugate.** In many minimization problems the concept of *convex duality* reveals important information about the problem. It is based on the *convex conjugate* F^* of a function F which is defined via

$$F^*(s) = \sup_{w \in \mathbb{R}^n} w \cdot s - F(w).$$

The convex conjugate is a convex function that is lower semicontinuous. The arguments of F^* are slopes related to F at w. To see this, assume that $F : \mathbb{R} \to \mathbb{R}$ is continously differentiable and F' is strictly monotone. Then, we may regard $s = F'(w)$ as the independent variable and the function

$$G(s) = w \cdot s - F(w)$$

satisfies $G'(s) = w$. A geometric interpretation of F^* is shown in Fig. 3, cf. [ZRM09]. The quantity $-F^*(s)$ is the value of the tangent to F in w at 0. If F is not differentible then the minimal one of all such value is chosen.

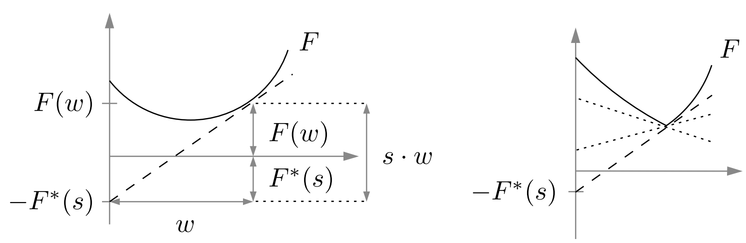

Figure 3. Geometric construction of the convex conjugate for differentiable (left) and nondifferentiable functions (right).

We have that $F = F^{**}$ if and only if F is convex and lower semicontinuous. We always have the *Fenchel inequality*

$$s \cdot w \le F(w) + F^*(s),$$

where equality holds if and only if $s \in \partial F(w)$, cf. Fig. 3. We also have the equivalence

$$s \in \partial F(w) \quad \Longleftrightarrow \quad w \in \partial F^*(s).$$

Formally, this means that $(\partial F^*)^{-1} = \partial F$.

Examples III.2. *(i) We have that*

$$I^*_{K_\gamma(0)}(s) = \gamma |s|.$$

(ii) For $1 < p, q < \infty$ *with* $1/p + 1/q = 1$ *we have*

$$\Big(\frac{1}{p}|\cdot|^p\Big)^* = \Big(\frac{1}{q}|\cdot|^q\Big).$$

III.C. **Proximity operators.** For a convex, proper, lower semicontinuous function $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ which is bounded from below, we have that the mapping

$$w \mapsto \frac{1}{2}|w - \widetilde{w}|^2 + F(w)$$

has a unique minimizer which we refer to by the *proximity operator* $\operatorname{prox}_F : \mathbb{R}^n \to \mathbb{R}^n$, i.e.,

$$\operatorname{prox}_F(\widetilde{w}) = \operatorname{argmin}_{w \in \mathbb{R}^n} \frac{1}{2}|w - \widetilde{w}|^2 + F(w).$$

The optimality condition for the minimizer reads

$$0 \in (w - \widetilde{w}) + \partial F(w) \quad \Longleftrightarrow \quad \widetilde{w} \in (I + \partial F)(w),$$

which motivates the notation $\operatorname{prox}_F(\widetilde{w}) = (I + \partial F)^{-1}(\widetilde{w})$. The proximity operator for a function F can be expressed in terms of its convex conjugate F^*.

Remarks III.3. *(i)* Moreau's identity *states that we have*

$$\operatorname{prox}_{\gamma F^*}(\widetilde{w}) = w - \gamma \operatorname{prox}_F(\widetilde{w}/\gamma).$$

(ii) Note that prox_F *can be equivalently defined via minimizing*

$$w \mapsto \frac{1}{2}|w|^2 - w \cdot \widetilde{w} + F(w).$$

Moreau's identity applies to indicator functionals and provides the proximity operator of the degree-one homogeneous function I_K^*.

Example III.4. *For* $F = I_{K_1(0)}$ *we have* $F^* = |\cdot|$ *and*

$$\begin{aligned}\operatorname{prox}_{\gamma|\cdot|}(\widetilde{w}) &= \widetilde{w} - \gamma \operatorname{prox}_{I_{K_1(0)}}(\widetilde{w}/\gamma) \\ &= \widetilde{w} - \gamma \begin{cases} \widetilde{w}/\gamma, & |\widetilde{w}|/\gamma \le 1, \\ \widetilde{w}/|\widetilde{w}|, & |\widetilde{w}|/\gamma \ge 1, \end{cases} \\ &= (|\widetilde{w}| - \gamma)_+ \widetilde{w}/|\widetilde{w}|,\end{aligned}$$

cf. Section 2.

III.D. **Duality.** We consider the minimization of the sum of two convex functions, i.e., the problem of minimizing

$$J(w) = F(Bw) + G(w)$$

on a Banach space X, e.g., $X = \mathbb{R}^n$, with a bounded linear operator $B : X \to Y$, for another Banach space Y, e.g., $Y = \mathbb{R}^m$. Noting that for proper, convex, lower semicontinuous functions we have $F^{**} = F$, we may replace

$F(Bw) = F^{**}(Bw)$ by a supremum to derive the identity

$$\begin{aligned}\inf_w J(w) &= \inf_w \big[F(Bw) + G(w)\big] \\ &= \inf_w \sup_s \big[(Bw)\cdot s - F^*(s) + G(w)\big] \\ &\ge \sup_s \inf_w \big[-F^*(s) + (Bw)\cdot s + G(w)\big] \\ &= \sup_s \big[-F^*(s) + \inf_w \big(w\cdot(B's) + G(w)\big)\big] \\ &= \sup_s \big[-F^*(s) - \sup_w \big(w\cdot(-B's) - G(w)\big)\big] \\ &= \sup_s \big[-F^*(s) - G^*(-B's)\big].\end{aligned}$$

In this derivation we used that exchanging the order of the infimum and the supremum results in an inequality, we wrote the infimum as the negative of a supremum, and we used the adjoint or transpose B' of the operator B. If, e.g., G is continuous then the inequality is an equality, i.e., we have *strong duality* in the sense that

$$\inf_w \big[F(Bw) + G(w)\big] = \sup_s \big[-F^*(s) - G^*(-B's)\big],$$

cf. [ET76] for details. We have expressed the minimization problem as a maximization problem of the functional

$$K(s) = -F^*(s) - G^*(-B's),$$

which is called the *dual problem* associated with the *primal problem* defined by the minimization of J. A geometric explanation of the duality relation with $B = I$ is provided in Fig. 4. The primal problem seeks a point w that minimizes the vertical distance between the graphs of F and $-G$. The dual problem seeks a slope s such that the distance between parallel tangents with slopes s to F and $-G$ is maximal. From the definitions of F^{**} and G^* we find that the optimal values s and w are related via the inclusions

$$Bw \in \partial F^*(s), \quad -B's \in \partial G(w),$$

which are in fact optimality conditions for the saddle-point problem occurring in the derivation of the dual problem.

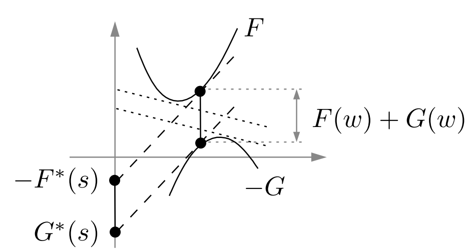

FIGURE 4. Geometric interpretation of the duality relation. The minimal vertical distance between F and $-G$ equals the maximal distance between parallel tangents to F and $-G$.

4. Iterative solution methods

IV.A. **Splitting method.** A general approach to the practical minimization of the functional

$$J(w) = F(Bw) + G(w)$$

is based on introducing the new variable $y = Bw$ and imposing this equality via a Lagrange multiplier and a regularizing term, i.e., considering the augmented Lagrange functional

$$\widetilde{L}_\tau(w, y; \lambda) = F(y) + G(w) + \lambda \cdot (Bw - y) + \frac{\tau}{2}\|Bw - y\|_Y^2.$$

Here we let $\tau > 0$ be a parameter and assume that Y is a Hilbert space. Minimizing J is now equivalent to finding a saddle point for $\widetilde{L}_\tau$, i.e.,

$$\inf_{w \in X} J(w) = \inf_{(w,y) \in X \times Y} \sup_{\lambda \in Y} \widetilde{L}_\tau(w, y; \lambda).$$

To determine a saddle point, we alternatingly minimize with respect to w and y and carry out an ascent step with respect to λ.

Algorithm IV.1 (Splitting method). *Choose $y^{-1}, \lambda^0 \in Y$ and $\tau > 0$ and set $j = 0$.*
(1) Compute the minimizer $w^j \in X$ for the mapping

$$w \mapsto \widetilde{L}_\tau(w, y^{j-1}; \lambda^j).$$

(2) Compute the minimizer $y^j \in Y$ for the mapping

$$y \mapsto \widetilde{L}_\tau(w^j, y; \lambda^j).$$

(3) Compute the maximizer $\lambda^{j+1} \in Y$ for the mapping

$$\lambda \mapsto \frac{-1}{2\tau}\|\lambda - \lambda^j\|_Y^2 + \widetilde{L}_\tau(w^j, y^j, \lambda),$$

i.e., set $\lambda^{j+1} = \lambda^j + \tau(Bw^j - y^j)$.
(4) Stop if $\|y^j - y^{j-1}\|_Y + \|Bw^j - y^j\|_Y \le \varepsilon_{stop}$; increase $j \to j+1$ and continue with (1) otherwise.

The algorithm always terminates; global convergence holds if F or G is strongly convex, cf. [FG83]. The important feature of the algorithm is that the minimization of F and G is decoupled. For certain simple functionals such as degree-one homogeneous functions or indicator functionals as discussed in Section 2, the minimization with respect to w or y can thus be computed explicitly, i.e., they are obtained via evaluating proximity operators. In the maximization with respect to λ the additional term

$$\frac{-1}{2\tau}\|\lambda - \lambda^j\|_Y^2$$

corresponds to an ascent or gradient method. Including this quadratic term is crucial for the convergence analysis. We remark that the splitting algorithm can also be used to solve the dual problem via minimizing $-K$.

IV.B. **Primal-dual method.** Another approach to solving the convex minimization problem defined by the functional J is based on considering the saddle-point problem

$$\inf_w \sup_s -F^*(s) + Bw \cdot s + G(w)$$

which occurs in the passage from the primal to the dual formulation. The following algorithm alternatingly carries out descent and ascent steps with respect to w and s, respectively. This corresponds to evaluating proximity operators related to F^* and G. To guarantee convergence under minimal conditions on the involved step size $\tau > 0$, the approximations of the variable w are extrapolated in the first step of the algorithm.

Algorithm IV.2 (Primal-dual method)**.** *Choose* $w^{-1} \in X$ *and set* $d_t w^{-1} = 0$ *and* $j = 0$.
(1) Define the extrapolated value

$$\widetilde{w}^j = w^{j-1} + d_t w^{j-1}.$$

(2) Compute a maximizer $s^j \in Y$ *for the mapping*

$$s \mapsto \frac{-1}{2\tau}\|s - s^{j-1}\|_Y^2 + B\widetilde{w}^j \cdot s - F^*(s).$$

(3) Compute a minimizer $w^j \in X$ *for the mapping*

$$w \mapsto \frac{1}{2\tau}\|w - w^{j-1}\|_X^2 + Bw \cdot s^j + G(w).$$

(4) Stop if $\|w^j - w^{j-1}\|_X \le \varepsilon_{stop}$*; set* $d_t w^j = (w^j - w^{j-1})/\tau$*, increase* $j \to j+1$*, and continue with (1) otherwise.*

Global convergence of the iteration can be proved if the step size τ is chosen sufficiently small so that $\tau\|B\|_{\mathrm{L}(X,Y)} < 1$, cf., e.g., [Bar15b].

IV.C. **Semismooth Newton method.** In some situations the optimality conditions related to a nonsmooth minimization problem can be reformulated as a nonlinear, nondifferentiable system of equations. A generalized notion of derivative allows us to formulate and analyze a generalized Newton scheme.

Definition IV.3. *The mapping* $N : X \to X$ *is called* Newton differentiable *at* $x \in X$ *if there exist* $\varepsilon > 0$ *and a function* $\delta N : B_\varepsilon(x) \to \mathrm{L}(X, X)$ *such that*

$$\lim_{z \to 0} \frac{\|N(x+z) - N(x) - \delta N(x+z)[z]\|_X}{\|z\|_X} = 0.$$

The function δN *is called* Newton derivative *of* N *at* x*.*

Note that in contrast to the definition of the classical derivative, the function δN is evaluated at the perturbed point $x + z$. This is exactly the quantity that arises in the analysis of the classical Newton scheme.

Example IV.4. *The function* $N : \mathbb{R} \to \mathbb{R}$, $x \mapsto \max\{x, 0\}$, *is Newton differentiable with Newton derivative* $\delta N(x) = 0$ *for* $x < 0$, $\delta N(0) = s$ *for arbitrary* $s \in [0, 1]$, *and* $\delta N(x) = 1$ *for* $x > 0$.

Assuming that N is Newton differentiable with invertible Newton derivative, we may formulate the following scheme.

Algorithm IV.5 (Newton method)**.** *Let* $x^0 \in X$ *and set* $j = 0$.
(1) Compute $d^{j+1} \in X$ *such that*

$$\delta N(x^j)[d^{j+1}] = -N(x^j).$$

(2) Stop if $\|d^{j+1}\|_X \le \varepsilon_{stop}$; *set* $x^{j+1} = x^j + d^{j+1}$, *increase* $j \to j + 1$, *and continue with (1) otherwise.*

The Newton method is locally superlinearly convergent if $N(\overline{x}) = 0$ and δN is boundedly invertible in a neighbourhood of $\overline{x}$. Note that the method coincides with the classical Newton scheme if N is differentiable.

5. Obstacle problem

V.A. **Mathematical model.** The obstacle problem models the constrained deflection $u : \Omega \to \mathbb{R}$ of an elastic membrane occupying the domain $\Omega \subset \mathbb{R}^2$ subject to a vertical force $f : \Omega \to \mathbb{R}$. The constraint is determined by a function $\chi : \Omega \to \mathbb{R}$ that models an obstacle, cf. Fig. 5.

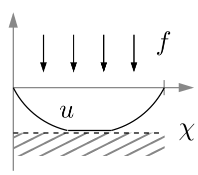

FIGURE 5. Schematical description of the obstacle problem.

The mathematical model assumes $|\nabla u| \ll 1$ and leads to the following minimization problem, cf., e.g., [Rod87]:

$$\begin{cases} \text{Minimize} \quad I(u) = \dfrac{1}{2} \displaystyle\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x \\ \text{subject to} \quad u \ge \chi \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega. \end{cases}$$

With the convex set

$$K = \{v \in H^1_0(\Omega) : v \ge \chi \text{ in } \Omega\}$$

the minimization problem can be recast as minimizing

$$J(u) = I(u) + I_K(u).$$

Since I and I_K are weakly lower semicontinuous on $H^1_0(\Omega)$, and since I is strictly convex, it follows that there exists a unique solution $u \in K$. It satisfies the optimality condition

$$0 \in DI(u) + \partial I_K(u),$$

i.e., there exists a *Lagrange multiplier* $\lambda \in \partial I_K(u)$ such that

$$0 = -\Delta u - f + \lambda.$$

We have that $\lambda \le 0$ in Ω and $\lambda = 0$ in the complement of the coincidence set $\{x \in \Omega : u(x) = \chi(x)\}$.

Remark V.1. *The solution $u \in K$ of the obstacle problem is equivalently characterized by the variational inequality*

$$\big(\nabla u, \nabla(v-u)\big) \ge \big(f, v-u\big)$$

for all $v \in K$ with the L^2 scalar product $(\cdot,\cdot)$. This is obtained, e.g., by computing the right-sided derivative at 0 of the mapping $t \mapsto J(u+t(v-u))$.

V.B. **Complementarity.** The properties of the Lagrange multiplier lead to the complementarity principle

$$\lambda(u-\chi) = 0, \quad \lambda \le 0, \quad u - \chi \ge 0.$$

Given an arbitrary number $c > 0$ these relations are satisfied if and only if the complementarity function

$$\mathcal{C}(u,\lambda) = \lambda - \min\{0, \lambda + c(u-\chi)\}$$

vanishes in Ω. The solution of the obstacle problem is thus equivalent to determining a pair (u,λ) such that

$$-\Delta u - f + \lambda = 0, \quad \mathcal{C}(u,\lambda) = 0.$$

V.C. **Discretization.** We approximate the obstacle problem with a finite element method, i.e., we look for a function u_h in the finite-dimensional subspace $\mathcal{S}^1_0(\mathcal{T}_h)$ of $H^1_0(\Omega)$ defined for a partition $\mathcal{T}_h$ of Ω into triangles or tetrahedra via

$$\mathcal{S}^1(\mathcal{T}_h) = \{v_h \in C(\overline{\Omega}) : v_h|_T \text{ affine for all } T \in \mathcal{T}_h\}$$

and

$$\mathcal{S}^1_0(\mathcal{T}_h) = \mathcal{S}^1(\mathcal{T}_h) \cap H^1_0(\Omega).$$

There exists a nodal basis $(\varphi_z : z \in \mathcal{N}_h)$ associated with the set $\mathcal{N}_h$ of vertices of triangles or tetrahedra such that

$$v_h = \sum_{z \in \mathcal{N}_h} v_h(z)\varphi_z$$

for every $v_h \in \mathcal{S}^1(\mathcal{T}_h)$. Choosing an approximation $\chi_h \in \mathcal{S}^1(\mathcal{T}_h)$ of the obstacle χ, the discretized obstacle problem reads:

$$\begin{cases} \text{Minimize} \quad I(u_h) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u_h|^2 \,\mathrm{d}x - \int_\Omega f u_h \,\mathrm{d}x \\ \text{subject to} \quad u_h \ge \chi_h \text{ in } \Omega,\ u_h = 0 \text{ on } \partial\Omega. \end{cases}$$

If Ω is convex and $\chi \in H^2(\Omega)$ then we have $u \in H^2(\Omega)$ and the quasi-optimal error estimate

$$\|\nabla(u-u_h)\|_{L^2(\Omega)} \le ch\|D^2 u\|_{L^2(\Omega)},$$

cf. [Cia78, GLT81], where $h > 0$ is the maximal mesh-size of the triangulation $\mathcal{T}_h$. Note that in general we have $u \notin H^3(\Omega)$ even if $\chi \in C^\infty(\overline{\Omega})$, cf. [KS80].

V.D. **Semismooth Newton scheme.** Arguing as in the continuous situation the discrete obstacle problem is equivalent to solving the system of equations

$$AU + \Lambda = B, \quad U \geq Z, \quad \Lambda \leq 0, \quad \Lambda_i (U - Z)_i = 0, \ i = 1, 2, \dots, L.$$

Here, A is the finite element stiffness matrix, $U, \Lambda, Z \in \mathbb{R}^L$ are vectors that contain the nodal values of u_h, λ_h, χ_h and B is an approximation of the right-hand side functional defined by f. The equations are equivalent to

$$AU + \Lambda = B, \quad \mathcal{C}(U, \Lambda) = \Lambda - \min\{0, \Lambda + c(U - Z)\} = 0,$$

where the minimum is taken componentwise. This can equivalently be written as $N(U, \Lambda) = 0$, where

$$N(U, \Lambda) = \begin{bmatrix} N_1(U,\Lambda) \\ N_2(U,\Lambda) \end{bmatrix} = \begin{bmatrix} AU + \Lambda - B \\ \mathcal{C}(U, \Lambda) \end{bmatrix}.$$

For a set $\mathcal{A} \subset \{1, 2, \dots, L\}$ we define $I_\mathcal{A} \in \mathbb{R}^{L \times L}$ for $1 \leq i, j \leq L$ by

$$(I_\mathcal{A})_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

We denote $\mathcal{A}^c = \{1, 2, \dots, L\} \setminus \mathcal{A}$ and note that $I_\mathcal{A} + I_{\mathcal{A}^c} = I_L$ is the identity matrix in $\mathbb{R}^{L \times L}$.

Proposition V.2. *The function N is Newton-differentiable at every $(U, \Lambda) \in \mathbb{R}^L \times \mathbb{R}^L$ and with the set*

$$\mathcal{A} = \{1 \leq i \leq L : \Lambda_i + c(U_i - Z_i) < 0\}$$

we have

$$\delta N(U, \Lambda) = \begin{bmatrix} \delta N_1(U,\Lambda) \\ \delta N_2(U,\Lambda) \end{bmatrix} = \begin{bmatrix} A & I_L \\ -cI_\mathcal{A} & I_{\mathcal{A}^c} \end{bmatrix}.$$

In particular, $\delta N(U, \Lambda)$ is regular and the semismooth Newton scheme for the iterative solution of $F(U, \Lambda) = 0$ is well-defined and locally superlinearly convergent.

One step of the semismooth Newton scheme, i.e., the linear system of equations

$$\delta N(\widetilde{U}, \widetilde{\Lambda})[\delta U, \delta \Lambda]^\top = -N(\widetilde{U}, \widetilde{\Lambda})$$

for a given iterate $(\widetilde{U}, \widetilde{\Lambda})$, is equivalent to the identity

$$\begin{bmatrix} A & I_L \\ -cI_{\widetilde{\mathcal{A}}} & I_{\widetilde{\mathcal{A}}^c} \end{bmatrix} \begin{bmatrix} \delta U \\ \delta \Lambda \end{bmatrix} = - \begin{bmatrix} A\widetilde{U} + \widetilde{\Lambda} - B \\ \widetilde{\Lambda} - \min\{0, \widetilde{\Lambda} + c(\widetilde{U} - Z)\} \end{bmatrix},$$

where $\widetilde{\mathcal{A}} = \{1 \leq i \leq L : \widetilde{\Lambda}_i + c(\widetilde{U}_i - Z_i) < 0\}$. This system can be written as

$$A(\widetilde{U} + \delta U) + (\widetilde{\Lambda} + \delta \Lambda) = B, \quad (\widetilde{\Lambda} + \delta\Lambda)_{\widetilde{\mathcal{A}}^c} = 0, \quad (\widetilde{U} + \delta U)|_{\widetilde{\mathcal{A}}} = Z|_{\widetilde{\mathcal{A}}}.$$

The semismooth Newton scheme can thus be formulated in the following form which is a version of a primal-dual active set method, cf. [HIK02] for details.

Algorithm V.3 (Primal-dual active set method)**.** *Let* $(U^0, \Lambda^0) \in \mathbb{R}^L \times \mathbb{R}^L$ *and* $c > 0$ *and set* $j = 0$.
(1) Compute (U^{j+1}, Λ^{j+1}) *via defining*

$$\mathcal{A}_j = \{1 \le i \le L : \Lambda_i^j + c(U_i^j - Z_i) < 0\}$$

and solving

$$\begin{bmatrix} A & I_L \\ I_{\mathcal{A}_j} & I_{\mathcal{A}_j^c} \end{bmatrix} \begin{bmatrix} U^{j+1} \\ \Lambda^{j+1} \end{bmatrix} = \begin{bmatrix} B \\ I_{\mathcal{A}_j} Z \end{bmatrix}.$$

(2) Stop the iteration if $\|\nabla(u_h^{j+1} - u_h^j)\|_{L^2(\Omega)} \le \varepsilon_{stop}$*; increase* $j \to j+1$ *and continue with (1) otherwise.*

Remarks V.4. *(i) The unknowns related to the active set* $\mathcal{A}_j$ *can be eliminated from the linear system of equations.*
(ii) Global convergence of the algorithm and monotonicity $U^{j+1} \ge U^j \ge Z$ *for* $j \ge 2$ *can be proved if* A *is an* M*-matrix.*
(iii) Classical active set strategies define $\mathcal{A}_j = \{1 \le i \le L : U_i^j \le Z_i^j\}$ *which corresponds to a formal limit* $c \to \infty$.

V.E. **Augmented Lagrange method.** We introduce the auxiliary variable $y = u$ and extend the functional J by adding a term that enforces this identity, i.e.,

$$\widetilde{L}_\tau(u, y; \mu) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x + I_K(y) + (\mu, u - y) + \frac{\tau}{2}\|u - y\|_{L^2(\Omega)}^2.$$

The obstacle problem is thus equivalent to determining a saddle point $(u, y; \mu)$ for $\widetilde{L}_\tau$, i.e., solving

$$\inf_{(u,y)} \sup_{\mu} \widetilde{L}_\tau(u, y; \mu).$$

The splitting method of Algorithm IV.1 can be applied to this formulation. In this way, the constraint $u \in K$ included via the indicator functional of K is decoupled from the Dirichlet energy. This is of practical interest since both functionals themselves can be treated efficiently. The splitting approach thus leads to a globally convergent iterative method that is straightforward to realize. In general, it is slower than the semismooth Newton method. If the conditions for global convergence of the Newton method are not satisfied, then the splitting method can be used to determine a good starting value.

V.F. **Numerical experiment.** Figure 6 shows two finite element approximations of an obstacle problem on $\Omega = (0, 1)^2$ with $f(x) = -5$ and $\chi(x) = -1/4$ for all $x \in \Omega$. For a better visualization of the contact set we displayed the negative of the numerical solutions. The solutions were obtained with the semismooth Newton scheme with stopping criterion defined by $\varepsilon_{stop} = 10^{-3}$. The required iteration numbers are shown in Table 1. For a comparison we also solved the obstacle problem with an augmented Lagrange method, where we always used the step size $\tau = 10$.

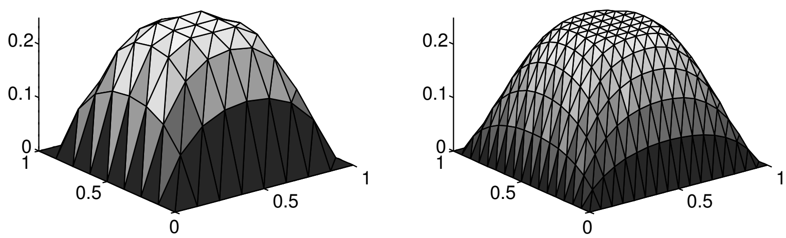

FIGURE 6. Numerical solutions $-u_h$ in an obstacle problem on two different triangulations.

Mesh size h	2^{-3}	2^{-4}	2^{-5}	2^{-6}
Semismooth Newton	4	6	8	13
Augmented Lagrange	22	31	30	37

TABLE 1. Iteration numbers for the semismooth Newton and the augmented Lagrange method in a two-dimensional obstacle problem.

6. Total-variation minimization

VI.A. **Mathematical model.** Given a noisy gray-scale image $g : \Omega \to \mathbb{R}$ we aim at removing the noise by solving a convex minimization problem. While high-frequency contributions should be eliminated, edges and other important features should be preserved. Classical Sobolev spaces are therefore too restrictive to provide meaningful image regularizations. The space of *functions of bounded variation* is an extension of the space $W^{1,1}(\Omega)$ and contains piecewise weakly differentiable functions which may be discontinuous across certain lower-dimensional interfaces. A seminorm in $BV(\Omega)$ is given by the total variation of the distributional gradient of a function $u \in L^1(\Omega)$, i.e.,

$$|Du|(\Omega) = \Big\{ -\int_\Omega u \operatorname{div} p \, \mathrm{d}x : p \in C_c^\infty(\Omega;\mathbb{R}^d),\ \|p\|_{L^\infty(\Omega)} \le 1 \Big\}.$$

If $u \in L^2(\Omega)$ then the vector fields p can be taken from the larger space $H_N(\mathrm{div};\Omega)$ of square integrable vector fields whose distributional divergence belongs to $L^2(\Omega)$ and which have a vanishing normal component on $\partial\Omega$. If $u = \chi_A$ is the characteristic function of a sufficiently regular set $A \subset \Omega$ then $|Du|$ is the perimeter of A which coincides with the length or surface measure of ∂A. The Rudin–Osher–Fatemi model [ROF92, CL97] for image denoising seeks a solution $u \in BV(\Omega) \cap L^2(\Omega)$ for the minimization problem:

$$\text{Minimize} \quad I(u) = |Du|(\Omega) + \frac{\alpha}{2}\|u-g\|^2_{L^2(\Omega)}.$$

Here, $\alpha > 0$ is a given parameter. The problem is illustrated in Fig. 7.

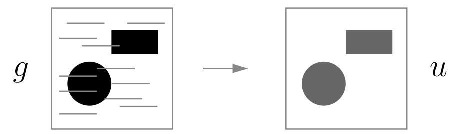

FIGURE 7. Illustration of the image denoising problem.

Existence of minimizers follows from appropriate weak compactness properties of the space $BV(\Omega)$, cf. [ABM06]. Since the functional I is strongly convex, the distance of any function to the minimizer is controlled by the difference of the values of the functional.

Lemma VI.1. *If $u \in BV(\Omega) \cap L^2(\Omega)$ minimizes I then we have*

$$\frac{\alpha}{2}\|u-v\|_{L^2(\Omega)}^2 \le I(v) - I(u)$$

for every $v \in BV(\Omega) \cap L^2(\Omega)$. In particular, minimizers are uniquely defined.

Proof. We define $F : BV(\Omega) \to \mathbb{R}$ and $G : L^2(\Omega) \to \mathbb{R}$ by

$$F(u) = |Du|(\Omega), \quad G(u) = \frac{\alpha}{2}\|u-g\|_{L^2(\Omega)}^2$$

and extend F by $+\infty$ to $L^2(\Omega)$. Then F is convex and G is strongly convex and Fréchet differentiable with $DG(u)[w] = \alpha(u-g, w)$, i.e., we have

$$\delta G(u)[v-u] + \frac{\alpha}{2}\|u-v\|_{L^2(\Omega)}^2 + G(u) = G(v)$$

for all $u, v \in L^2(\Omega)$. Since $u \in BV(\Omega) \cap L^2(\Omega)$ is a minimizer, we have

$$0 \in \partial I(u) = \partial F(u) + DG(u),$$

or equivalently $-DG(u) \in \partial F(u)$, i.e.,

$$-DG(u)[v-u] + F(u) \le F(v).$$

The strong convexity of G yields that

$$\frac{\alpha}{2}\|u-v\|_{L^2(\Omega)}^2 + G(u) - G(v) + F(u) \le F(v),$$

which proves the assertion. □

VI.B. **Discretization.** To approximate minimizers of I numerically, we note that continuous, piecewise affine functions belong to $BV(\Omega)$ and restrict the minimization to $\mathcal{S}^1(\mathcal{T}_h) \subset W^{1,1}(\Omega)$, i.e., we aim at solving:

$$\text{Minimize} \quad I(u_h) = |Du_h|(\Omega) + \frac{\alpha}{2}\|u_h - g\|_{L^2(\Omega)}^2$$

in the set of functions $u_h \in \mathcal{S}^1(\mathcal{T}_h)$. The existence of a unique solution is a consequence of the continuity and strong convexity of I. Lemma VI.1 implies an error estimate in the $L^2(\Omega)$ norm, i.e., we have, cf. [Bar12, BNS14],

$$\frac{\alpha}{2}\|u-u_h\|_{L^2(\Omega)}^2 \le ch^{1/2}.$$

This estimate does not assume additional regularity properties of the exact solution $u \in BV(\Omega) \cap L^2(\Omega)$ but uses the fact that we have $u \in L^\infty(\Omega)$.

Remarks VI.2. *(i) Since for $u \in BV(\Omega) \cap L^2(\Omega)$ the best approximation in $\mathcal{S}^1(\mathcal{T}_h)$ satisfies $\inf_{v_h \in \mathcal{S}^1(\mathcal{T}_h)} \|u - v_h\|_{L^2(\Omega)} \le h^{1/2}$, the convergence rate of the theorem is suboptimal. Numerical experiments indicate that the optimal convergence rate $\mathcal{O}(h^{1/2})$ in $L^2(\Omega)$ is in general not attained.*
(ii) The discretization with piecewise constant functions on a sequence of triangulations does in general not lead to a convergent numerical method.

VI.C. **Primal-dual method.** To develop an iterative solution method for solving the nondifferentiable minimization problem defined by I, we derive a related saddle-point formulation. For this we define the set of elementwise constant vector fields by

$$\mathcal{L}^0(\mathcal{T}_h)^d = \big\{ q_h \in L^1(\Omega;\mathbb{R}^d) : q_h|_T \text{ constant for every } T \in \mathcal{T}_h \big\}$$

and note that due to the fact that $\nabla u_h \in \mathcal{L}^0(\mathcal{T}_h)^d$ for $u_h \in \mathcal{S}^1(\mathcal{T}_h)$ we have

$$\begin{aligned}
&\inf_{u_h \in \mathcal{S}^1(\mathcal{T}_h)} \int_\Omega |\nabla u_h| \,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2_{L^2(\Omega)} \\
&= \inf_{u_h \in \mathcal{S}^1(\mathcal{T}_h)} \sup_{p_h \in \mathcal{L}^0(\mathcal{T}_h)^d} \int_\Omega p_h \cdot \nabla u_h \,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2_{L^2(\Omega)} - I_{K_1(0)}(p_h) \\
&= \inf_{u_h \in \mathcal{S}^1(\mathcal{T}_h)} \sup_{p_h \in \mathcal{L}^0(\mathcal{T}_h)^d} L_h(u_h, p_h).
\end{aligned}$$

Here, $I_{K_1(0)}$ is the indicator functional of the set $K_1(0) = \{p \in L^1(\Omega;\mathbb{R}^d) : |p| \le 1 \text{ a.e. in } \Omega\}$. A saddle point (u_h, p_h) satisfies $|p_h| \le 1$ in Ω and the equations

$$(p_h, \nabla v_h) = -\alpha(u_h - g, v_h), \quad (\nabla u_h, q_h - p_h) \le 0$$

for all $(v_h, q_h) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h)^d$ with $|q_h| \le 1$ in Ω. With an appropriate step size $\tau > 0$ and a discrete inner product $(\cdot,\cdot)_{h,s}$ on $\mathcal{S}^1(\mathcal{T}_h)$ that may differ from the L^2 inner product, we obtain the following variant of the primal-dual iteration.

Algorithm VI.3 (Primal-dual iteration)**.** *Let $(\cdot,\cdot)_{h,s}$ be an inner product on $\mathcal{S}^1(\mathcal{T}_h)$, $\tau > 0$, $(u_h^0, p_h^0) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h)^d$, set $d_t u_h^0 = 0$ and $j = 0$.*
(1) Set $\widetilde{u}_h^j = u_h^{j-1} + \tau d_t u_h^{j-1}$.
(2) Compute the maximizer $p_h^j \in \mathcal{L}^0(\mathcal{T}_h)^d$ for the mapping

$$p_h \mapsto \frac{-1}{2\tau}\|p_h - p_h^{j-1}\|^2_{L^2(\Omega)} + \int_\Omega p_h \cdot \nabla \widetilde{u}_h^j \,\mathrm{d}x - I_{K_1(0)}(p_h).$$

(3) Compute the minimizer $u_h^j \in \mathcal{S}^1(\mathcal{T}_h)$ for the mapping

$$u_h \mapsto \frac{1}{2\tau}\|u_h - u_h^{j-1}\|^2_{h,s} + \int_\Omega p_h^j \cdot \nabla u_h \,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2_{L^2(\Omega)}.$$

(4) Stop if $\|u_h^{j-1} - u_h^j\|_{h,s} \le \varepsilon_{stop}$; set $d_t u_h^j = (u_h^j - u_h^{j-1})/\tau$, increase $j \to j+1$, and continue with (1) otherwise.

Remark VI.4. *Note that p_h^j in Step (2) is given by the truncation operation*

$$p_h^j = \big(p_h^{j-1} + \tau\nabla\widetilde{u}_h^j\big)/\max\{1, |p_h^{j-1} + \tau\nabla\widetilde{u}_h^j|\}$$

which can be computed elementwise.

The iterates of the primal-dual algorithm converge to a stationary point if τ is sufficiently small, e.g., $\tau = O(h)$ if $(\cdot,\cdot)_{h,s}$ is the inner product in $L^2(\Omega)$, cf. [CP11]. Other useful choices of the inner product $(\cdot,\cdot)_{h,s}$ are weighted combinations of the inner product in $L^2(\Omega)$ and the semi-inner product in $H^1(\Omega)$, e.g.,

$$(v_h, w_h)_{h,s} = (v_h, w_h) + h^{(1-s)/s}(\nabla v_h, \nabla w_h),$$

for $s = 1/2$ this is a discrete version of the inner product in the broken Sobolev space $H^{1/2}(\Omega)$. It provides a good choice since this norm remains bounded for bounded sequences in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$, cf. [Bar15b, Bar15a] for details.

VI.D. **Splitting method.** Splitting methods are attractive for minimization problems involving total variation since these are directly obtained from introducing the variable $r_h = \nabla u_h$ and considering the augmented Lagrange functional

$$\widetilde{L}_{\tau,h}(u_h, r_h; \mu_h) = \int_\Omega |r_h| \,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2_{L^2(\Omega)} + (\mu_h, \nabla u_h - r_h)_h + \frac{\tau}{2}\|\nabla u_h - r_h\|_h^2.$$

Note that the minimization with respect to r_h can be done explicitly elementwise if $\|\cdot\|_h$ is a multiple of the L^2 norm. Often, the norm and inner product in $L^2(\Omega)$ are chosen to define $\widetilde{L}_{\tau,h}$. This can in general not be expected to lead to a uniformly stable iterative scheme since typically we have $\nabla u \notin L^2(\Omega;\mathbb{R}^d)$ and

$$\|\nabla u_h\|_{L^2(\Omega)} \sim h^{-d/2}$$

as $h \to 0$. Instead one may use a weighted inner product and corresponding norm, e.g., defined by

$$(r_h, s_h)_h = h^d \int_\Omega r_h \cdot s_h \,\mathrm{d}x$$

The weighting factor h^d compensates the growth of the L^2 norm of ∇u_h as $h \to 0$. This has the interpretation of a softened penalty method.

VI.E. **Numerical experiment.** We set $\Omega = (0,1)^2$ and let $g = \chi_{B_r(m)}$ be the characteristic function of the ball of radius $r = 1/5$ around the midpoint $m = (1/2, 1/2)$. The regularization parameter is chosen as $\alpha = 100$. Figure 8 displays numerical solutions on two different triangulations. These were obtained with the primal-dual method for $s = 1/2$, step size $\tau = h^{1/2}/10$, and stopping criterion $\varepsilon_{stop} = 10^{-2}$. Table 2 shows the corresponding iteration numbers on different triangulations together with those needed in the splitting method. We used the weighted L^2 norm to define the penalty term and always set $\tau = 10$.

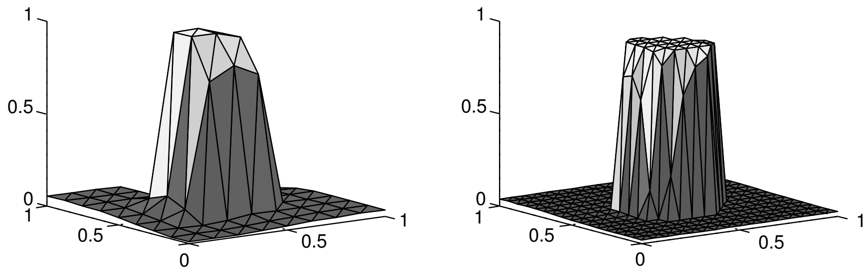

FIGURE 8. Numerical solutions u_h for a total-variation regularized problem on two different triangulations.

Mesh size h	2^{-3}	2^{-4}	2^{-5}	2^{-6}
Primal-dual	173	265	450	677
Augmented Lagrange	67	326	599	2197

TABLE 2. Iteration numbers for the primal-dual and the augmented Lagrange method in a two-dimensional total-variation regularized problem.

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