APPROXIMATION OF FRACTIONAL HARMONIC MAPS

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Abstract. This paper addresses the approximation of fractional harmonic maps. Besides a unit-length constraint, one has to tackle the difficulty of nonlocality. We establish weak compactness results for critical points of the fractional Dirichlet energy on unit-length vector fields. We devise and analyze numerical methods for the approximation of various partial differential equations related to fractional harmonic maps. The compactness results imply the convergence of numerical approximations. Numerical examples on spin chain dynamics and point defects are presented to demonstrate the effectiveness of the proposed methods.

1. Introduction

A fundamental problem in the calculus of variations concerns critical points of energy functionals subject to pointwise constraints. Related applications arise in ferromagnetism to model magnetization fields, liquid crystal theories defining orientations of rod-like molecules, continuum mechanics for describing inextensible rods and unshearable plates, and in quantum mechanics for spin systems. We refer the reader to the articles [5, 18, 17, 13, 6, 36, 14, 35, 37, 23, 45] for corresponding mathematical models with numerical methods and to [38, 32, 24] for recent analytical results.

In this article we consider the case of energies related to the fractional Laplace operator. Fractional operators are nonlocal and enable long range interactions. They enforce less smoothness in comparison to their classical counterparts. These features make them attractive for applications leading to certain singularities such as defects in the mathematical description of liquid crystals, which are often modeled by harmonic maps. While some ideas from the treatment of standard, local differential operators can be employed to define stable numerical schemes, new ideas are required to establish the convergence of discrete stationary configurations.

Our starting point is a fractional Dirichlet energy

$$I[u] = \frac{1}{2}\int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2 \,\mathrm{d}x \tag{1}$$

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for an appropriate definition of the fractional Laplace operator $(-\Delta)^{\frac{s}{2}}$ with $0 < s < 1$. Here $\Omega \subset \mathbb{R}^d$ is an open bounded domain with Lipschitz boundary $\partial\Omega$. We then consider stationary points for I subject to a unit-length constraint, i.e., in the set

$$\mathcal{A} = \{v - \vec{N} \in \widetilde{H}^s(\Omega;\mathbb{R}^N) : |v(x)|^2 = 1 \text{ for a.e. } x \in \Omega\},$$

where

$$\widetilde{H}^s(\Omega;\mathbb{R}^N) = \{f \in H^s(\mathbb{R}^d;\mathbb{R}^N) \,:\, f = 0 \quad \text{in } \mathbb{R}^d \setminus \Omega\},$$

and $\vec{N} \in C^\infty(\mathbb{R}^d;\mathbb{R}^N)$ is a fixed vector field that defines a unit-length exterior Dirichlet condition on $\mathbb{R}^d \setminus \Omega$. Obviously, a homogeneous boundary condition is incompatible with the unit-length constraint.

Stationary points for I in $\mathcal{A}$ are called *fractional harmonic maps* and are formally characterized by the Euler–Lagrange equations

$$(-\Delta)^s u = \lambda u \quad \text{in } \Omega, \quad |u|^2 = 1 \quad \text{in } \Omega, \quad u|_{\mathbb{R}^d\setminus\Omega} = \vec{N}, \tag{2}$$

where $\lambda \in L^1(\Omega)$ is a Lagrange multiplier related to the pointwise unit-length constraint. The function λ depends nonlinearly on the vector field u, e.g., in the classical case $s = 1$ we have that $\lambda = |\nabla u|^2$. This critical nonlinear dependence requires appropriate arguments to show that accumulation points of bounded sequences of solutions are again solutions of the nonlinear equation. Such stability results are crucial for showing that numerical approximations converge to fractional harmonic maps.

A useful equivalent characterization of fractional harmonic maps is the weak formulation

$$\big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v\big) = 0 \tag{3}$$

for all $v \in \widetilde{H}^s(\Omega;\mathbb{R}^N)$ satisfying the pointwise orthogonality relation $u \cdot v = 0$ almost everywhere in Ω. We refer the reader to section 2 below for a specification of the bilinear form in (3). This characterization states that critical points are stable with respect to tangential perturbations. If $N = 3$ then the latter equation is equivalent to the identity

$$\big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} (u \times \phi)\big) = 0 \tag{4}$$

for all $\phi \in C_c^\infty(\Omega;\mathbb{R}^3)$. Observe that for $|u| = 1$ and $v \cdot u = 0$ we have $v = u \times (v \times u)$. We further note that the latter identity can be generalized to other target dimensions $N \neq 3$ by considering $v = Xu$ with a skew-symmetric matrix valued mapping $X : \Omega \to so(N)$ whose pointwise application to u is identified with a product $\phi \wedge u$, where $\phi(x)$ is for almost every $x \in \Omega$ a skew-symmetric bilinear form that is identified with a vector $\phi(x) \in \mathbb{R}^{N'}$; for ease of readability we also write in this case $u \times \phi$.

It turns out that a limit passage in the nonlinear equation (4) is possible. In particular, in section 3, we shall establish that if $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$, such that $u_j \rightharpoonup u$ in an appropriate fractional order Sobolev space, as $j \to \infty$, then

$$\big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}} (u_j \times \phi)\big) \to \big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} (u \times \phi)\big) \tag{5}$$

for all $\phi \in C_c^\infty(\Omega;\mathbb{R}^{N'})$. The key challenge here is the fact that due to nonlocality of $(-\Delta)^s$, the standard arguments from the classical case of $s = 1$ cannot be applied. Our proof uses a localization argument combined with properties of the Hardy-Littlewood maximal function. Simpler arguments lead to this result when the fractional Laplace operator is defined via a Fourier transformation. We use (5) to carry out critical limit passages in the justification of three numerical problems related to fractional harmonic maps. We refer the reader to the pioneering work [27] and to the contributions [44, 47, 49, 43, 40, 41, 42] for various properties of minimizing, stationary, and critical fractional harmonic maps.

Discrete fractional harmonic maps. The first numerical problem concerns the convergence of discrete fractional harmonic maps as the mesh-sizes of underlying triangulations tend to zero. We consider a sequence $\{\mathcal{T}_h\}_{h>0}$ of uniformly shape regular triangulations of the polygonal or polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$ with maximal mesh-sizes $h \to 0$. Discrete fractional harmonic maps belong to the discrete admissible set

$$\mathcal{A}_h = \{v_h - \widetilde{\mathcal{I}}_h \vec{N} \in \mathcal{S}^1_0(\mathcal{T}_h)^N : |v_h(z)|^2 = 1 \text{ for all } z \in \mathcal{N}_h\},$$

where $\mathcal{S}^1_0(\mathcal{T}_h)$ is the space of piecewise linear, globally continuous functions for a triangulation $\mathcal{T}_h$ of $\widetilde{\Omega}$ vanishing in the exterior $\widetilde{\Omega} \setminus \Omega$; the set $\mathcal{N}_h$ contains the vertices of elements inside Ω at which the unit-length constraint is imposed, $\mathcal{I}_h$ and $\widetilde{\mathcal{I}}_h$ are the nodal interpolation operators on $\mathcal{T}_h$ and an extension $\widetilde{\mathcal{T}}_h$ which provides a triangulation of a domain $\widetilde{\Omega}$ such that $\overline{\Omega} \subset \widetilde{\Omega}$ and the support of $\vec{N}$ is contained in $\widetilde{\Omega}$.

We then define discrete fractional harmonic maps as vector fields $u_h \in \mathcal{A}_h$ with the property

$$\big((-\Delta)^{\frac{s}{2}} u_h, (-\Delta)^{\frac{s}{2}} v_h\big) = 0 \tag{6}$$

for all $v_h \in \mathcal{F}_h[u_h]$, where $\mathcal{F}_h[u_h]$ is defined as

$$\mathcal{F}_h[u_h] = \big\{v_h \in \mathcal{S}^1_0(\mathcal{T}_h)^N : v_h(z) \cdot u_h(z) = 0 \text{ for all } z \in \mathcal{N}_h\big\}.$$

If $N = 3$ then the vector fields $v_h \in \mathcal{F}_h[u_h]$ are represented by

$$v_h = \mathcal{I}_h[u_h \times \phi]$$

for $\phi \in C^\infty_c(\Omega; \mathbb{R}^3)$. In section 5.1, we will show that if $\{u_h\}_{h>0}$ is a bounded sequence of discrete fractional harmonic maps then every weak limit $u \in \widetilde{H}^s(\Omega; \mathbb{R}^N)$ as $h \to 0$ is a fractional harmonic map. Compact perturbations R_h that model solution errors or consistency terms can be included on the right-hand side of (6) and incorporated in the analysis provided that $\|R_h\|_{\widetilde{H}^s(\Omega;\mathbb{R}^N)'} \to 0$ as $h \to 0$. For ease of presentation, we assume an exact discretization of the bilinear form associated with the fractional Laplace operator. We refer the reader to [2, 3, 9] for corresponding results in the context of the linear fractional Poisson problem. In our experiments we follow [2] and [7] for finite element and spectral method implementations, respectively. We also refer to [20, 8] for other efficient approaches to implement integral fractional Laplacian. Some other applications of fractional operators include, imaging [7], geophysics [54], and optimal control [12].

Fractional harmonic map heat flow. The second application addresses a parabolic evolution defined by the L^2-gradient flow for I given in (1); it was studied analytically in [53, 46, 51]. Its discretization or discretizations of gradient flows for other metrics define fully practical methods to determine discrete fractional harmonic maps. The L^2-flow of fractional harmonic maps is formally given by the partial differential equation

$$\partial_t u = -(-\Delta)^s u + \lambda u, \quad |u|^2 = 1,$$

where again λ is the Lagrange multiplier subject to the unit-length constraint. Rigorously, we define solutions of the fractional harmonic map heat flow as maps $u : (0,T) \times \Omega \to \mathbb{R}^N$ with

$$u - \vec{N} \in H^1(0,T; L^2(\Omega;\mathbb{R}^N)) \cap L^\infty(0,T; \widetilde{H}^s(\Omega;\mathbb{R}^N))$$

that satisfy $u(0) = u_0$ for a given vector field $u_0 \in \mathcal{A}$, the constraint $|u(t,x)|^2 = 1$ almost everywhere in $(0,T) \times \Omega$, and with the inner product $(\cdot,\cdot)$ in $L^2(\Omega;\mathbb{R}^N)$

$$(\partial_t u, v) + \big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v\big) = 0 \tag{7}$$

for all vector fields $v \in \widetilde{H}^s(\Omega;\mathbb{R}^N)$ and almost every $t \in (0,T)$ with the orthogonality relation

$$u(t,x) \cdot v(x) = 0$$

for almost every $(t,x) \in (0,T) \times \Omega$, we furthermore require solutions to satisfy an energy-decay property. Our numerical scheme adopts ideas from [5, 15, 16] and imposes the orthogonality condition at the nodes of a triangulation in an explicit way while the evolution equation is discretized implicitly with the backward difference quotient operator

$$d_t u^k = \tau^{-1}(u^k - u^{k-1})$$

for a step size $\tau > 0$. We hence compute a sequence

$$\{u_h^k\}_{k=0,\dots,K} \in \vec{N} + \mathcal{S}^1_0(\mathcal{T}_h)^N$$

such that $u_h^0 = u_{0,h}$ and $d_t u_h^k \in \mathcal{F}_h[u_h^{k-1}]$ is for $k = 1, 2, \dots, K$ such that

$$(d_t u_h^k, v_h) + \big((-\Delta)^{\frac{s}{2}} u_h^k, (-\Delta)^{\frac{s}{2}} v_h\big) = 0 \tag{8}$$

for all $v_h \in \mathcal{F}_h[u_h^{k-1}]$, i.e., for all $v_h \in \mathcal{S}^1_0(\mathcal{T}_h)^N$ with

$$u_h^{k-1}(z) \cdot v_h(z) = 0$$

for all $z \in \mathcal{N}_h$. Note that the problems in the time steps are linear systems with unique solutions and testing with $v_h = d_t u_h^k$ shows the energy monotonicity

$$\|d_t u_h^k\|^2 + \frac{d_t}{2}\|(-\Delta)^{\frac{s}{2}} u_h^k\|^2 + \frac{\tau}{2}\|d_t u_h^k\|^2 = 0. \tag{9}$$

The linearized, explicit treatment of the constraint leads to a violation that is controlled by the step-size $\tau > 0$, i.e., since $d_t u_h^k(z) \cdot u_h^{k-1}(z) = 0$ and $u_h^k = u_h^{k-1} + \tau d_t u_h^k$ we have

$$|u_h^k(z)|^2 = |u_h^{k-1}(z)|^2 + \tau^2 |d_t u_h^k(z)|^2 = \dots = |u_h^0(z)|^2 + \tau^2 \sum_{\ell=1}^k |d_t u_h^k(z)|^2.$$

By carrying out a discrete integration, i.e., multiplying by local volumes β_z and summing over $z \in \mathcal{N}_h$, and noting $|u_h^0(z)|^2 = 1$, we find that

$$\big\||u_h^k|^2 - 1\big\|_{L^1_h(\Omega)} \le \tau^2 \sum_{\ell=1}^k \|d_t u_h^k\|^2_{L^2_h(\Omega)}.$$

The right-hand side is of order $\mathcal{O}(\tau)$ owing to (9), and we have

$$\|v\|_{L^p_h(\Omega)} = \int_\Omega \mathcal{I}_h |v|^p \,\mathrm{d}x = \sum_{z \in \mathcal{N}_h} \beta_z |v(z)|^p, \quad \beta_z = \int_\Omega \varphi_z \,\mathrm{d}x,$$

with the nodal basis functions $\{\varphi_z\}_{z\in\mathcal{N}_h}$.

Hyperbolic system for spin dynamics. The third numerical problem is a hyperbolic evolution equation determined by the force balance

$$\partial_t u = \delta I[u] \times u = (-\Delta)^s u \times u, \quad |u|^2 = 1, \tag{10}$$

which has been used to model nonlocal effects in spin chains, cf. [55, 32, 38]. This evolution is constraint and energy preserving which follows directly from testing the equation with u and $(-\Delta)^s u$, respectively. To obtain these features for a discretization, we follow [36, 14] and use Crank-Nicolson type midpoint approximations, i.e., we consider the time stepping scheme

$$d_t u^k = -u^{k-1/2} \times (-\Delta)^s u^{k-1/2},$$

with the average

$$u^{k-1/2} = \frac{1}{2}(u^{k-1} + u^k).$$

A binomial formula then implies discrete energy and constraint preservation, e.g., testing with $u^{k-1/2}$ implies that

$$d_t|u^k|^2 = d_t u^k \cdot u^{k-1/2} = 0,$$

so that $|u^k|^2 = |u^{k-1}|^2 = \cdots = |u^0|^2$ almost everywhere in Ω. A spatial discretization uses quadrature to allow for a localization of the preservation properties, i.e.,

$$(d_t u_h^k, v_h)_h = \big((-\Delta)_h^s u_h^{k-1/2}, \mathcal{I}_h[u_h^{k-1/2} \times v_h]\big)_h \tag{11}$$

where the discrete inner product is consistent with the norm $\|\cdot\|_{L_h^2(\Omega)}$ and given by

$$(y_h, v_h)_h = \int_\Omega \mathcal{I}_h[y_h \cdot v_h]\,\mathrm{d}x = \sum_{z\in\mathcal{N}_h} \beta_z y_h(z)\cdot v_h(z),$$

and $y_h = (-\Delta)_h^s w_h$ is for given $w_h \in \mathcal{S}_0^1(\mathcal{T}_h)^3$ the uniquely defined function $y_h \in \mathcal{S}_0^1(\mathcal{T}_h)^3$ with

$$(y_h, v_h)_h = \big((-\Delta)^{\frac{s}{2}} w_h, (-\Delta)^{\frac{s}{2}} v_h\big)$$

for all $v_h \in \mathcal{S}_0^1(\mathcal{T}_h)^3$. Choosing the test function $v_h = u_h^{k-1/2}(z)\varphi_z$ with the hat function $\varphi_z \in \mathcal{S}_0^1(\mathcal{T}_h)$ associated with a node $z \in \mathcal{N}_h$ in (11) then leads to the discrete constraint preservation property

$$\beta_z d_t |u_h^k(z)|^2 = 0.$$

Analogously, by choosing $v_h = (-\Delta)_h^s u_h^{k-1/2}$ we find with the definition of $(-\Delta)_h^s$ that

$$\begin{aligned} d_t \frac{1}{2}\big\|(-\Delta)^{\frac{s}{2}} u_h^k\big\|_h^2 &= \big((-\Delta)^{\frac{s}{2}} d_t u_h^k, (-\Delta)^{\frac{s}{2}} u_h^{k-1/2}\big) \\ &= \big(d_t u_h^k, (-\Delta)_h^s u_h^{k-1/2}\big)_h = 0, \end{aligned}$$

i.e., the preservation of the discrete fractional Dirichlet energy. The scheme (11) requires the iterative solution of a nonlinear system of equations in every time step. A simple fixed-point iteration is constraint preserving and convergent provided that the step-size condition $\tau = O(h^{2s})$ is satisfied.

Outline. The remainder of the paper is organized as follows: In section 2, we introduce definitions and notation related to fractional Sobolev spaces. Section 3 is devoted to the weak compactness result for fractional harmonic maps defined via the integral representation of the fractional Laplace operator; a corresponding result for a spectral version of the fractional Laplacian is provided in Appendix A. Required finite element spaces are defined in section 4 along with a general application of the the continuous compactness result to the discrete setting. Section 5 specifies the three numerical algorithms described above and provides corresponding stability and convergence results. In section 6, we provide numerical experiments which illustrate the good approximation properties of our numerical schemes.

2. Fractional Sobolev spaces

Without any specific mention, we use $(\cdot,\cdot)$ to denote the L^2-scalar product and $\|\cdot\|$ the L^2-norm. The scalar product is typically defined over Ω or $\mathbb{R}^d$ with an appropriate interpretation in case of the fractional Laplacian. For a Banach space X, we denote its topological dual by X' and the pairing between X' and X by $\langle\cdot,\cdot\rangle_{X',X}$. Moreover, we use $\to$ and $\rightharpoonup$ to indicate strong and weak convergence, respectively. We occasionally denote a relation $A \le CB$, with C being a non-essential constant, by $A \precsim B$. The set B_ρ denotes the open ball of radius ρ centered at 0.

To define the fractional Laplace operator we consider the weighted Lebesgue space

$$\mathbb{L}^1_s(\mathbb{R}^d) = \left\{ f:\mathbb{R}^d \to \mathbb{R} \text{ measurable },\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^{d+2s}}\,\mathrm{d}x < \infty \right\}$$

and first define for $f \in \mathbb{L}^1_s(\mathbb{R}^d)$, $\varepsilon > 0$, and $x \in \mathbb{R}^d$ the quantity

$$(-\Delta)^s_\varepsilon f(x) = C_{d,s} \int_{\{y\in\mathbb{R}^d, |y-x|>\varepsilon\}} \frac{f(x)-f(y)}{|x-y|^{d+2s}}\,\mathrm{d}y$$

where the constant $C_{d,s} = \left(s2^{2s}\Gamma\left(\frac{2s+d}{2}\right)\right)/\left(\pi^{\frac{d}{2}}\Gamma(1-s)\right)$ is obtained with Euler's Gamma function. We then define the *integral version* of the fractional Laplace operator for $s \in (0,1)$ via a limit passage for $\varepsilon \to 0$, i.e.,

$$(-\Delta)^s f(x) = C_{d,s}\text{P.V.} \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+2s}}\,\mathrm{d}y = \lim_{\varepsilon\to 0}(-\Delta)^s_\varepsilon f(x), \tag{12}$$

where P.V. indicates the Cauchy principal value. Note that this definition for the full space $\mathbb{R}^d$ coincides with the spectral definition of the fractional Laplacian obtained using Fourier transform [29, Proposition 3.4], see also [25]. Such an equivalence also holds in case of periodic boundary conditions [1, Eq. (2.53)].

Remark 2.1. If we replace the integration domain $\mathbb{R}^d$ in (12) by an open set Ω we obtain the so-called *regional fractional Laplacian*. All arguments given in this paper can be adapted to that setting by minor modifications provided the boundary conditions are meaningful, e.g., in the case $s > 1/2$.

Based on the definition of the operator $(-\Delta)^s$ we introduce fractional order Sobolev spaces $H^s(\mathbb{R}^d)$ for $s \in (0,1)$ by setting

$$[f]_{H^s(\mathbb{R}^d)} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{2}},$$
$$\|f\|_{H^s(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)} + [f]_{H^{s,2}(\mathbb{R}^d)}.$$

Then the Sobolev space $H^s(\mathbb{R}^d)$ is defined as

$$H^s(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \, : \, \|f\|_{H^s(\mathbb{R}^d)} < +\infty \right\}$$

which is a Hilbert space. The set of vectorial functions $f : \mathbb{R}^d \to \mathbb{R}^N$ whose components belong to $H^s(\mathbb{R}^d)$ is denoted by $H^s(\mathbb{R}^d; \mathbb{R}^N)$. We will omit dependence on N while writing corresponding norms when it is clear from the context.

For bounded open sets $\Omega \subset \mathbb{R}^d$ and parameters $s \in (0,1)$ we define Sobolev spaces $\widetilde{H}^s(\Omega)$ by considering trivial extensions to $\mathbb{R}^d$, i.e., we set

$$\widetilde{H}^s(\Omega) = \{ f \in L^2(\mathbb{R}^d) \, : \, (-\Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R}^d), \quad f \equiv 0 \text{ in } \mathbb{R}^d \setminus \Omega \}.$$

We recall the following density result for $s \in (0,1]$ and $\Omega \subset \mathbb{R}^d$ for domains with Lipschitz boundary [31]

$$\widetilde{H}^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{\widetilde{H}^s(\Omega)}}$$

and by a Poincaré type inequality, which is a consequence of Hölder's inequality and Sobolev imbedding theorem, [4, Theorem 3.1.4.], a norm on $\widetilde{H}^s(\Omega; \mathbb{R}^N)$ is given by

$$\|f\|_{\widetilde{H}^s(\Omega)} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)}.$$

We refer the reader to [29, Theorem 7.1] for boundedness and compactness results of embeddings of $\widetilde{H}^s(\Omega)$ into $L^2(\Omega)$. Following [30], an integration-by-parts formula can be established for the fractional Laplace operator, i.e., for all $f, g \in \widetilde{H}^s(\Omega)$ we have

$$\begin{aligned} \big((-\Delta)^{\frac{s}{2}} f, (-\Delta)^{\frac{s}{2}} g\big) &= \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+2s}} \, \mathrm{d}y \, \mathrm{d}x \\ &= \langle (-\Delta)^s f, g \rangle_{\widetilde{H}^s(\Omega)', \widetilde{H}^s(\Omega)}. \end{aligned} \tag{13}$$

In the proofs of section 3 we will also encounter the Hardy-Littlewood maximal function $\mathcal{M}$. It is defined as

$$\mathcal{M}f(x) = \sup_{r>0} |B_r(x)|^{-1} \int_{B_r(x)} |f(y)| dy.$$

The maximal theorem, see [52], states that for $p \in (1, \infty]$, the (sub-linear) operator $\mathcal{M}$ is bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$. That is, there exists a constant $C > 0$ such that for all $f \in L^p(\mathbb{R}^d)$ we have

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^d)} \leq C \, \|f\|_{L^p(\mathbb{R}^d)}.$$

The maximal function of a derivative controls the Hölder or Lipschitz-constant of a function. More precisely the following inequality holds true, see [19, 34]

$$\frac{|f(x)-f(y)|}{|x-y|} \precsim \mathcal{M}|\nabla f|(x) + \mathcal{M}|\nabla f|(y).$$

A similar estimate holds for the fractional Laplacian, which was shown in [50, Proposition 6.6.] (but may have been known before): for $\alpha \in (0,1)$ we have

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \precsim \mathcal{M}|(-\Delta)^{\frac{\alpha}{2}} f|(x) + \mathcal{M}|(-\Delta)^{\frac{\alpha}{2}} f|(y).$$

Here, by an abuse of notation we will write $\mathcal{M}$ for the square of the maximal function $\mathcal{M}\circ\mathcal{M}$.

3. Weak compactness for integral fractional Laplacian

The goal of this section is to identify a weak compactness property for fractional harmonic maps. This result is critical to establish convergence of our numerical approximations. In particular, we establish that if $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $u_j \rightharpoonup u$ in $\widetilde{H}^s(\Omega;\mathbb{R}^N)$ as $j\to\infty$ then we have

$$\big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}}(u_j \times \phi)\big) \to \big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}}(u \times \phi)\big) \tag{14}$$

for any $\phi \in C_c^\infty(\Omega;\mathbb{R}^{N'})$, where we recall that fractional harmonic maps fulfill (4). In the classical setting for $s=1$ the result is a direct consequence of the product rule and properties of the cross produt.

To generalize the critical limit passage we begin by rewriting the nonlinear term as follows

$$\begin{aligned}
\big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}}(u_j \times \phi)\big) &= \big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}} u_j \times \phi\big) \\
&\quad + \big((-\Delta)^{\frac{s}{2}} u_j, u_j \times (-\Delta)^{\frac{s}{2}}\phi\big) \\
&\quad + \big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}}(u_j\times\phi) - (-\Delta)^{\frac{s}{2}} u_j \times \phi - u_j \times (-\Delta)^{\frac{s}{2}}\phi\big) \\
&= \big((-\Delta)^{\frac{s}{2}} u_j, u_j \times (-\Delta)^{\frac{s}{2}}\phi\big) \\
&\quad + \big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}}(u_j\times\phi) - (-\Delta)^{\frac{s}{2}} u_j \times \phi - u_j \times (-\Delta)^{\frac{s}{2}}\phi\big) \\
&= a_j + b_j
\end{aligned}$$

where we used that $\big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}} u_j \times \phi\big) = 0$. Since $\widetilde{H}^s(\Omega;\mathbb{R}^N)$ is compactly embedded in $L^2(\Omega;\mathbb{R}^N)$ for $s>0$, by a weak-strong limiting argument we conclude that

$$a_j \to \big((-\Delta)^{\frac{s}{2}} u, u \times (-\Delta)^{\frac{s}{2}}\phi\big).$$

It thus remains to show that

$$b_j \to \big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}}(u\times\phi) - u \times (-\Delta)^{\frac{s}{2}}\phi\big)$$

to deduce the convergence result (14).

For ease of notation we abbreviate the second argument in the inner products defining the quantities b_j, i.e., we define a bilinear operator H_s via

$$H_s(f,g) = (-\Delta)^{\frac{s}{2}}(fg) - f(-\Delta)^{\frac{s}{2}} g - ((-\Delta)^{\frac{s}{2}} f) g.$$

H_s measures the error term in the fractional Leibniz rule. Since the fractional Laplace operator is applied componentwise we can extend the definition of H_s to products of vector fields. For this, we represent the linear cross product operation $z \mapsto u \times z$ for $z \in \mathbb{R}^{N'}$ by the a matrix multiplication $z \mapsto A_u z$ for a suitably defined matrix A_u. In particular, for a matrix-valued mapping $A(x) \in \mathbb{R}^{N\times N'}$ and a vector-valued function $v(x) \in \mathbb{R}^N$ we write

$$H_s(A,v)(x) = \Big(\sum_{k=1}^{N'} H_s(A^{i,k}, v^k)(x)\Big)_{i=1}^{N} \in \mathbb{R}^N.$$

With this preparation, the sequence $\{b_j\}_{j\in\mathbb{N}}$ is represented as

$$b_j = \big((-\Delta)^{\frac{s}{2}} u_j, H_s(u_j \times, \phi)\big).$$

The following proposition provides a strong continuity property of the operator H_s that implies the main result.

Proposition 3.1. Let $s \in (0,1)$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and assume that the sequence $\{f_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ converges to $f \in L^2(\mathbb{R}^d)$ locally in $\mathbb{R}^d$, that is

$$\forall K \subset \mathbb{R}^d \text{ compact: } \lim_{j\to\infty} \|f_j - f\|_{L^2(K)} = 0$$

and

$$\sup_{j\in\mathbb{N}} \|f_j\|_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\frac{s}{2}} f_j\|_{L^2(\mathbb{R}^d)} < \infty.$$

Then for every fixed $\varphi \in C_c^\infty(\Omega)$

$$\|H_s(f_j,\varphi) - H_s(f,\varphi)\|_{L^2(\mathbb{R}^d)} \xrightarrow{j\to\infty} 0.$$

Remark 3.2 ($s \geq 1$)**.** The result of Proposition 3.1 directly works for $s \in (0,2)$ with $s < d$. Further it can be extended to the case $s \geq 2$ by using that the classical Laplace operator satisfies a product rule.

Proof. Abbreviating $\tilde g_j = f_j - f$ we need to show that for any fixed $\varphi \in C_c^\infty(\Omega)$,

$$\|H_s(\tilde g_j, \varphi)\|_{L^2(\mathbb{R}^d)} \to 0.$$

Let $\eta_R \in C_c^\infty(B_{2R})$, $\eta \equiv 1$ in B_R, where we assume that R is large so that $B_{R/2} \supset \operatorname{supp}\varphi$. Then

$$H_s(\tilde g_j, \varphi) = H_s(\eta_R \tilde g_j, \varphi) + H_s((1-\eta_R)\tilde g_j, \varphi).$$

Observe that $g_j := \eta_R \tilde g_j$ satisfies the same assumptions as g_j and additionally it has compact support.

We are going to show that

$$\lim_{j\to\infty} \|H_s(\eta_R \tilde g_j, \varphi)\|_{L^2(\mathbb{R}^d)} = 0 \quad \forall R > 0 \tag{15}$$

and

$$\lim_{R\to\infty} \limsup_{j\to\infty} \|H_s((1-\eta_R)\tilde g_j, \varphi)\|_{L^2(\mathbb{R}^d)} = 0 \tag{16}$$

Together, (15) and (16) imply the claim.

Proof of (16)*:* Observe that by disjoint support of φ and $(1-\eta_R)$

$$H_s((1-\eta_R)\tilde g_j, \varphi) = \varphi(-\Delta)^{\frac{s}{2}}((1-\eta_R)\tilde g_j) + (-\Delta)^{\frac{s}{2}}\varphi((1-\eta_R)\tilde g_j). \tag{17}$$

Using the integral representation of $(-\Delta)^{\frac{s}{2}}$ we have for any $x \in \operatorname{supp} \varphi$ (and thus $1-\eta_R(x) = 0$),

$$(-\Delta)^{\frac{s}{2}} \left((1-\eta_R)\tilde{g}_j\right)(x) = c \int_{\mathbb{R}^d} |x-y|^{-s-d}(1-\eta_R(y))\tilde{g}_j(y)\,\mathrm{d}y.$$

Since $|x-y|^{-s-d}$ is smooth and bounded whenever $x \in \operatorname{supp}\varphi$ and $y \in \operatorname{supp}(1-\eta_R)$ we find that

$$\|(-\Delta)^{\frac{s}{2}} \left((1-\eta_R)\tilde{g}_j\right)\|_{L^\infty(\operatorname{supp}\varphi)} \precsim C(R)\|\tilde{g}_j\|_{L^2(\mathbb{R}^d)}$$

and

$$\|\nabla(-\Delta)^{\frac{s}{2}} \left((1-\eta_R)\tilde{g}_j\right)\|_{L^\infty(\operatorname{supp}\varphi)} \precsim C(R)\|\tilde{g}_j\|_{L^2(\mathbb{R}^d)}.$$

That is, by the assumptions on L^2-boundedness of $\tilde{g}_j$, the Lipschitz norm of $(-\Delta)^{\frac{s}{2}} \left((1-\eta_R)\tilde{g}_j\right)$ is uniformly bounded in $\operatorname{supp}\varphi$. On the other hand, by weak convergence we have for almost every x

$$(-\Delta)^{\frac{s}{2}} \left((1-\eta_R)\tilde{g}_j\right)(x) \to 0$$

as $j \to \infty$. Since a.e. limits and uniform limits must coincide, we can argue by Arzela-Ascoli, and conclude that

$$\limsup_{j\to\infty} \|(-\Delta)^{\frac{s}{2}} \left((1-\eta_R)\tilde{g}_j\right)\|_{L^\infty(\operatorname{supp}\varphi)} = 0.$$

For the other term in (17) we observe that for $x \in \operatorname{supp}(1-\eta_R)$

$$|(-\Delta)^{\frac{s}{2}}\varphi(x)| \precsim \int_{\mathbb{R}^d} |x-y|^{-s-d}|\varphi(y)|\,\mathrm{d}y \le \operatorname{dist}(x, \operatorname{supp}\varphi)^{-s-d}\|\varphi\|_{L^1(\mathbb{R}^d)}.$$

By assumption on R we have $\operatorname{dist}(\mathbb{R}^d\backslash B_R, \operatorname{supp}\varphi) \ge \frac{R}{2}$. Thus

$$\|(-\Delta)^{\frac{s}{2}}\varphi((1-\eta_R)\tilde{g}_j)\|_{L^2(\mathbb{R}^d)} \le R^{-s-d}\|\varphi\|_{L^1(\mathbb{R}^d)}\,\|\tilde{g}_j\|_{L^2(\mathbb{R}^d)}.$$

By the $L^2(\mathbb{R}^d)$-boundedness of $(\tilde{g}_j)_j$ and since $\varphi \in C_c^\infty(\mathbb{R}^d)$, we conclude that

$$\limsup_{j\to\infty} \|(-\Delta)^{\frac{s}{2}}\varphi((1-\eta_R)\tilde{g}_j)\|_{L^2(\mathbb{R}^d)} \precsim C(\varphi)R^{-s-d}$$

so that

$$\lim_{R\to\infty} \limsup_{j\to\infty} \|(-\Delta)^{\frac{s}{2}}\varphi((1-\eta_R)\tilde{g}_j)\|_{L^2(\mathbb{R}^d)} = 0.$$

Proof of (15)*:* It remains to show that for any fixed $R > 0$, setting $g_j := \eta_R\tilde{g}_j$

$$\|H_s(g_j, \varphi)\|_{L^2(\mathbb{R}^d)} \to 0.$$

Since $s \in (0,1)$, a direct calculation as in [48] or [28] yields that

$$|H_s(g_j, \varphi)(x)| \precsim \left| \int_{\mathbb{R}^d} \frac{(g_j(x)-g_j(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+s}}\,\mathrm{d}y \right|,$$

with a constant that depends on d and s. Our strategy is to show that for any $t \in (0,1)$, $t > s-1$

$$\|H_s(g_j, \varphi)\|_{L^2(\mathbb{R}^d)} \precsim \left(\|\varphi\|_{L^\infty(\mathbb{R}^d)} + \|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}\right)\left(\|g_j\|_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\frac{t}{2}}g_j\|_{L^2(\mathbb{R}^d)}\right). \tag{18}$$

A compact embedding property $\widetilde{H}^s(\Omega) \to \widetilde{H}^t(\Omega)$ proved in Proposition 3.3 below then implies the statement.

It thus remains to prove (18). Since Ω is bounded there is $R > 0$ such that $\overline{\Omega} \subset B_{R/2}$, where B_ρ denotes the ball of radius ρ centered at 0. We partition $\mathbb{R}^d = (\mathbb{R}^d \setminus B_R) \cup B_R$ and thereby obtain the estimate

$$\|H_s(g_j,\varphi)\|^2_{L^2(\mathbb{R}^d)} \precsim I + II + III + IV,$$

where

$$\begin{aligned}
I &= \int_{\mathbb{R}^d\setminus B_R} \left| \int_{\mathbb{R}^d\setminus B_R} \frac{(g_j(x)-g_j(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+s}} \,\mathrm{d}y \right|^2 \mathrm{d}x,\\
II &= \int_{B_R} \left| \int_{\mathbb{R}^d\setminus B_R} \frac{(g_j(x)-g_j(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+s}} \,\mathrm{d}y \right|^2 \mathrm{d}x\\
&= \int_{B_{R/2}} \left| \int_{\mathbb{R}^d\setminus B_R} \frac{g_j(x)\varphi(x)}{|x-y|^{d+s}} \,\mathrm{d}y \right|^2 \mathrm{d}x,\\
III &= \int_{\mathbb{R}^d\setminus B_R} \left| \int_{B_{R/2}} \frac{g_j(y)\varphi(y)}{|x-y|^{d+s}} \,\mathrm{d}y \right|^2 \mathrm{d}x,\\
IV &= \int_{B_R} \left| \int_{B_R} \frac{(g_j(x)-g_j(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+s}} \,\mathrm{d}y \right|^2 \mathrm{d}x.
\end{aligned}$$

We will show that the terms I, II, III, IV are bounded in such a way that we can deduce (18).
Estimate for I. Noting that $\operatorname{supp}\varphi \cup \operatorname{supp} g_j \subset B_{\frac{1}{2}R}$ we find that

$$I = 0.$$

Estimate for II. Observe that if $x \in B_{R/2}$ and $y \in \mathbb{R}^d\setminus B_R$ then $|x-y| \approx 1+|y|$, with a constant depending on R. Thus

$$\begin{aligned}
II &\precsim \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} \int_{B_{R/2}} |g_j(x)|^2 \left| \int_{\mathbb{R}^d\setminus B_R} \frac{1}{(1+|y|)^{d+s}} \,\mathrm{d}y \right|^2 \mathrm{d}x\\
&\precsim \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} \|g_j\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Estimate for III. Similarly as for II, for $x \in \mathbb{R}^d\setminus B_R$ and $y \in B_{R/2}$ we have $|x-y| \approx 1+|x|$, and thus

$$\begin{aligned}
III &\precsim \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d\setminus B_R} \frac{1}{(1+|x|)^{d+s}} \left| \int_{B_{R/2}} |g_j(y)| \,\mathrm{d}y \right|^2 \mathrm{d}x\\
&\precsim \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} \|g_j\|^2_{L^1(B_{R/2})}\\
&\precsim \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} \|g_j\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Estimate for IV. Recall that we have [50, Proposition 6.6.] for any $t \in (0,1)$,

$$|g_j(x)-g_j(y)| \precsim |x-y|^t \left(|\mathcal{M}(-\Delta)^{\frac{t}{2}} g_j(x)| + |\mathcal{M}(-\Delta)^{\frac{t}{2}} g_j(y)| \right).$$

Here $\mathcal{M}$ is a finite power of the Hardy-Littlewood maximal function. Then

$$\begin{aligned}IV \precsim&\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^d)}\int_{B_R}\left(\int_{B_R}\left(|\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j(x)|+|\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j(y)|\right)\,\frac{1}{|x-y|^{d+s-1-t}}\,\mathrm{d}y\right)^2\mathrm{d}x\\ \precsim&\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^d)}\int_{B_R}|\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j(x)|^2\left(\int_{B_R}|x-y|^{1+t-s-d}\,\mathrm{d}y\right)^2\,\mathrm{d}x\\ &+\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^d)}\int_{B_R}\left(\int_{B_R}|\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j(y)|\,|x-y|^{1+t-s-d}\,\mathrm{d}y\right)^2\,\mathrm{d}x.\end{aligned}$$

Since $1+t-s>0$ we have for $|x|\leq R$ that

$$\int_{B_R}|x-y|^{1+t-s-d}\,\mathrm{d}y\leq C(R).$$

On the other hand recall the Riesz potential $I^\sigma=(-\Delta)^{-\sigma/2}$ which for $\sigma\in(0,d)$ is defined as

$$I^\sigma f(x)=c_{d,\sigma}\int_{\mathbb{R}^d}|x-y|^{\sigma-d}f(y)\,\mathrm{d}y.$$

Then,

$$\begin{aligned}\int_{B_R}\left(\int_{B_R}|\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j(y)|\,|x-y|^{1+t-s-d}\,\mathrm{d}y\right)^2\,\mathrm{d}x\precsim&\left\|I^{1+t-s}\left(\chi_{B_R}\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j\right)\right\|^2_{L^2(B_R)}\\ \precsim&\left\|I^{1+t-s}\left(\chi_{B_R}\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j\right)\right\|^2_{L^p(\mathbb{R}^d)}\end{aligned}$$

for any $p\in[2,\infty)$. Observe that $\chi_{B_R}\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j\in L^q(\mathbb{R}^d)$ for any $q\in[1,2]$. Indeed, by Hölder's inequality and maximal theorem, cf. [52],

$$\|\chi_{B_R}\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j\|_{L^q(\mathbb{R}^d)}\precsim C(R)\|\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j\|_{L^2(\mathbb{R}^d)}\precsim\|(-\Delta)^{\frac{t}{2}}g_j\|_{L^2(\mathbb{R}^d)}.$$

Since $1+t-s\in(0,d)$ there are $p\in[2,\infty)$ and $q\in[1,2]$ with

$$1+t-s-\frac{d}{q}=-\frac{d}{p}.$$

For such p and q, by Sobolev embedding the operator $I^{1+t-s}:L^q(\mathbb{R}^d)\to L^p(\mathbb{R}^d)$ is bounded. Consequently, for that choice of p and q we have

$$\begin{aligned}\left\|I^{1+t-s}\left(\chi_{B_R}\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j\right)\right\|_{L^p(\mathbb{R}^d)}\precsim&\|\mathcal{M}(-\Delta)^{\frac{t}{2}}g_j\|_{L^q(B_R)}\\ \precsim&\|(-\Delta)^{\frac{t}{2}}g_j\|_{L^2(B_R)}\precsim\|(-\Delta)^{\frac{t}{2}}g_j\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

On combining previous estimates we find that

$$IV\precsim\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^d)}\,\|(-\Delta)^{\frac{t}{2}}g_j\|^2_{L^2(\mathbb{R}^d)}.$$

The estimates for I,II,III,IV imply (18). □

The following embedding result is used in the proof of Proposition 3.1.

Proposition 3.3. Let $s \in (0,1)$ and let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Assume $\{g_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^d;\mathbb{R}^N)$ strongly converges to 0 with $\operatorname{supp} g_j \subset \overline{\Omega}$ and

$$\sup_{j\in\mathbb{N}} \|(-\Delta)^{\frac{s}{2}} g_j\|_{L^2(\mathbb{R}^d)} < \infty. \tag{19}$$

Then for any $t \in (0,s)$

$$\lim_{j\to\infty} \|(-\Delta)^{\frac{t}{2}} g_j\|_{L^2(\mathbb{R}^d)} = 0. \tag{20}$$

Proof. It suffices to show that there is a subsequence that satisfies (20), since then any cluster point of $(\|(-\Delta)^{\frac{t}{2}} g_j\|_{L^2(\mathbb{R}^d)})_{j\in\mathbb{N}}$ is zero, and thus the whole sequence converges. First we observe that g_j weakly converges to 0 in $W^{s,2}(\mathbb{R}^d)$. Indeed, by assumption

$$\sup_j \left(\|g_j\|_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\frac{s}{2}} g_j\|_{L^2(\mathbb{R}^d)}\right) < \infty. \tag{21}$$

Thus up to taking a subsequence g_j converges to some $g \in W^{s,2}(\mathbb{R}^d)$ with $g \equiv 0$ in $\mathbb{R}^d\backslash\Omega$. Since g_j converges to zero strongly in L^2, we know that g vanishes identically.

Noting $t < s$ we can use Sobolev embedding and have for some $p > 2$ (if $s < d/2$ we can take $p = 2d/(d-2s)$, otherwise any $p > 2$ is permitted) from (21)

$$\sup_j \left(\|(-\Delta)^{\frac{t}{2}} g_j\|_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\frac{t}{2}} g_j\|_{L^p(\mathbb{R}^d)}\right) \precsim \sup_j \left(\|g_j\|_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\frac{s}{2}} g_j\|_{L^2(\mathbb{R}^d)}\right). \tag{22}$$

As in [10] we split $\mathbb{R}^d$ into two sets. For $\varepsilon > 0$ we define

$$\Omega_{o,\varepsilon} = \{x \in \mathbb{R}^d : \operatorname{dist}(x,\Omega) > \varepsilon\}, \quad \Omega_{i,\varepsilon} = \mathbb{R}^d\backslash\Omega_{o,\varepsilon}.$$

Estimate on $\Omega_{o,\varepsilon}$. Since $\operatorname{supp} g \subset \overline{\Omega}$ and $\operatorname{dist}(\Omega,\Omega_{o,\varepsilon}) \succsim \varepsilon$ we have for $x \in \Omega_{o,\varepsilon}$

$$(-\Delta)^{\frac{t}{2}} g_j(x) = \int_\Omega g_j(y)|x-y|^{-d-t}\, \mathrm{d}y \precsim (1+|x|)^{-d-t} \int_\Omega |g_j(y)|\, \mathrm{d}y.$$

Using Hölder's inequality we then find

$$\|(-\Delta)^{\frac{t}{2}} g_j\|_{L^2(\Omega_{o,\varepsilon})} \precsim \ C(\varepsilon)\|g_j\|_{L^2(\mathbb{R}^d)} \to 0. \tag{23}$$

Estimate on $\Omega_{i,\varepsilon}$. Observe that if we set $f_j := (-\Delta)^{\frac{t}{2}} g_j$ we have from (21) and (22)

$$\sup_{j\in\mathbb{N}} \left(\|(-\Delta)^{\frac{s-t}{2}} f_j\|_{L^2(\mathbb{R}^d)} + \|f_j\|_{L^2(\mathbb{R}^d)}\right) < \infty.$$

From [10, Proposition 3.2] we obtain that f_j converges strongly to some f in $L^2(\Omega_{i,\varepsilon})$ (since $\Omega_{i,\varepsilon}$ is bounded). Since on the other hand f_j weakly converges to zero we have $f \equiv 0$ and thus

$$\lim_{j\to\infty} \|(-\Delta)^{\frac{t}{2}} g_j\|_{L^2(\Omega_{i,\varepsilon})} = \lim_{j\to\infty} \|f_j\|_{L^2(\Omega_{i,\varepsilon})} = 0. \tag{24}$$

On combining (23), (24) we infer (20), which completes the proof. □

Remark 3.2, also directly applies to Proposition 3.3. The propositions imply the main result of this section.

Theorem 3.4 (Weak compactness)**.** Let $\{u_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^d;\mathbb{R}^N)$ be a sequence such that

$$\sup_{j\in\mathbb{N}} \|u_j\|_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\Omega)} < \infty \tag{25}$$

$|u_j(x)|^2 \to 1$ as $j \to \infty$ for almost every $x \in \Omega$. For every accumulation point $u \in L^2(\mathbb{R}^d;\mathbb{R}^N)$ we have $|u(x)|^2 = 1$ for almost every $x \in \Omega$. Moroever if $u_j \rightharpoonup u$ in $L^2(\mathbb{R}^d,\mathbb{R}^N)$ as $j \to \infty$ then

$$\big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}}(u_j \times \phi)\big) \to \big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}}(u \times \phi)\big)$$

for every $\phi \in C_c^\infty(\Omega;\mathbb{R}^{N'})$. If $u_j \in \vec{N} + \widetilde{H}^s(\Omega;\mathbb{R}^N)$ for all $j \in \mathbb{N}$ then also $u \in \vec{N} + \widetilde{H}^s(\Omega;\mathbb{R}^N)$.

Proof. Set

$$\Gamma_j := \big((-\Delta)^{\frac{s}{2}} u_j, (-\Delta)^{\frac{s}{2}}(u_j \times \phi)\big)$$

and

$$\Gamma := \big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}}(u \times \phi)\big).$$

Since u_j is by assumption uniformly bounded in $H^s(\mathbb{R}^d)$, by Rellich's theorem, up to taking a subsequence u_{j_k} converges strongly to u in $L^2(K)$ for any compact set K.

Splitting as described in the beginning of this section Γ_j into a_j and b_j, we obtain from Proposition 3.1 that a subsequence of b_{j_k} converges to b.

That is, we have

$$\Gamma_{j_k} \to \Gamma.$$

We can make this argument for any subsequence of $(\Gamma_j)_{j\in\mathbb{N}}$ and obtain a subsubsequence which converges to Γ. This implies that any cluster point of $(\Gamma_j)_{j\in\mathbb{N}}$ must actually be Γ, which implies that Γ is indeed the limit of the whole sequence Γ_j. □

Remark 3.5. The assumed uniform bound $\|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\Omega)}$ in Theorem 3.4 can equivalently be replaced by a bound for $\|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^d)}$.

4. Finite element setting

We consider sequences of uniformly shape regular and conforming triangulations $\{\mathcal{T}_h\}_{h>0}$ of the bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$ consisting of triangles or tetrahedra; the parameter $h > 0$ represents a maximal mesh-size. The space of continuous, piecewise affine finite element functions is defined via

$$\mathcal{S}^1(\mathcal{T}_h) = \{v_h \in C(\overline{\Omega}) : v_h|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h\}.$$

We let $\mathcal{N}_h$ be the set of vertices of elements, which are the nodes of the finite element space. The set $\{\varphi_z : z \in \widetilde{\mathcal{N}}_h\}$ is the nodal basis consisting of hat functions $\varphi_z \in \mathcal{S}^1(\mathcal{T}_h)$ associated with vertices $z \in \mathcal{N}_h$. The corresponding nodal interpolation operator $\mathcal{I}_h : C(\overline{\Omega}) \to \mathcal{S}^1(\mathcal{T}_h)$ is given by

$$\mathcal{I}_h v = \sum_{z\in\mathcal{N}_h} v(z)\varphi_z.$$

We note the classical nodal interpolation estimates

$$h_T^{-1}\|(v - \mathcal{I}_h v)\|_{L^2(T)} + \|\nabla(v - \mathcal{I}_h v)\|_{L^2(T)} \lesssim h_T\|D^2 v\|_{L^2(T)}$$

for $v \in H^2(T)$ with the diameter $h_T > 0$ of an element $T \in \mathcal{T}_h$ and a constant $c > 0$ that is independent of $h > 0$. We remark that the discrete L^p norms defined via

$$\|v\|_{L_h^p(\Omega)}^p = \int_\Omega \mathcal{I}_h |v|^p \,\mathrm{d}x = \sum_{z \in \mathcal{N}_h} \beta_z |v(z)|^p, \quad \beta_z = \int_\Omega \varphi_z \,\mathrm{d}x,$$

for $v \in C(\overline{\Omega})$ are equivalent to L^p norms on the space $\mathcal{S}^1(\mathcal{T}_h)$. If the triangulations are quasiuniform then for given $u_h \in \mathcal{S}^1(\mathcal{T}_h)^N$ and $\phi \in C_c^\infty(\Omega; \mathbb{R}^{N'})$ we have for $0 < s \le 1$ that

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}}(u_h \times \phi - \mathcal{I}_h[u_h \times \phi])\| & \\ &\lesssim \|\nabla(u_h \times \phi - \mathcal{I}_h[u_h \times \phi]\| \\ &\lesssim h\big(\|u_h\|\|D^2\phi\|_{L^\infty(\Omega)} + \|\nabla u_h\|\|\nabla \phi\|_{L^\infty(\Omega)}\big) \\ &\lesssim h\|\phi\|_{W^{2,\infty}(\Omega)}\big(\|u_h\| + h^{s-1}\|(-\Delta)^{\frac{s}{2}} u_h\|\big) \\ &\lesssim ch^s\|\phi\|_{W^{2,\infty}(\Omega)}\big(\|u_h\| + \|(-\Delta)^{\frac{s}{2}} u_h\|\big). \end{aligned} \tag{26}$$

Here, we used the inequality $\|(-\Delta)^{\frac{s}{2}} v\| \lesssim \|\nabla v\|$ for $v \in H^1(\Omega)$ and the inverse estimate

$$\|\nabla v_h\| \lesssim h^{s-1}\|(-\Delta)^{\frac{s}{2}} v_h\| \tag{27}$$

for $v_h \in \mathcal{S}^1(\mathcal{T}_h)^N$, cf., e.g., [21, Prop. 3.1]. Below we also use the inverse estimate, which for quasi-uniform meshes immediately follows from [22, Eq. (3.2)], however, a similar expression can also be derived for just the shape regular meshes following the proof of [22, Lemma 5.2],

$$\|(-\Delta)^{\frac{s}{2}} v_h\| \lesssim h^{-s}\|v_h\|. \tag{28}$$

To impose exterior Dirichlet conditions and to approximate the fractional Laplace operator we consider a larger domain $\widetilde{\Omega} \subset \mathbb{R}^d$ with $\overline{\Omega} \subset \widetilde{\Omega}$ and a triangulation $\widetilde{\mathcal{T}}_h$ of $\widetilde{\Omega}$ that extends $\mathcal{T}_h$. We then let $\widetilde{\mathcal{I}}_h \vec{N} \in \mathcal{S}^1(\widetilde{\mathcal{T}}_h)^N$ be the nodal interpolant of $\vec{N} \in C_c^\infty(\widetilde{\Omega}; \mathbb{R}^N)$ on $\widetilde{\mathcal{T}}_h$. With this we obtain the following discrete variant of Theorem 3.4.

Corollary 4.1 (Discrete weak compactness)**.** Let $\{u_h\}_{h>0} \subset L^2(\widetilde{\Omega}; \mathbb{R}^N)$ be a sequence of finite element functions $u_h \in \mathcal{S}^1(\widetilde{\mathcal{T}}_h)^N$ subordinated to a sequence of quasiuniform triangulations $\{\mathcal{T}_h\}_{h>0}$ such that $u_h = \widetilde{\mathcal{I}}_h \vec{N}$ in $\widetilde{\Omega} \setminus \Omega$,

$$\|u_h\|_{L^2(\Omega)} + \|(-\Delta)^{\frac{s}{2}} u_h\| \le c$$

for all $h > 0$ and $|u_h(x)|^2 \to 1$ as $h \to 0$ for almost every $x \in \Omega$. For every accumulation point $u \in L^2(\widetilde{\Omega}; \mathbb{R}^N)$ we have $|u(x)|^2 = 1$ for almost every $x \in \Omega$ and if $u_{h'} \rightharpoonup u$ in $L^2(\widetilde{\Omega}; \mathbb{R}^N)$ for a subsequence $h' \to 0$ then we have

$$\big((-\Delta)^{\frac{s}{2}} u_{h'}, (-\Delta)^{\frac{s}{2}} \mathcal{I}_{h'}[u_{h'} \times \phi]\big) \to \big((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}}[u \times \phi]\big)$$

for every $\phi \in C_c^\infty(\Omega; \mathbb{R}^{N'})$ as $h' \to 0$.

Proof. The result follows from applying Theorem 3.4 to the corrected sequence $\{\widetilde{u}_h\}_{h>0} \subset L^2(\widetilde{\Omega}; \mathbb{R}^N)$ defined via $\widetilde{u}_h = u_h - \mathcal{I}_h \vec{N} + \vec{N}$, which satisfies $\widetilde{u}_h = \vec{N}$ in $\widetilde{\Omega} \setminus \Omega$, noting that $\mathcal{I}_h \vec{N} - \vec{N} \to 0$ in $H^1(\Omega; \mathbb{R}^N)$, and incorporating the estimate (26). ☐

5. Numerical schemes and convergence

In this section we devise numerical schemes for prototypical problems related to fractional harmonic maps into spheres and show that they approximate corresponding continuous objects. Throughout the following we use the definitions

$$\mathcal{A}_h = \{u_h - \mathcal{I}_h[\vec{N}] \in \mathcal{S}^1_0(\mathcal{T}_h)^N : |u_h(z)|^2 = 1 \text{ f.a. } z \in \mathcal{N}_h\},$$
$$\mathcal{F}_h[u_h] = \{v_h \in \mathcal{S}^1_0(\mathcal{T}_h)^N : v_h(z)\cdot u_h(z) = 0 \text{ f.a. } z \in \mathcal{N}_h\}.$$

5.1. **Fractional harmonic maps.**

We consider the problem of finding critical points for the fractional Dirichlet energy subject to a sphere constraint and Dirichlet exterior conditions in $\mathbb{R}^d \setminus \Omega$ determined by a suitable vector field $\vec{N} \in C^\infty(\mathbb{R}^d;\mathbb{R}^N)$. We recall that the problem is equivalent to determining $u \in \mathcal{A}$ such that

$$((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v) = 0 \tag{29}$$

for all $v \in C^\infty_c(\Omega;\mathbb{R}^N))$ with $u \cdot v = 0$ in Ω. A *discrete fractional harmonic map* $u_h \in \mathcal{A}_h$ satisfies the equation

$$((-\Delta)^{\frac{s}{2}} u_h, (-\Delta)^{\frac{s}{2}} v_h) = 0 \tag{30}$$

for all $v_h \in \mathcal{F}_h[u_h]$. Bounded sequences of discrete fractional harmonic maps weakly accumulate at fractional harmonic maps.

Proposition 5.1. Let $N \geq 2$ and $\{u_h\}_{h>0} \subset L^2(\Omega;\mathbb{R}^N)$ be a sequence of discrete fractional harmonic maps on a sequence of quasiuniform triangulations with $\|(-\Delta)^{\frac{s}{2}} u_h\| \leq c$ for all $h > 0$. Then every accumulation point $u \in \vec{N} + \widetilde{H}^s(\Omega;\mathbb{R}^N)$ satisfies $u \in \mathcal{A}$ and (29).

Proof. The statement is an immediate consequence of Corollary 4.1 together with the nodal interpolation estimate

$$\begin{aligned}\||u_h|^2 - 1\|_{L^1(\Omega)} &= \||u_h|^2 - \mathcal{I}_h[|u_h|^2]\|_{L^1(\Omega)} \\ &\lesssim h^2 \|D^2_h |u_h|^2\|_{L^1(\Omega)} \leq c h^2 \|\nabla u_h\|^2_{L^2(\Omega)},\end{aligned}$$

where D^2_h denotes the elementwise application of the second derivative. With the compact embedding $\widetilde{H}^s(\Omega;\mathbb{R}^N) \hookrightarrow L^1(\Omega;\mathbb{R}^N)$ and the inverse estimate (27) we deduce that $|u|^2 = 1 = \lim_{h'\to 0} |u_{h'}|^2$ almost everywhere in Ω. ∎

5.2. **Fractional harmonic map heat flow.**

We next discuss the convergence of numerical approximations of the L^2-gradient flow of the constrained fractional Dirichlet energy, i.e., suitable solutions $u : (0,T) \times \Omega \to S^{N-1}$ of the evolution equation

$$(\partial_t u, v) + ((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v) = 0$$

for all $v \in \widetilde{H}^s(\Omega;\mathbb{R}^N)$ with $u(t,x)\cdot v(x) = 0$. The problem is complemented by the Dirichlet exterior condition $u(t,\cdot) - \vec{N} \in \widetilde{H}^s(\Omega;\mathbb{R}^N)$ for all $t \in (0,T)$ and the initial condition $u(0,\cdot) = u_0$ in Ω. The following numerical scheme uses a semi-implicit time discretization with an explicit treatment of the linearized length constraint. Note that we follow [16] and avoid a correction step which leads to a progressive constraint violation.

Algorithm 5.2 (Discrete L^2-flow)**.** Let $\tau > 0$ and $u_h^0 \in \mathcal{I}_h \vec{N} + \mathcal{S}_0^1(\mathcal{T}_h)^N$ with $|u_h^0(z)|^2 = 1$ for all $z \in \mathcal{N}_h$. For $k = 0, 1, \dots, K$ compute $d_t u_h^k \in \mathcal{F}_h[u_h^{k-1}]$ such that

$$(d_t u_h^k, v_h) + \big((-\Delta)^{\frac{s}{2}}[u_h^{k-1} + \tau d_t u_h^k], (-\Delta)^{\frac{s}{2}} v_h\big) = 0$$

for all $v_h \in \mathcal{F}_h[u_h^{k-1}]$, and define $u_h^k = u_h^{k-1} + \tau d_t u_h^k$.

The algorithm is unconditionally stable and convergent; the violation of the constraint is bounded independently of the number of iterations.

Proposition 5.3. There exist uniquely defined iterates $\{u_h^k\}_{k=0,\dots,K} \in \mathcal{S}^1(\mathcal{T}_h)^N$ with $u_h^k|_{\widetilde{\Omega}\setminus\Omega} = \mathcal{I}_h \vec{N}]_{\widetilde{\Omega}\setminus\Omega}$ and for all $K' \le K$

$$\frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u_h^{K'}\|^2 + \tau \sum_{k=1}^{K'} \|d_t u_h^k\|^2 \le \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u_h^0\|^2.$$

Moreover, letting $e_{h,0}$ denote the discrete initial energy on the right-hand side of the inequality we have that

$$\|\mathcal{I}_h |u_h^k|^2 - 1\|_{L^1(\Omega)} \precsim \tau e_{h,0}.$$

If the triangulations are quasiuniform then every weak accumulation point

$$u \in \vec{N} + H^1(0,T; L^2(\Omega;\mathbb{R}^N)) \cap L^\infty(0,T; \widetilde{H}^s(\Omega;\mathbb{R}^N))$$

for $(h,\tau) \to 0$ of the sequence of linear interpolants $\{\widehat{u}_h\}_{h>0}$ of the iterates $\{u_h^k\}_{k=0,\dots,K}$ solves the fractional harmonic map heat flow problem.

Proof. For every $k = 1, 2, \dots, K$ we have by the Lax–Milgram lemma that there exists a unique solution $d_t u_h^k \in \mathcal{F}_h[u_h^{k-1}]$ for $k = 1, 2, \dots, K$. By choosing $v_h = d_t u_h^k$ in the discrete equation and using the binomial formula $2b \cdot (b-a) = (b-a)^2 + (b^2 - a^2)$ we find that

$$\|d_t u_h^k\|^2 + \frac{d_t}{2}\|(-\Delta)^{\frac{s}{2}} u_h^k\|^2 + \frac{\tau}{2}\|(-\Delta)^{\frac{s}{2}} d_t u_h^k\|^2 = 0.$$

A summation over $k = 1, 2, \dots, K'$ yields the asserted identity. The orthogonality relation $d_t u_h^k \cdot u_h^{k-1} = 0$ at the nodes in $\mathcal{N}_h$ shows that for all $z \in \mathcal{N}_h$ we have

$$|u_h^k(z)|^2 = |u_h^{k-1}(z)|^2 + \tau^2 |d_t u_h^k(z)|^2 = \dots = |u_h^0(z)|^2 + \tau^2 \sum_{\ell=1}^{k} |d_t u_h^\ell(z)|^2.$$

By using $|u_h^0(z)|^2 = 1$ and summing over the nodes $z \in \mathcal{N}_h$ and using the equivalence of discrete and continuous L^p norms we deduce the estimate for the constraint violation. We let $\widehat{u}_h$ and $u_h^\pm$ denote the piecewise linear and constant interpolants of $(u_h^k)_{k=0,\dots,K}$. In particular, $u_h^- = u_h^{k-1} + \tau d_t u_h^k$ and $u_h^+ = u_h^{k-1}$. For almost every $t \in (0,T)$ we have that

$$(\partial_t \widehat{u}_h, v_h) + \big((-\Delta)^{\frac{s}{2}} u_h^-, (-\Delta)^{\frac{s}{2}} v_h\big) = 0$$

for all $v_h \in \mathcal{S}_0^1(\mathcal{T}_h)^N$ satisfying $v_h \cdot u_h^+(t,\cdot) = 0$. With the help of Corollary 4.1 we may pass to a limit in this equation as $(h,\tau) \to 0$. ☐

5.3. **Spin dynamics.** We finally address the approximation of solutions of the unconstrained but length-preserving evolution equation

$$\partial_t u = (-\Delta)^s u \times u$$

for a given initial state u_0, which describes the physical principle that the rate of change of angular momentum equals torque. For simplicity, we consider here periodic or homogeneous Neumann boundary conditions on $\mathbb{R}^d \setminus \Omega$. The evolution equation is length and energy preserving which is also satisfied by the following numerical scheme. For this, the use of midpoint values

$$u_h^{k-1/2}(z) = \frac{1}{2}(u_h^k(z) + u_h^{k-1}(z))$$

is essential, we follow [36, 14].

Algorithm 5.4 (Discrete spin dynamics)**.** Let $\tau > 0$ and $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^3$ with $|u_h^0(z)|^2 = 1$ for all $z \in \mathcal{N}_h$. For $k = 1, 2, \dots, K$ compute $u_h^k \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that

$$(d_t u_h^k, v_h)_h = \big((-\Delta)^{\frac{s}{2}} u_h^{k-1/2}, (-\Delta)^{\frac{s}{2}} \mathcal{I}_h[u_h^{k-1/2} \times v_h]\big)$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$.

A useful representation of the scheme is obtained with the discrete fractional Laplacian $(-\Delta)_h^s : \mathcal{S}^1(\mathcal{T}_h)^3 \to \mathcal{S}^1(\mathcal{T}_h)^3$ obtained as the representative of the corresponding bilinear form with the discrete inner product, i.e., $(-\Delta)_h^s u_h \in \mathcal{S}_0^1(\mathcal{T}_h)^3$ is defined as the unique function $y_h \in \mathcal{S}_0^1(\mathcal{T}_h)^3$ with

$$(y_h, v_h)_h = \big((-\Delta)^{\frac{s}{2}} u_h, (-\Delta)^{\frac{s}{2}} v_h\big)$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. With this discrete operator we have

$$(d_t u_h^k, v_h)_h = \big(((-\Delta)_h^s u_h^{k-1/2}, \mathcal{I}_h[u_h^{k-1/2} \times v_h]\big)_h$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. Owing to the use of the discrete inner product this is equivalent to the equality of nodal values, i.e.,

$$d_t u_h^k(z) = u_h^{k-1/2}(z) \times (-\Delta)_h^s u_h^{k-1/2}(z)$$

for all $z \in \mathcal{N}_h$. With these preparations we deduce the constraint and energy preservation properties.

Proposition 5.5. There exists a sequence $\{u_h^k\}_{k=0,\dots,K}$ that satisfies the nonlinear discrete system of Algorithm 5.4 for $k = 1, 2, \dots, K$. Every solution $\{u_h^k\}_{k=0,\dots,K}$ satisfies $|u_h^k(z)|^2 = 1$ for all $z \in \mathcal{N}_h$ and $k = 0, 1, \dots, K$ and

$$\frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u_h^k\|^2 = \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u_h^0\|^2$$

for $k = 1, 2, \dots, K$.

Proof. Given u_h^{k-1} the average $\overline{u}_h^k = (u_h^k + u_h^{k-1})/2$ is required to satisfy $\Phi_h(\overline{u}_h^k)[v_h] = 0$ for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$, where

$$\Phi_h(\overline{u}_h^k)[v_h] = \frac{2}{\tau}(\overline{u}_h^k - u_h^{k-1}, v_h)_h - \big((-\Delta)_h^s \overline{u}_h^k, \mathcal{I}_h[\overline{u}_h^k \times v_h]\big)_h.$$

By choosing $v_h = \overline{u}_h^k$ we find that

$$\Phi_h(\overline{u}_h^k)[\overline{u}_h^k] = \frac{2}{\tau}(\overline{u}_h^k - u_h^{k-1}, \overline{u}_h^k) \geq \frac{2}{\tau}\big(\|\overline{u}_h^k\|^2 - \|\overline{u}_h^k\|\|u_h^{k-1}\|\big)$$
$$\geq \frac{1}{\tau}\left(\|\overline{u}_h^k\|^2 - \|\overline{u}_h^{k-1}\|^2\right),$$

i.e., $\Phi_h(\overline{u}_h^k)[\overline{u}_h^k] \geq 0$ for $\|\overline{u}_h^k\| \geq \|\overline{u}_h^{k-1}\|$. Hence, Brouwer's fixed-point theorem implies the existence of a solution $\Phi_h(\overline{u}_h^k)[v_h] = 0$ for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. If $\{u_h^k\}_{k=0,\dots,K}$ is an arbitrary sequence satisfying the equations of Algorithm 5.4 then choosing $v_h = u_h^{k-1/2}(z)\varphi_z$ implies that

$$\beta_z d_t u_h^k(z) \cdot u_h^{k-1/2}(z) = 0,$$

i.e., $\beta_z d_t |u_h^k(z)|^2 = 0$ and hence $|u_h^k(z)|^2 = 1$ for all $z \in \mathcal{N}_h$ and $k = 1, 2, \dots, K$. By choosing $v_h = (-\Delta)_h^s u_h^{k-1/2}$ we find that

$$0 = (d_t u_h^k, (-\Delta)_h^s u_h^{k-1/2})_h = ((-\Delta)^{\frac{s}{2}} d_t u_h^k, (-\Delta)^{\frac{s}{2}} u_h^{k-1/2}) = 0,$$

i.e., $d_t \|(-\Delta)^{\frac{s}{2}} u_h^k\|^2 = 0$. □

If the step size is sufficiently small then the nonlinear systems of equations that arise in the steps of Algorithm 5.4 have unique solutions which can be computed with a simple fixed-point iteration.

Proposition 5.6. Given $u_h^{k-1} \in \mathcal{S}^1(\mathcal{T}_h)^3$ with $\|u_h^{k-1}\|_{L^\infty(\Omega)} = 1$ a solution $u_h^k \in \mathcal{S}^1(\mathcal{T}_h)^3$ is determined via $u_h^k = 2r_h - u_h^{k-1}$, where $r_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ is a fixed point of the iteration

$$r_h^\ell = u_h^{k-1} + \frac{\tau}{2}\mathcal{I}_h[r_h^\ell \times (-\Delta)_h^s r_h^{\ell-1}]$$

for $\ell = 1, 2, \dots$ with arbitrary $r_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^3$. The iteration is globally convergent provided that $\tau < c_{\text{inv}}^{-2} h^{2s} |\Omega|^{-1/2}$.

Proof. The Lax–Milgram lemma implies the existence of uniquely defined iterates $(r_h^\ell)_{\ell=1,2,\dots}$ which are equivalently characterized via

$$(r_h^\ell, v_h)_h = (u_h^{k-1}, v_h)_h + \frac{\tau}{2}(r_h^\ell \times (-\Delta)_h^s r_h^{\ell-1}, v_h)_h$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. By choosing $v_h = r_h^\ell$ we find that $\|r_h^\ell\|_h \leq \|u_h^{k-1}\|_h \leq |\Omega|^{1/2}$ for $\ell = 1, 2, \dots$. The difference $\delta_h^\ell = r_h^\ell - r_h^{\ell-1}$ of two iterates satisfies

$$\delta_h^\ell = \frac{\tau}{2}\mathcal{I}_h[\delta_h^\ell \times (-\Delta)_h^s r_h^{\ell-1}] + \frac{\tau}{2}\mathcal{I}_h[r_h^{\ell-1} \times (-\Delta)_h^s \delta_h^{\ell-1}]$$

using the inverse estimate (28) we find that $\|(-\Delta)_h^s r_h^{\ell-1}\|_h \leq c_{\text{inv}}^2 h^{-2s} |\Omega|^{1/2}$ and

$$2\|\delta_h^\ell\|_h \leq \tau c_{\text{inv}}^2 h^{-2s} |\Omega|^{1/2} \big(\|\delta_h^\ell\|_h + \|\delta_h^{\ell-1}\|_h\big).$$

Hence, if $q = \tau c_{\text{inv}}^2 h^{-2s} |\Omega|^{1/2} < 1$ we find that $\|\delta_h^\ell\|_h \leq q^{\ell-1} \|\delta_h^1\|_h$ and hence that r_h^ℓ converges as $\ell \to \infty$. □

Remark 5.7. With the linear interpolants $(\widehat{u}_{h,\tau})$ and the piecewise averages $(\overline{u}_{h,\tau})$ of the iterates $(u_h^k)_{k=0,\dots,K}$ the numerical scheme can be written as

$$(\partial_t \widehat{u}_{h,\tau}, v_h) + ((-\Delta)^{\frac{s}{2}} \overline{u}_h, (-\Delta)^{\frac{s}{2}} \mathcal{I}_h[\overline{u}_h \times v_h]) = 0$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. Weak accumulation points of the sequence $(\widehat{u}_{h,\tau})$ as $(h,\tau) \to 0$ for sequences of quasiuniform triangulations then satisfy the equation

$$-\int_0^T (u, \partial_t \phi)\,\mathrm{d}t + \int_0^T ((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}}[u \times \phi])\,\mathrm{d}t = (u_0, \phi(0))$$

for all $\phi \in C^\infty([0,T]; C_c^\infty(\Omega;\mathbb{R}^3))$ with $\phi(T,\cdot) = 0$. This follows from an application of Corollary 4.1.

6. Numerical experiments

In this section we illustrate the performance of the numerical methods via numerical experiments for one-dimensional spin chain dynamics and the fractional harmonic map heat flow. Fractional harmonic maps arise here as stationary limiting points of the fractional harmonic map heat flow.

6.1. **Spin dynamics.** We consider the spin system from [55]

$$\partial_t u = -u \times (-\Delta)^s u, \quad u(0) = u_0, \tag{31}$$

in a one-dimensional periodic setting, i.e., we use periodic boundary conditions on $\Omega = (0, 2\pi)$ and write $\Omega = \mathbb{T}$. This allows us to approximate the fractional Laplace operator via a Fourier sum, i.e., given a continuous function $w \in C(\mathbb{T})$ we define its discrete Fourier transform via the coefficients

$$\widetilde{v}_k = \frac{2\pi}{M} \sum_{j=0}^{M} e^{-\mathrm{i}kx_j} v(x_j)$$

for $k = -M/2, -M/2+1, \dots, M/2-1$, $M \in \mathbb{N}$ even, and with $x_j = j2\pi/M$, $j = 0, 1, \dots, M$, we refer the reader to [7] for details. The coefficients are obtained from standard implementations of the FFT method. The span of the trigonometric basis functions $\varphi^k(x) = e^{\mathrm{i}kx}$, $x \in \mathbb{T}$, $k = -M/2, \dots, M/2+1$, defines the discrete space $\mathcal{S}_M$. For $v \in \mathcal{S}_M$ we have the representation

$$v = \frac{1}{2\pi} \sum_{k=-M/2}^{M/2-1} \widetilde{v}_k \varphi^k.$$

The discrete fractional Laplace operator $(-\Delta)_M^s$ is for $v \in C(\mathbb{T})$ defined as

$$(-\Delta)_M^s v = \frac{1}{2\pi} \sum_{k=-M/2}^{M/2-1} |k|^{2s} \widetilde{v}_k \varphi^k.$$

We remark that for functions $v, w \in \mathcal{S}_M$ quadrature is exact in the approximation of the L^2 inner product of complex valued functions, i.e., we have

$$(v, w)_{L^2(\mathbb{T};\mathbb{C})} = \frac{2\pi}{M} \sum_{j=0}^{M} v(x_j)\overline{w(x_j)} = (v, \overline{w})_h.$$

With these settings we replace Algorithm 5.4 by the following iteration in which $\mathcal{T}_h$ is a uniform partition of $\mathbb{T}$ into M intervals $T_j = [x_{j-1}, x_j]$, $j = 1, 2, \dots, M$ of length $h = 2\pi/M$

Algorithm 6.1 (Discrete spin dynamics)**.** Let $\tau > 0$ and $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^3$ with $|u_h^0(z)|^2 = 1$ for all $z \in \mathcal{N}_h$. For $k = 1, 2, \dots, K$ compute $u_h^k \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that

$$d_t u_h^k = -\mathcal{I}_h\big[u_h^{k-1/2} \times (-\Delta)_M^s u_h^{k-1/2}\big].$$

Our first example leads to a solitary traveling wave solution given via the simplest Blaschke function $\mathcal{B}(z) = z$, cf. [38].

Example 6.2. Let $s = 1/2$, $T = 4\pi$, $v = 1/2$, and for $x \in \mathbb{T}$ define

$$u^0(x) = \big[v, (1-v^2)^{1/2}\cos(x), (1-v^2)^{1/2}\sin(x)\big]^{\mathsf{T}}.$$

Then $u(t, x) = u^0(x - vt)$ solves the spin dynamics system (31).

Figure 1 shows snapshots of the evolution computed with Algorithm 6.1. We observe that the initial state re-occurs when the time horizon $T = 4\pi$ is reached by the time stepping scheme. The nonlinear systems of equations in the time steps of the algorithm were approximately solved with the fixed-point iteration specified in the proof of Proposition 5.6. Our overall observation is that a few iterations are sufficient to decrease the L^2 difference of two iterates below the tolerance τ^2. Nearly no variations of the discrete energies and lengths of the vectors were observed.

The initial data in the second experiment are a perturbation of a harmonic map. We let $\Pi_{S^2} : \mathbb{R}^3 \setminus \{0\} \to S^2$ denote the orthogonal projection onto the unit sphere.

Example 6.3. Let $s = 1/2$, $T = 4$, and for $\xi : \mathbb{T} \to \mathbb{R}^3$ with $\|\xi\|_{L^\infty(\mathbb{T})} \le 1/2$ define

$$u^0(x) = \Pi_{S^2}\big[\xi_1(x), \cos(x) + \xi_2(x), \sin(x) + \xi_3(x)\big]^{\mathsf{T}}.$$

We used a perturbation ξ satisfying $\|\xi\|_{L^\infty(\mathbb{T})} \le 0.05$. Some iterates of the discrete evolution defined by the time-stepping scheme of Algorithm 6.1 are displayed in Figure 2. Due to the presence of the perturbation the solution oscillates between perturbations of the stationary states $u_\pm(x) = \pm\big[0, \cos(x), \sin(x)\big]^{\mathsf{T}}$. Because of the less regular solution compared with the example considered above, slightly more iterations are needed to solve the nonlinear systems of equations and a corresponding moderately increased violation of the energy conservation property is observed. For the tested discretizations with $M = 64, 128, 256$, $h = 2\pi/M$, and $\tau = h/10$, these violations were smaller than $\delta = 10^{-3}$ and decayed super linearly as $h \to 0$.

6.2. **Fractional harmonic map heat flow.** We next experimentally investigate the fractional harmonic map heat flow in two-dimensional domains. We consider the integral fractional Laplacian as defined in (12). For its discretization we follow [2] and replace the unbounded domain $\mathbb{R}^d$ by a bounded set $\widetilde{\Omega}$ with $\overline{\Omega} \subset \widetilde{\Omega}$, this defines a discrete fractional Dirichlet energy $E_{s,h}$ and a corresponding bilinear form. Our example enforces a singularity via smooth but topologically nontrivial boundary conditions which is implemented via an additive decomposition of the unknown. Alternative approaches for imposing the exterior boundary condition are discussed in [9, 11, 3]. The treatment of the linearized constraints

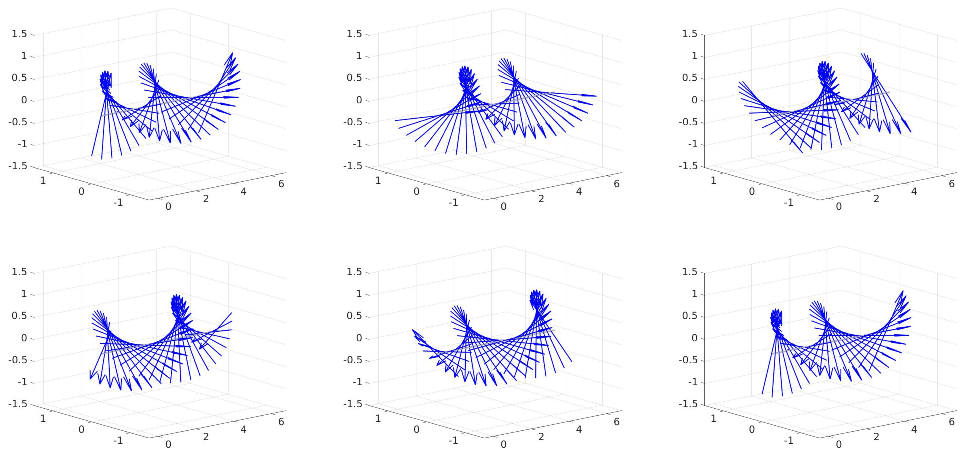

FIGURE 1. Snapshots of approximations of a solitary wave in a periodic spin chain for $t_\ell = (\ell/5)T$, $\ell = 0, 1, \ldots, 5$ (left to right, top to bottom), obtained with Algorithm 6.1 for $M = 32$, $h = 2\pi/M$, and $\tau = h/10$.

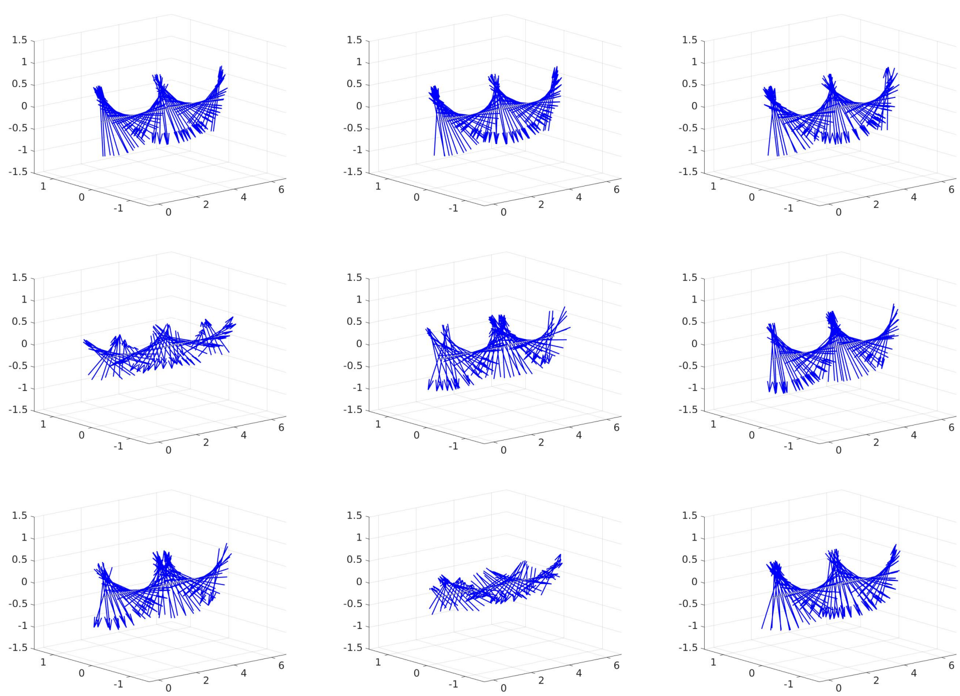

FIGURE 2. Snapshots of approximations at $t_\ell = (\ell/10)T$, $\ell = 0, 1, \ldots, 8$ (left to right, top to bottom), of an evolution resulting from a perturbed harmonic map as initial data. The approximations oscillate between nearly stationary states. The approximations were obtained with Algorithm 6.1 for $M = 64$, $h = 2\pi/M$, and $\tau = h/10$.

follows [15, section 7.2.5]. In both examples below, as the exterior data we use a function $\vec{N}$ with

$$\vec{N}(x) = \frac{x}{|x|}$$

for $x \in \partial\Omega$. Our initial vector fields where obtained via normalizations of certain random vectors at the inner nodes of the triangulations. We always used the step size $\tau = 2h$ and as stopping criterion the condition $\|d_t u_h^k\|_{s,h} < 10^{-6}$.

Example 6.4. We let $d = N = 2$, and consider the square $\Omega = (-0.5, 0.5)^2$ or the disk $\Omega = B_{0.5}$. As extended domain $\widetilde{\Omega}$ we choose a ball of radius $r = 1.5$ centered at the origin. We use an unstructured triangulation for $\widetilde{\Omega}$ generated using the package Gmsh [33] which extends an unstructrued triangulation of Ω.

Figure 3 displays the discrete energies $E_{s,h}[u_h^k]$, $k = 0, 1, \ldots, K$, of the iterates $u_h^k \in \mathcal{S}^1(\mathcal{T}_h)^N$ for different fractional parameters $0 < s < 1$ and fixed mesh size $h = 0.025$. The results confirm the theoretically established energy decay property of Algorithm 5.2. In particular, a rapid initial energy decay is followed by a slower further reduction of the energy before the process becomes nearly stationary. Figure 4 illustrates a corresponding discrete evolution for the case $s = 0.2$ and the mesh size $h = 0.025$ via snapshots of the iterates provided by Algorithm 5.2. We observe that the initial discontinuity of the initial function along parts of the boundary is quickly removed and a slightly diffused point defect develops which moves towards the center of the domain during the evolution.

Figures 5 and 6 show nearly stationary configurations u_h, i.e., nearly discrete fractional harmonic maps on the square and on the disk, for the values $s = 0.2$, $s = 0.4$, and $s = 0.6$, as well as decreasing mesh sizes $h = 0.048$, $h = 0.033$, and $h = 0.025$ in the case of the square. For larger values of s, we clearly observe well localized point defects. For the choice $s = 0.2$ we find that the defect is smeared out over a neighborhood of the origin in which the numerical solution is irregular and whose diameter appears to decay to zero as $h \to 0$. Owing to limitations in the spatial resolution and the occurring topological singularity we are unable to identify an experimental convergence behavior to the canonical solution candidate $u(x) = x/|x|$, cf. [39]. However, for both cases of domains we obtain numerical solutions that appear to be very close to this vector field.

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Appendix A. Spectral fractional Laplacian

In this section, we will prove results analogous to section 3 but for a different definition of fractional Laplacian. For any $s \geq 0$, consider the fractional order Sobolev space

$$\mathbb{H}^s(\Omega) = \big\{u = \sum_{k=1}^{\infty} u_k \varphi_k \in L^2(\Omega) : \ \ \|u\|^2_{\mathbb{H}^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^s u_k^2 < \infty\big\}, \quad u_k = \int_\Omega u \varphi_k \,\mathrm{d}x.$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ are the eigenvalues and corresponding normalized eigenfunctions of the standard Laplacian for homogeneous Dirichlet boundary conditions.

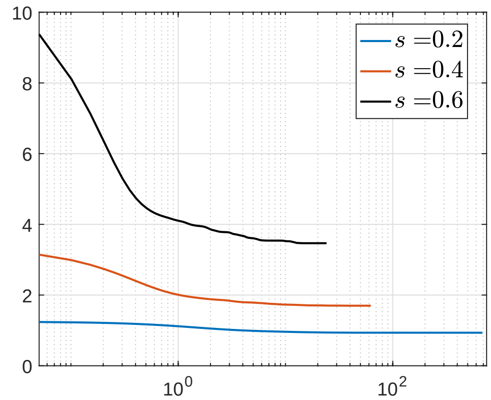

FIGURE 3. Discrete fractional energies $E_{s,h}[u_h^k]$, $k = 0, 1, 2, \ldots$, with respect to the time t^k, for a fixed mesh with meshsize $h = 0.025$, and different values of s in Example 6.4 with $\Omega = (0.5, 0.5)^2$. For all values of s an energy decay property is confirmed.

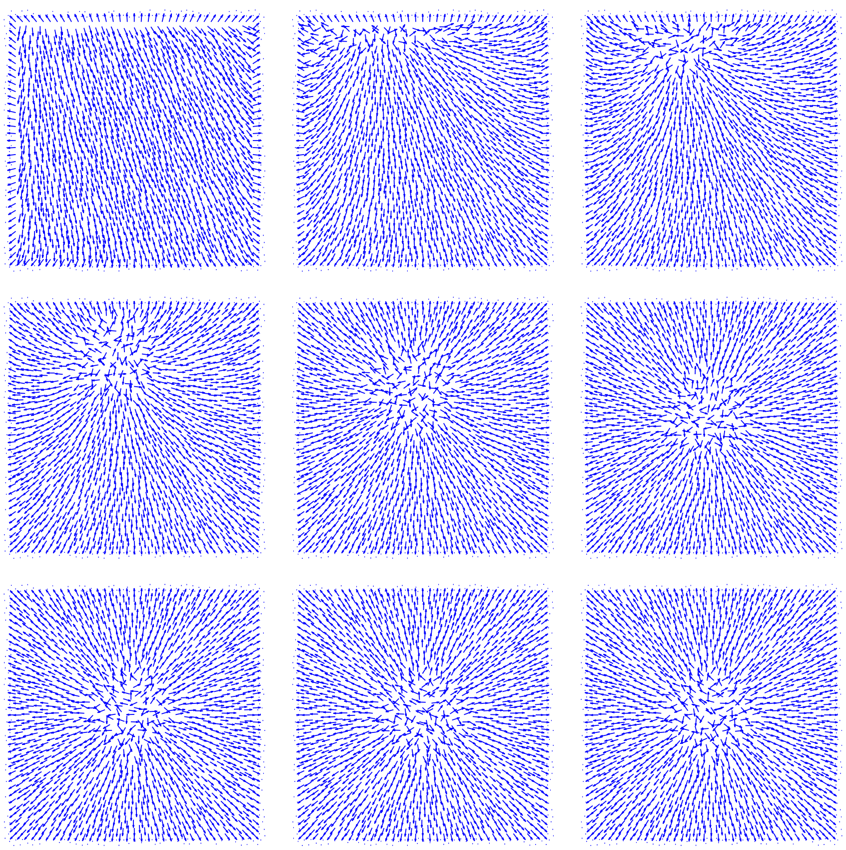

FIGURE 4. Snapshots of a discrete fractional harmonic map heat flow with $s = 0.2$ and $h = 0.025$ via approximations u_h^k for $t^k = 0.51, 3.03, 6.07, 12.13, 22.74, 45.49, 253.21, 510.96, 685.32$ (left to right, top to bottom) in Example 6.4. An initial discontinuity at the boundary is regularized and the formation of a point defect that moves to the origin is observed.

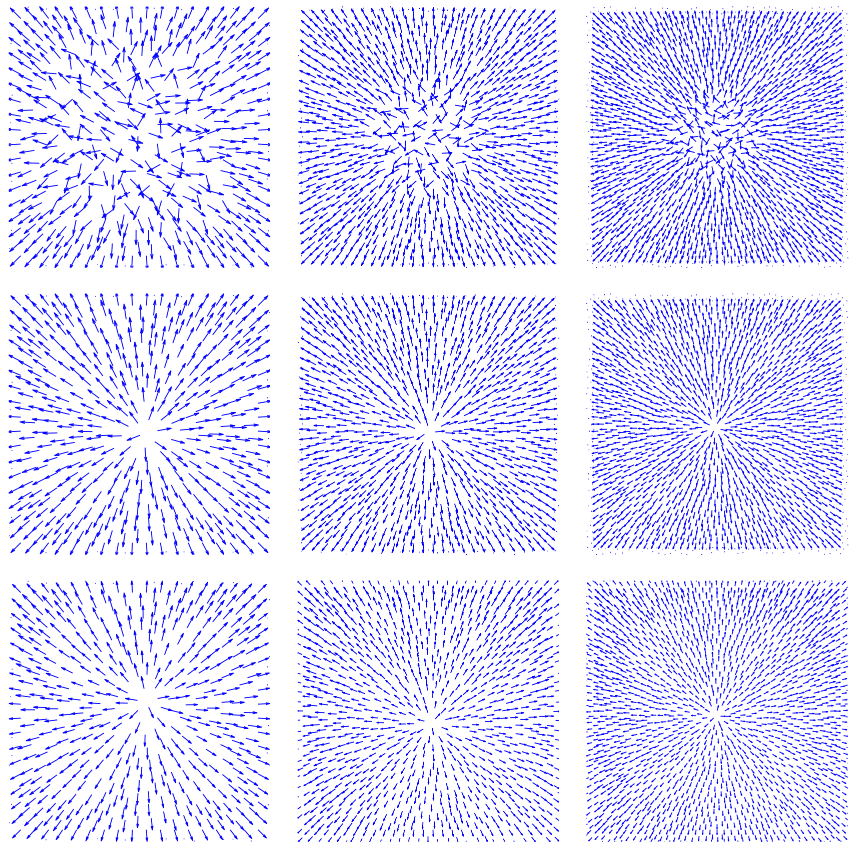

FIGURE 5. (Ω square) Approximate discrete fractional harmonic maps u_h in Example 6.4 for fractional parameters $s = 0.2$ (top), $s = 0.4$ (middle), $s = 0.6$ (bottom), and mesh sizes meshsizes $h = 0.048$ (left), $h = 0.033$ (middle), and $h = 0.025$ (right). In all cases a point defect is approximated which is more localized for larger values of s and smaller values of h.

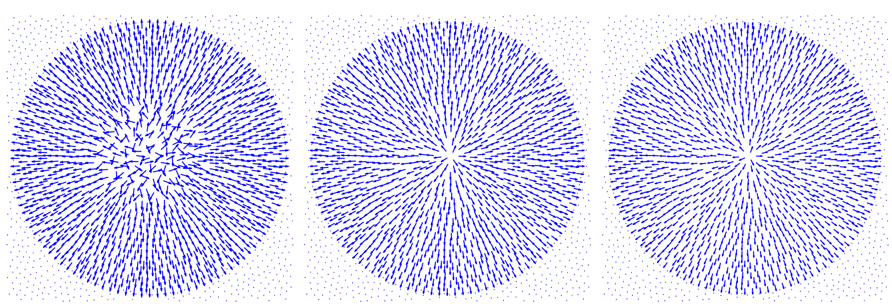

FIGURE 6. (Ω disc) Approximate discrete fractional harmonic maps u_h in Example 6.4 for fractional parameters $s = 0.2$ (left), $s = 0.4$ (middle), $s = 0.6$ (right), for a fixed mesh size $h = 0.028$. In all cases a point defect is approximated which is more localized for larger values of s and smaller values of h.

The spectral fractional Dirichlet Laplacian is defined on the space $\mathbb{H}^s(\Omega)$ by

$$(-\Delta_\Omega)^s u = \sum_{k=1}^{\infty} \lambda_k^s u_k \varphi_k\,, \qquad u_k = \int_\Omega u \varphi_k \,\mathrm{d}x.$$

We have that $\|u\|_{\mathbb{H}^s(\Omega)} = \|(-\Delta_\Omega)^{\frac{s}{2}} u\|_{L^2(\Omega)}$. An integral representation of the operator $(-\Delta_\Omega)^s$ from [26, Eq. (1.3)] states that for almost every $x \in \Omega$ we have

$$(-\Delta_\Omega)^s u(x) = \mathrm{P.V.} \int_\Omega [u(x) - u(y)]\, J(x,y)\,\mathrm{d}y + \kappa(x) u(x). \tag{32}$$

Letting $K_\Omega(t,x,y)$ denote the heat kernel of the semigroup generated by standard Laplace operator on $L^2(\Omega)$ and Γ be the usual Gamma function we have

$$J(x,y) = \frac{s}{\Gamma(1-s)} \int_0^\infty \frac{K_\Omega(t,x,y)}{t^{1+s}}\,\mathrm{d}t,$$
$$\kappa(x) = \frac{s}{\Gamma(1-s)} \int_0^\infty \Big(1 - \int_\Omega K_\Omega(t,x,y)\,\mathrm{d}y\Big)\frac{\mathrm{d}t}{t^{1+s}}.$$

From the properties of K_Ω we have that J is symmetric and nonnegative and that κ is nonnegative. Moreover, we have the estimate [26, Theorem 2.3]

$$0 \le J(x,y) \precsim |x-y|^{-d-s}. \tag{33}$$

As in section 3 we define for $f, g \in \mathbb{H}^s(\Omega)$

$$H_{s,\Omega}(f,g) = (-\Delta_\Omega)^{\frac{s}{2}}(fg) - f(-\Delta_\Omega)^{\frac{s}{2}} g - ((-\Delta_\Omega)^{\frac{s}{2}} f) g$$

We note that the operator extends to general bilinear operations on vector fields f and g in a canonical way. Analogously to Proposition 3.1 we have the following weak continuity property of the operator $H_{s,\Omega}$.

Proposition A.1. Let $\Omega \subset \mathbb{R}^d$ be bounded Lipschitz, $s \in (0,2)$ and $s < d$. Assume that

$$\sup_j \|f_j\|_{L^\infty(\Omega)} + \|f_j\|_{\mathbb{H}^s(\Omega)} < \infty$$

and

$$\lim_{j\to\infty} \|f_j - f\|_{L^2(\Omega)} = 0.$$

Given any $\varphi \in C_c^\infty(\Omega)$ we have

$$\|H_{s,\Omega}(f_j,\varphi) - H_{s,\Omega}(f,\varphi)\|_{L^2(\Omega)} \to 0$$

as $j \to \infty$.

Proof. We abbreviate $g_j = f_j - f$ and note that using the integral representation of the operator $(-\Delta_\Omega)^{\frac{s}{2}}$ we have

$$H_{s,\Omega}(g_j,\varphi) = \int_\Omega J(x,y)\,(g_j(x) - g_j(y))\,(\varphi(x) - \varphi(y))\,\mathrm{d}y\,\mathrm{d}x.$$

Since $\varphi \in C_c^\infty(\Omega)$ is fixed, we can choose $\varepsilon > 0$ and $\eta \in C_c^\infty(\Omega)$ such that $\eta(x) \equiv 1$ whenever $\mathrm{dist}\,(x, \mathrm{supp}\,\varphi) \le \varepsilon$ and $\eta \equiv 0$ whenever $\mathrm{dist}\,(x, \partial\Omega) \le \varepsilon$. Let $K = \{x \in \Omega : \eta(x) = 1\}$.

Set $\tilde{g}_j = \eta g_j$. We then have by Lemma A.2 below that

$$\|(-\Delta)^{\frac{s}{2}}\tilde{g}_j\|_{L^2(\mathbb{R}^d)} + \|\tilde{g}_j\|_{L^2(\mathbb{R}^d)} \leq C(\varepsilon)\|g_j\|_{\mathbb{H}^s(\Omega)}. \tag{34}$$

We split integrals using the partition of Ω into $\Omega \setminus K$ and K to obtain the estimate

$$\|H_{s,\Omega}(g_j,\varphi)\|^2_{L^2(\Omega)} \precsim I + II + III + IV,$$

where

$$\begin{aligned}
I &= \int_{\Omega\setminus K}\left|\int_{\Omega\setminus K} J(x,y)\,(g_j(x)-g_j(y))\,(\varphi(x)-\varphi(y))\,dy\right|^2 \mathrm{d}x,\\
II &= \int_{K}\left|\int_{\Omega\setminus K} J(x,y)\,(\tilde{g}_j(x)-\tilde{g}_j(y))\,\varphi(x)dy\right|^2 \mathrm{d}x,\\
III &= \int_{\Omega\setminus K}\left|\int_{K} J(x,y)\,(\tilde{g}_j(x)-\tilde{g}_j(y))\,\varphi(y)dy\right|^2 \mathrm{d}x,\\
IV &= \int_{K}\left|\int_{K} J(x,y)\,(\tilde{g}_j(x)-\tilde{g}_j(y))\,(\varphi(x)-\varphi(y))\,dy\right|^2 \mathrm{d}x.
\end{aligned}$$

We show that the terms I, II, III, IV converge to zero as $j \to \infty$ to deduce the asserted result.

Estimate for I. Recalling that $\operatorname{supp}\varphi \subset K$ we find that $I = 0$.

Estimate of II and III. We observe that if $\varphi(x) = 0$ then $\operatorname{dist}(x, \Omega\setminus K) \geq \varepsilon$. Thus if $x \in \Omega\setminus K$ and $y \in \operatorname{supp}\varphi$ (or $y \in \Omega\setminus K$ and $x \in \operatorname{supp}\varphi$) then $|x-y| \succsim \varepsilon$ and thus by (33) and thus

$$J(x,y) \leq C(\Omega,\varepsilon,s)\,\min\{1+|x|, 1+|y|\}^{-d-s}.$$

We then argue exactly as in the proof of Proposition 3.1 to obtain

$$II + III \precsim \|\varphi\|^2_{L^\infty(\mathbb{R}^d)}\|g_j\|^2_{L^2(\Omega)}.$$

Estimate of IV. As in the proof of Proposition 3.1 we obtain for some $t < s$ that

$$IV \precsim \|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^d)}\,\|(-\Delta)^{\frac{t}{2}}\tilde{g}_j\|^2_{L^2(\mathbb{R}^d)}.$$

Combining the estimates for I, II, III, IV we obtain

$$\|H_{s,\Omega}(g_j,\varphi)\|_{L^2(\Omega)} \precsim \left(\|\varphi\|_{L^\infty(\mathbb{R}^d)} + \|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}\right)\,\left(\|g_j\|_{L^2(\Omega)} + \|(-\Delta)^{\frac{t}{2}}\tilde{g}_j\|_{L^2(\mathbb{R}^d)}\right)$$

By assumption we have $\|g_j\|_{L^2(\Omega)} \to 0$ as $j \to \infty$ and in view of (34) and Proposition 3.3 that

$$\|(-\Delta)^{\frac{t}{2}}\tilde{g}_j\|_{L^2(\mathbb{R}^d)} \to 0.$$

This concludes the proof. □

The following auxiliary estimate is needed in the proof of Proposition A.1.

Lemma A.2. Let $\Omega \subset \mathbb{R}^d$ be a bounded set and $\eta \in C_c^\infty(\Omega)$. Then for any $g \in \mathbb{H}^s(\Omega)$ we have $\eta g \in H^s(\mathbb{R}^d)$ with the estimate

$$\|(-\Delta)^{\frac{s}{2}}(\eta g)\|_{L^2(\mathbb{R}^d)} + \|\eta g\|_{L^2(\mathbb{R}^d)} \leq C(\eta)\|g\|_{\mathbb{H}^s(\Omega)}.$$

Proof. The estimate follows by an interpolation argument of the mapping $T_\eta g = \eta g$ as a bounded linear operator $L^2(\Omega) \to L(\mathbb{R}^d)$ and $W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^d)$ in combination with the fact that the spaces $\mathbb{H}^s(\Omega)$ and $W^{s,2}(\mathbb{R}^d)$ are equivalently obtained via interpolation. □

References

[1] N. Abatangelo and E. Valdinoci. Getting acquainted with the fractional Laplacian. In *Contemporary research in elliptic PDEs and related topics*, volume 33 of *Springer INdAM Ser.*, pages 1–105. Springer, Cham, 2019.

[2] G. Acosta, F. M. Bersetche, and J. P. Borthagaray. A short FE implementation for a 2d homogeneous Dirichlet problem of a fractional Laplacian. *Comput. Math. Appl.*, 74(4):784–816, 2017.

[3] G. Acosta, J. P. Borthagaray, and N. Heuer. Finite element approximations of the nonhomogeneous fractional Dirichlet problem. *IMA J. Numer. Anal.*, 39(3):1471–1501, 2019.

[4] D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[5] F. Alouges. A new algorithm for computing liquid crystal stable configurations: the harmonic mapping case. *SIAM J. Numer. Anal.*, 34(5):1708–1726, 1997.

[6] F. Alouges. A new finite element scheme for Landau-Lifchitz equations. *Discrete Contin. Dyn. Syst. Ser. S*, 1(2):187–196, 2008.

[7] H. Antil and S. Bartels. Spectral approximation of fractional PDEs in image processing and phase field modeling. *Comput. Methods Appl. Math.*, 17(4):661–678, 2017.

[8] H. Antil, P. Dondl, and L. Striet. Approximation of integral fractional laplacian and fractional pdes via sinc-basis. *arXiv preprint arXiv:2010.06509*, 2020.

[9] H. Antil, R. Khatri, and M. Warma. External optimal control of nonlocal PDEs. *Inverse Problems*, 35(8):084003, 35, 2019.

[10] H. Antil, C. N. Rautenberg, and A. Schikorra. On a fractional version of a murat compactness result and applications. *To appear in SIAM J. of Math. Anal.*, 2021.

[11] H. Antil, D. Verma, and M. Warma. External optimal control of fractional parabolic PDEs. *ESAIM Control Optim. Calc. Var.*, 26, 2020.

[12] H. Antil and M. Warma. Optimal control of fractional semilinear PDEs. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 5, 30, 2020.

[13] J. W. Barrett, S. Bartels, X. Feng, and A. Prohl. A convergent and constraint-preserving finite element method for the p-harmonic flow into spheres. *SIAM J. Numer. Anal.*, 45(3):905–927, 2007.

[14] S. Bartels. Fast and accurate finite element approximation of wave maps into spheres. *ESAIM Math. Model. Numer. Anal.*, 49(2):551–558, 2015.

[15] S. Bartels. *Numerical methods for nonlinear partial differential equations*, volume 47 of *Springer Series in Computational Mathematics*. Springer, Cham, 2015.

[16] S. Bartels. Projection-free approximation of geometrically constrained partial differential equations. *Math. Comp.*, 85(299):1033–1049, 2016.

[17] S. Bartels, X. Feng, and A. Prohl. Finite element approximations of wave maps into spheres. *SIAM J. Numer. Anal.*, 46(1):61–87, 2007/08.

[18] S. Bartels and A. Prohl. Convergence of an implicit finite element method for the Landau-Lifshitz-Gilbert equation. *SIAM J. Numer. Anal.*, 44(4):1405–1419, 2006.

[19] B. Bojarski and P. Hajlasz. Pointwise inequalities for Sobolev functions and some applications. *Studia Math.*, 106(1):77–92, 1993.

[20] A. Bonito, W. Lei, and J. E. Pasciak. Numerical approximation of the integral fractional laplacian. *Numerische Mathematik*, 142(2):235–278, 2019.

[21] J. P. Borthagaray and P. Ciarlet, Jr. On the convergence in H^1-norm for the fractional Laplacian. *SIAM J. Numer. Anal.*, 57(4):1723–1743, 2019.

[22] J. P. Borthagaray, D. Leykekhman, and R. H. Nochetto. Local energy estimates for the fractional laplacian. *arXiv preprint arXiv:2005.03786*, 2020.

[23] J. P. Borthagaray, R. H. Nochetto, and S. W. Walker. A structure-preserving FEM for the uniaxially constrained Q-tensor model of nematic liquid crystals. *Numer. Math.*, 145(4):837–881, 2020.

[24] L. Bugiera, E. Lenzmann, A. Schikorra, and J. Sok. On symmetry of traveling solitary waves for dispersion generalized NLS. *Nonlinearity*, 33(6):2797–2819, 2020.

[25] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[26] L. Caffarelli and P. Stinga. Fractional elliptic equations, Caccioppoli estimates and regularity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(3):767–807, 2016.

[27] F. Da Lio and T. Rivière. Three-term commutator estimates and the regularity of $\frac{1}{2}$-harmonic maps into spheres. *Anal. PDE*, 4(1):149–190, 2011.

[28] P. D'Ancona. A short proof of commutator estimates. *J. Fourier Anal. Appl.*, 25(3):1134–1146, 2019.

[29] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[30] S. Dipierro, X. Ros-Oton, and E. Valdinoci. Nonlocal problems with Neumann boundary conditions. *Rev. Mat. Iberoam.*, 33(2):377–416, 2017.

[31] A. Fiscella, R. Servadei, and E. Valdinoci. Density properties for fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.*, 40(1):235–253, 2015.

[32] P. Gérard and E. Lenzmann. A Lax pair structure for the half-wave maps equation. *Lett. Math. Phys.*, 108(7):1635–1648, 2018.

[33] C. Geuzaine and J.-F. Remacle. Gmsh: A 3-d finite element mesh generator with built-in pre-and post-processing facilities. *International journal for numerical methods in engineering*, 79(11):1309–1331, 2009.

[34] P. Hajlasz. Sobolev spaces on an arbitrary metric space. *Potential Anal.*, 5(4):403–415, 1996.

[35] G. Hrkac, C.-M. Pfeiler, D. Praetorius, M. Ruggeri, A. Segatti, and B. Stiftner. Convergent tangent plane integrators for the simulation of chiral magnetic skyrmion dynamics. *Adv. Comput. Math.*, 45(3):1329–1368, 2019.

[36] T. K. Karper and F. Weber. A new angular momentum method for computing wave maps into spheres. *SIAM J. Numer. Anal.*, 52(4):2073–2091, 2014.

[37] J. Kraus, C.-M. Pfeiler, D. Praetorius, M. Ruggeri, and B. Stiftner. Iterative solution and preconditioning for the tangent plane scheme in computational micromagnetics. *J. Comput. Phys.*, 398:108866, 27, 2019.

[38] E. Lenzmann and A. Schikorra. On energy-critical half-wave maps into $\mathbb{S}^2$. *Invent. Math.*, 213(1):1–82, 2018.

[39] F.-H. Lin. A remark on the map $x/|x|$. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(12):529–531, 1987.

[40] K. Mazowiecka and A. Schikorra. Fractional div-curl quantities and applications to nonlocal geometric equations. *J. Funct. Anal.*, 275(1):1–44, 2018.

[41] V. Millot and M. Pegon. Minimizing 1/2-harmonic maps into spheres. *Calc. Var. Partial Differential Equations*, 59(2):Paper No. 55, 37, 2020.

[42] V. Millot, M. Pegon, and A. Schikorra. Partial regularity for fractional harmonic maps into spheres, 2020.

[43] V. Millot and Y. Sire. On a fractional Ginzburg-Landau equation and 1/2-harmonic maps into spheres. *Arch. Ration. Mech. Anal.*, 215(1):125–210, 2015.

[44] R. Moser. Intrinsic semiharmonic maps. *J. Geom. Anal.*, 21(3):588–598, 2011.

[45] R. H. Nochetto, S. W. Walker, and W. Zhang. A finite element method for nematic liquid crystals with variable degree of orientation. *SIAM Journal on Numerical Analysis*, 55(3):1357–1386, 2017.

[46] X. Pu and B. Guo. The fractional Landau-Lifshitz-Gilbert equation and the heat flow of harmonic maps. *Calc. Var. Partial Differential Equations*, 42(1-2):1–19, 2011.

[47] J. Roberts. A regularity theory for intrinsic minimising fractional harmonic maps. *Calc. Var. Partial Differential Equations*, 57(4):Paper No. 109, 68, 2018.

[48] A. Schikorra. Interior and Boundary-Regularity for Fractional Harmonic Maps on Domains. *arxiv, unpublished*, page arXiv:1103.5203, Mar 2011.

[49] A. Schikorra. ε-regularity for systems involving non-local, antisymmetric operators. *Calc. Var. Partial Differential Equations*, 54(4):3531–3570, 2015.
[50] A. Schikorra. Boundary equations and regularity theory for geometric variational systems with Neumann data. *Arch. Ration. Mech. Anal.*, 229(2):709–788, 2018.
[51] A. Schikorra, Y. Sire, and C. Wang. Weak solutions of geometric flows associated to integro-differential harmonic maps. *Manuscripta Math.*, 153(3-4):389–402, 2017.
[52] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
[53] M. Struwe. On the evolution of harmonic maps in higher dimensions. *J. Differential Geom.*, 28(3):485–502, 1988.
[54] C. J. Weiss, B. G. van Bloemen Waanders, and H. Antil. Fractional operators applied to geophysical electromagnetics. *Geophysical Journal International*, 220(2):1242–1259, 2020.
[55] T. Zhou and M. Stone. Solitons in a continuous classical Haldane-Shastry spin chain. *Phys. Lett. A*, 379(43-44):2817–2825, 2015.

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