ERROR ANALYSIS FOR THE NUMERICAL APPROXIMATION OF THE HARMONIC MAP HEAT FLOW WITH NODAL CONSTRAINTS

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ABSTRACT. An error estimate for a canonical discretization of the harmonic map heat flow into spheres is derived. The numerical scheme uses standard finite elements with a nodal treatment of linearized unit-length constraints. The analysis is based on elementary approximation results and only uses the discrete weak formulation.

1. INTRODUCTION

The harmonic map heat flow into spheres is obtained as the L^2 gradient flow of the Dirichlet energy on vector fields satisfying a pointwise unit length condition, i.e., for the functional

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x, \quad u \in H^1(\Omega, \mathbb{R}^m), \quad |u|^2 = 1.$$

Given initial data $u^0 \in H^1(\Omega, \mathbb{R}^m)$ with $|u^0(x)|^2 = 1$ for almost every $x \in \Omega$ the evolution problem reads in strong form

$$(1) \qquad \partial_t u - \Delta u = |\nabla u|^2 u, \quad u(0) = u^0, \quad |u|^2 = 1, \quad \nabla u|_{\partial\Omega} \cdot n = 0.$$

The problem admits possibly non-unique weak solutions which satisfy the energy decay property

$$(2) \qquad E(u(t)) + \int_0^t \|\partial_t u\|^2 \, \mathrm{d}s \leq E(u^0),$$

and the weak formulation of the evolution problem which is given by

$$(3) \qquad (\partial_t u, \phi) + (\nabla u, \nabla \phi) = 0,$$

for all $t \in [0, T]$ with test functions $\phi \in H^1(\Omega, \mathbb{R}^m)$ satisfying the orthogonality condition

$$\phi(x) \cdot u(t, x) = 0,$$

which arises as a linearization of the unit-length constraint. The time derivative $\partial_t u$ satisfies this condition, i.e., we have

$$\partial_t u(t, x) \cdot u(t, x) = 0.$$

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The well-posedness of the problem and properties of solutions have been investigated in, e.g., [22, 20]. For the developement of numerical methods it is attractive to exploit the elementary fact that the orthogonality implies the preservation of the unit-length constraint, i.e., the identity $\partial_t |u|^2 = 2\partial_t u \cdot u$ or equivalently,

$$(4) \qquad |u(t,x)|^2 - |u^0(t,x)|^2 = 2 \int_0^t \partial_t u(s,x) \cdot u(s,x) \, \mathrm{d}s,$$

yields that $|u|^2 = 1$ almost everywhere provided that the initial data has this property and the pointwise orthogonality $\partial_t u \cdot u = 0$ is fulfilled. An important observation for the derivation of error estimates is that regular solutions are unique and that a local stability result holds. Instead of deriving such a result from the full Euler–Lagrange equations (1) we follow the novel approach from [1] and consider the tangent space formulation

$$\partial_t u = P(u)\Delta u$$

with the tangential projection $P(s) = I - ss^{\mathsf{T}}$ at $s \in \mathbb{R}^m$. If u_* is an approximate solution satisfying $|u_*|^2 = 1$ almost everywhere, we define its defect d_* and residual r via

$$\partial_t u_* = P(u_*)\Delta u_* + d_* = P(u)\Delta u_* + r,$$

where $r = d_* - (P(u) - P(u_*))\Delta u_*$. This relation allows us to compare the equations and obtain an evolution equation for the error $e = u - u_*$, i.e.,

$$\partial_t e = P(u)\Delta e - r,$$

or in weak form

$$(\partial_t e, \phi) + (\nabla e, \nabla \phi) = -(r, \phi)$$

for all $\phi \in H^1(\Omega, \mathbb{R}^m)$ with $\phi \cdot u(t, \cdot) = 0$. The function $\partial_t e$ is an attractive test function to obtain an error estimate but may not be admissible. We thus consider its projection and note that, using, e.g., $P(u)\partial_t u = \partial_t u$,

$$\phi = P(u)\partial_t e = P(u)\partial_t u - P(u)\partial_t u_* = \partial_t e - q$$

with $q = -(P(u) - P(u_*))\partial_t u_*$. This implies that

$$\|\partial_t e\|^2 + (\nabla e, \nabla \partial_t e) = -(r, \partial_t e + q) - (\partial_t e, q) - (\nabla e, \nabla q).$$

Local Lipschitz estimates for the projection operator, cf. formula (16) below, yield that

$$\|q\|_{H^k} \le c_{a,k} \|e\|_{H^k}, \quad \|r\| \le c_{b,0} \|e\| + \|d_*\|,$$

where the constants $c_{a,k}$, $k = 1, 2$, and $c_{b,0}$ depend on higher order norms of u_*. We thus arrive at

$$\|\partial_t e\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla e\|^2 \le c_1 \|e\|_{H^1}^2 + c_2 \|d_*\|^2.$$

Using that $\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \le \|\partial_t e\|^2 + \|e\|^2$ a Gronwall argument proves

$$\sup_{t \in [0,T]} \|e(t)\|_{H^1}^2 \le \left(\|e(0)\|_{H^1}^2 + c_2 \int_0^T \|d_*\|^2 \, \mathrm{d}t \right) \exp\left((c_1 + 1)T \right).$$

In a semi- or fully discrete setting the approximations u_* are suitable inter-polants of a sufficiently regular exact solution of (3) and u is replaced by the solution of the numerical scheme.

As an example we consider the semi-discrete scheme which, following an idea by Alouges [2], determines for a step size $\tau > 0$ the sequence $(u^n)_{n=0,\dots,N} \subset H^1(\Omega, \mathbb{R}^m)$ via computing for given u^{n-1} the function $d_t u^n \in H^1(\Omega, \mathbb{R}^m)$ satisfying $d_t u^n \cdot u^{n-1} = 0$ in Ω and the linear system

$$(d_t u^n, \phi) + (\nabla u^n, \nabla \phi) = 0$$

for all $\phi \in H^1(\Omega, \mathbb{R}^m)$ satisfying $\phi \cdot u^{n-1} = 0$ in Ω. The new approximation u^n is given by $u^n = u^{n-1} + \tau d_t u^n$, in particular $d_t u^n$ is the backward difference quotient. By choosing $\phi = d_t u^n$ we find the unconditional energy stability

$$\frac{1}{2}\|\nabla u^{N'}\|^2 + \tau \sum_{n=1}^{N'} \|d_t u^n\|^2 \le \frac{1}{2}\|\nabla u^0\|^2.$$

for all $N' = 1, 2, \dots, N$. Furthermore, as observed in [12] the orthogonalities lead to the discrete version of relation (4) given by

$$|u^n|^2 = |u^{n-1}|^2 + \tau^2 |d_t u^{n-1}|^2 = \dots = 1 + \tau^2 \sum_{j=1}^n |d_t u^j|^2,$$

so that $|u^n|^2 \ge 1$ and $\||u^n|^2 - 1\|_{L^1} \le (\tau/2)\|\nabla u^0\|^2$, i.e., the constraint-violation is of order $O(\tau)$. The pointwise normalization of u^n, given by

$$\widetilde{u}_{\text{nor}}^n = \frac{u^n}{|u^n|},$$

is well defined and energy-decreasing which motivates considering $\widetilde{u}_{\text{nor}}^n$ as the new approximation. The energy-decreasing property of the normalization is however in general not satisfied in fully discrete settings, cf. [13], and therefore omitted. Moreover, including the projection in the scheme makes the numerical analysis more complicated as, e.g., $d_t u^n$ is not the backward difference quotient anymore if u^n is replaced by $\widetilde{u}_{\text{nor}}^n$. Nevertheless, our analysis shows that $\widetilde{u}_{\text{nor}}^n$ approximates the exact solution $u(t_n)$ with the same order as u^n which justifies the normalization as a postprocessing procedure.

With the auxiliary variable $\widehat{u}^u = u^{n-1}/|u^{n-1}|$ the iterates of the semi-discrete scheme satisfy

$$d_t u^n = P(\widehat{u}^n)\Delta u^n,$$

which leads to defining the defects d^n of the time-step evaluations $u_*^n = u(t_n)$, $t_n = n\tau$, via

$$d^n = P(\widehat{u}_*^n)(d_t u_*^n - \Delta u_*^n).$$

Since $P(u_*^n)(\partial_t u(t_n) - \Delta u_*^n) = 0$ we find that

$$d^n = P(\widehat{u}_*^n)\big(d_t u_*^n - \partial_t u(t_n)\big) + \big(P(\widehat{u}_*^n) - P(u_*^n)\big)(\partial_t u(t_n) - \Delta u_*^n),$$

and hence, with $I_n = [t_{n-1}, t_n]$, for u sufficiently regular, using a local Lipschitz estimate for P,

$$\|d^n\| \leq \|d_t u_*^n - \partial_t u(t_n)\| + c_u \|\widehat{u}_*^n - u_*^n\| \|\partial_t u(t_n) - \Delta u(t_n)\|_{L^\infty}$$
$$\leq \tau \|\partial_t^2 u\|_{C^0(I_n, L^2)} + c_u \tau \|\partial_t u\|_{C^0(I_n, L^2)} \|\partial_t u - \Delta u\|_{C^0(I_n, L^\infty)},$$

and thus

$$\tau \sum_{k=1}^{n} \|d^k\|^2 \leq c\tau^2 \|\partial_t^2 u\|_{L^2([0,T], L^2)}^2.$$

Using the strategy of the continuous perturbation result described above, we obtain the error estimate

$$\max_{n=0,...,N} \|u(t_n) - u^n\|_{H^1} \leq c_{\mathrm{sd}} \tau,$$

provided that u is sufficiently regular.

The fully discrete numerical scheme analyzed in [1] imposes the orthogonality condition in an averaged sense, i.e., via $\Pi_h(\widehat{u}_h^n \cdot \phi_h) = 0$ with the L^2 projection Π_h onto the underlying scalar finite element space. This definition of a discrete tangent space gives rise to a selfadjoint projection operator with suitable stability and approximation properties. Practically more efficient, in particular in view of the development of efficient iterative solvers [25] and the generalization of the methods to pointwise constraints in mechanical applications [11, 17], a nodewise variant of orthogonality appears attractive. In [1] it is argued that this can be analyzed by considering a suitable correction term. Here, we aim at a direct numerical analysis for the nodal variant of the constraint. We avoid the use of mass lumping which would provide a selfadjoint projection operator but would restrict the analysis to lowest order methods. Our results follow from basic estimates for nodal interpolation with and H^1 projection onto finite element spaces working only within the discrete weak formulation.

The finite element scheme computes iterates $(u_h^n)_{n=0,...,N}$ in a finite element space V_h and imposes the orthogonality conditions in the nodes of the underlying element. Hence, for given $\widehat{u} \in C(\overline{\Omega}; \mathbb{R}^m)$ we consider the discrete tangent space

$$(5) \qquad\qquad T_h(\widehat{u}) = \big\{ \phi_h \in V_h : \mathcal{I}_h(\phi_h \cdot \widehat{u}) = 0 \big\},$$

where $\mathcal{I}_h$ is the nodal interpolation operator associated with V_h. Note that $T_h(\widehat{u})$ only depends on the directions of the nodal values of $\widehat{u}$. Given an initial value $u_h^0 \in V_h$ we compute the sequence $(u_h^n)_{n=0,...,N} \in V_h$ by successively computing discrete time derivatives $d_t u_h^n \in T_h(\widehat{u}_h^n)$ such that

$$(6) \qquad\qquad (d_t u_h^n, \phi_h) + (\nabla u_h^n, \nabla \phi_h) = 0$$

for all $\phi_h \in T_h(\widehat{u}_h^n)$, where $\widehat{u}_h^n = u_h^{n-1}/|u_h^{n-1}|$. The error analysis becomes substantially more involved as, e.g., boundedness of the iterates away from zero has to be guaranteed. Our main result is the following variant of the results from [1].

Theorem 1.1 (Error estimate). *Assume that $\|u_h^0 - u^0\|_{H^1} \leq ch$ and let $(u_h^n)_{n=0,\ldots,N}$ be the continuous, piecewise linear finite element approximations obtained by* (6) *on a family of regular and quasi-uniform triangulations of Ω with mesh-sizes $h > 0$. Suppose that the exact solution u of the harmonic map heat flow* (1) *satisfies*

(7)
$$u \in C^2([0,T], H^1(\Omega)) \cap C^1([0,T], H^2(\Omega) \cap W^{1,\infty}(\Omega))$$
$$\cap C^0([0,T], W^{2,\infty}(\Omega)).$$

Then, for $h, \tau > 0$ sufficiently small we have that

$$\max_{n=0,\ldots,N} \|u_h^n - u(t_n)\|_{H^1} \leq c_{\mathrm{fd}}(\tau + h),$$

provided that $\tau \leq c_{\mathrm{m}} h^{1/2}$ with $c_{\mathrm{m}} > 0$ sufficiently small.

The regularity condition can be weakened to $u \in H^2([0,T], H^1(\Omega)) \cap H^1([0,T], H^2(\Omega) \cap W^{1,\infty}) \cap C^0([0,T], W^{2,\infty}(\Omega))$. This requirement and (7) can in general only be expected locally or for initial data with small initial energies. However, the optimal convergence of the numerical scheme in the case of smooth solutions underlines its efficiency and thereby complements weak convergence theories under minimal regularity assumptions.

Since our error analysis only uses elementary approximation results for Lagrange finite element methods, it directly extends to higher-order finite element methods of polynomial degree $r \geq 1$ with the convergence rate $O(\tau + h^r)$ under suitable regularity conditions. Moreover, by combining our consistency and stability bounds with the multiplier techniques used in [1], our error estimates extend to discretizations using linearly implicit backward difference formulae up to order $k \leq 5$ with the convergence rate $O(\tau^k + h^r)$ under suitable regularity conditions and in fact a weaker step-size condition. A crucial modification arises in the stability estimate for the error equation which is proved using multiplier techniqe based energy estimates, cf., e.g., [1, Appendix].

Various convergence theories for the numerical approximations obtained with variants of the iteration (6) have been established for the harmonic map heat flow and the closely related Landau–Lifshitz–Gilbert equations. Motivated by understanding the occurrence of singular solutions as in [19], the weak convergence of subsequences to weak solutions of the evolution problems has been established in [4, 16, 10, 15, 3, 14, 9, 5, 27, 26]. Error estimates with specific convergence rates have been proved under suitable regularity conditions, cf. [28, 21, 23, 6]. The method and estimates considered here are a variant of the arguments given in [1] using a nodal treatment of orthogonalities but also avoiding a projection step. Recently, error estimates for schemes that include such a step to guarantee the unit length condition in the nodes of a finite element space have been derived in [7, 8] and [24]. While they also lead to optimal error estimates they require more restrictive conditions and the methods may not be convergent in the absence

of a regular solution. A byproduct of our analysis shows that a postprocessing of our numerical solutions leads to approximations that obey the length constraint in the nodes and are quasi-optimal approximations of the exact solution.

The outline of this article is as follows. We specify notation and state some preliminary results in Section 2. In Section 3 we devise the fully discrete numerical scheme and derive properties of a projection operator related to the discrete tangent spaces. Section 4 provides stability bounds in terms of consistency terms. These lead to the main error estimate derived in Section 5.

2. PRELIMINARIES

We collect in this section some elementary facts about the harmonic map heat flow and numerical concepts for discretizing parabolic partial differential equations. We use standard notation for function spaces but often omit domains and target spaces when this is clear from the context. We abbreviate by $(\cdot, \cdot)$ and $\| \cdot \|$ the inner product and norm in $L^2(\Omega, \mathbb{R}^m)$. Throughout the article a factor c denotes a constant that may depend on regularity properties of an exact solution.

2.1. **Harmonic map heat flow.** For a bounded domain $\Omega \subset \mathbb{R}^{n_\Omega}$, with $n_\Omega = 1, 2, 3$, a time horizon $T > 0$, and given initial data $u^0 \in H^1(\Omega, \mathbb{R}^m)$ with $|u^0|^2 = 1$ almost everywhere in Ω we say that $u \in H^1((0, T), L^2(\Omega, \mathbb{R}^m) \cap L^\infty((0, T), H^1(\Omega, \mathbb{R}^m)$ is a weak solution of the harmonic map heat flow if $|u(t, x)|^2 = 1$ for almost every $(t, x) \in [0, T] \times \Omega$, and $u(t, 0) = u^0$, and

$$(\partial_t u, \phi) + (\nabla u, \nabla \phi) = 0$$

holds for almost every $t \in [0, T]$ and all $\phi \in H^1(\Omega, \mathbb{R}^m)$ with $\phi(x) \cdot u(t, x) = 0$ for almost every $x \in \Omega$. Defining the tangential projection operator $P(s) : \mathbb{R}^m \to \mathbb{R}^m$ for $s \in \mathbb{R}^m$ via

$$P(s) = I - ss^\mathsf{T},$$

we may state the the strong form (1) of the evolution problem as

$$\partial_t u = P(u)\Delta u, \quad |u|^2 = 1, \quad u(0, \cdot) = u^0, \quad \nabla u|_{\partial\Omega} \cdot n = 0.$$

For a function $\widehat{u} \in H^1(\Omega, \mathbb{R}^m)$ satisfying $|\widehat{u}|^2 = 1$ we define a tangent space $T(\widehat{u})$ relative to the unit sphere as

$$T(\widehat{u}) = \left\{ \phi \in H^1(\Omega, \mathbb{R}^m) : \phi \cdot \widehat{u} = 0 \right\}.$$

We note that we formally have $\partial_t u \in T(u)$ and that the weak formulation uses functions $\phi \in T(u)$. We also note the formal energy law (2) which follows from choosing $\phi = \partial_t u$ for regular solutions in the weak formulation of the flow; it can be rigorously established for suitably constructed weak solutions, cf. [29].

2.2. **Time discretization.** Given a step size $\tau > 0$ we define the time steps $t_n = n\tau$, $n = 0, 1, \ldots, N$, with $N \geq 0$ maximal such that $t_N \leq T$. We also define the time intervals $I_n = [t_{n-1}, t_n]$, $n = 1, 2, \ldots, N$. For a sequence $(a^n)_{n=0,\ldots,N}$ we define the backward difference quotient operator d_t via

$$d_t a^n = \tau^{-1}(a^n - a^{n-1})$$

for $n = 1, 2, \ldots, N$. If $u \in H^2(0, T; V)$ and $u^n = u(t_n)$ we have that

$$(8) \qquad \|d_t u^n\|_V = \left\| \tau^{-1} \int_{I_n} \partial_t u(s) \, ds \right\|_V \leq \|\partial_t u\|_{L^2_{\mathrm{av}}(I_n, V)},$$

where we use the abbreviation $\|\phi\|_{L^2_{\mathrm{av}}(I_n)} = \tau^{-1/2} \|\phi\|_{L^2(I_n)}$. Obviously, we have $\|\phi\|_{L^2_{\mathrm{av}}(I_n)} \leq \|\phi\|_{C^0(I_n)}$ if $\phi \in C^0(I_n; V)$. Moreover, we have

$$(9) \quad \|d_t u^n - \partial_t u(t_n)\|_V = \left\| \frac{1}{\tau} \int_{I_n} (t_{n-1} - s) \partial_t^2 u(s) \, ds \right\|_V \leq \frac{\tau}{\sqrt{3}} \|\partial_t^2 u\|_{L^2_{\mathrm{av}}(I_n, V)}.$$

2.3. **Space discretization.** For a regular and quasi-uniform triangulation $\mathcal{T}_h$ of the simplicial domain Ω with mesh-size $h > 0$ we denote the lowest order C^0-conforming finite element space of piecewise linear functions by $\mathcal{S}^1(\mathcal{T}_h)$ and abbreviate the corresponding vectorial finite element space by

$$V_h = \mathcal{S}^1(\mathcal{T}_h)^m.$$

We let $\mathcal{N}_h$ be the set of vertices of elements and denote the nodal interpolation operator applied to scalar or vector-valued functions by

$$\mathcal{I}_h : C(\overline{\Omega}; \mathbb{R}^\ell) \to \mathcal{S}^1(\mathcal{T}_h)^\ell, \quad \mathcal{I}_h v = \sum_{z \in \mathcal{N}_h} v(z) \varphi_z,$$

where $(\varphi_z : z \in \mathcal{N}_h)$ is the scalar nodal basis for $\mathcal{S}^1(\mathcal{T}_h)$. We let D_h^2 denote the elementwise defined Hessian and note that we have for $k = 0, 1$

$$\|v - \mathcal{I}_h v\|_{H^k} \leq c h^{2-k} \|\mathrm{D}_h^2 v\|$$

for $v \in H^1(\Omega)$ with $v|_K \in H^2(K)$ for all $K \in \mathcal{T}_h$. We make repeated use of inverse estimates, cf. [18, Section 4.5], which read for $v_h \in V_h$

$$(10) \qquad \|\nabla v_h\|_{L^p} \leq c h^{-1} \|v_h\|_{L^p}$$

and, incorporating a Sobelev inequality for $q = 2n_\Omega$,

$$(11) \qquad \|v_h\|_{L^\infty} \leq c h^{-n_\Omega/q} \|v_h\|_{L^q} \leq c h^{-1/2} \|v_h\|_{H^1}.$$

For $n_\Omega = 2$ the factor $h^{-1/2}$ can be replaced by $1 + |\log h|$, cf., e.g., [11], if $n_\Omega = 1$ it can be entirely omitted. We also make use of a mean-preserving Ritz projection $R_h : H^1(\Omega) \to V_h$, defined by

$$(12) \qquad (\nabla R_h v, \nabla w_h) + (R_h v, 1)(w_h, 1) = (\nabla v, \nabla w_h) + (v, 1)(w_h, 1),$$

for all $w_h \in V_h$. The element $R_h v \in V_h$ is uniquely defined by the Lax–Milgram lemma and choosing a constant function w_h yields that $(R_h v, 1) = (v, 1)$. We thus have that

$$(\nabla R_h v, \nabla w_h) = (\nabla v, \nabla w_h)$$

for all $w_h \in V_h$. If $\nabla v \cdot n = 0$ on $\partial \Omega$ we have that

$$(\nabla R_h v, \nabla w_h) = -(\Delta v, w_h).$$

Besides the standard H^1 error estimate for $v \in H^2(\Omega)$

(13) $$\|v - R_h v\|_{H^1} \leq ch \|v\|_{H^2},$$

we have the (generally suboptimal) L^∞ error estimate

(14) $$\|v - R_h v\|_{L^\infty} \leq ch \|v\|_{W^{2,\infty}}$$

for all $v \in W^{2,\infty}(\Omega)$ and that R_h is $W^{1,\infty}$ stable, cf. [18, Section 8.1].

2.4. **Normalization estimates.** It will be necessary to normalize vector fields that are uniformly bounded away from zero, i.e., for $u \in H^1(\Omega, \mathbb{R}^m)$ with $|u| \geq c_\ell > 0$ we define $N(u) \in H^1(\Omega, \mathbb{R}^m)$ via

$$N(u) = \widehat{u} = \frac{u}{|u|}.$$

Our first estimates concern stability properties of N.

Lemma 2.1 (Normalization bounds). *Let $u \in W^{1,\infty}(\Omega)$ and $u_h \in V_h$ with $0 < c_\ell \leq |u|, |u_h| \leq c_\ell^{-1}$. We then have that*

$$\|\nabla N(u)\|_{L^\infty} \leq c\|\nabla u\|_{L^\infty}, \quad \|\mathrm{D}_h^2 N(u_h)\|_{L^\infty} \leq c\|\nabla u_h\|_{L^\infty}^2.$$

Proof. The first estimate follows from the bound

$$\left| \partial_j \left(\frac{u}{|u|} \right) \right| \leq \left| \frac{\partial_j u}{|u|} \right| + \left| \frac{u(\partial_j u \cdot u)}{|u|^3} \right|.$$

Noting that $\partial_i \partial_j u_h = 0$ on every $K \in \mathcal{T}_h$ we verify that

$$\left| \partial_i \partial_j \left(\frac{u_h}{|u_h|} \right) \right| \leq \left| \frac{\partial_i u_h (\partial_j u_h \cdot u_h)}{|u_h|^3} \right| + \left| \frac{\partial_j u_h (\partial_i u_h \cdot u_h) + u_h (\partial_j u_h \cdot \partial_i u_h)}{|u_h|^3} \right|$$
$$+ \left| \frac{u_h (u_h \cdot \partial_i u_h)(u_h \cdot \partial_j u_h)}{|u_h|^5} \right|,$$

and deduce the second bound. □

The operator N is locally Lipschitz continuous.

Lemma 2.2 (Local Lipschitz estimate). *Let $k \in \{0, 1\}$ and $1 \leq p \leq \infty$. For all $u, \widetilde{u} \in W^{k,p}(\Omega, \mathbb{R}^m)$ with $0 < c_\ell \leq |u|, |\widetilde{u}| \leq c_\ell^{-1}$ in Ω we have*

$$\|N(u) - N(\widetilde{u})\|_{W^{k,p}} \leq c\|u - \widetilde{u}\|_{W^{k,p}}.$$

Proof. The estimate for $k = 0$ follows from the inequality

$$\left| \frac{u}{|u|} - \frac{\widetilde{u}}{|\widetilde{u}|} \right| = \left| \frac{u(|\widetilde{u}| - |u|) + |u|(u - \widetilde{u})}{|u||\widetilde{u}|} \right| \leq 2 \min\{|u|^{-1}, |\widetilde{u}|^{-1}\}|u - \widetilde{u}|.$$

If $k = 1$ we use $\partial_i N(u) = \partial_i u / |u| - u(\partial_i u \cdot u)/|u|^3$ to deduce the bound. □

Stability properties of the nodal interpolation of normalized vector fields are provided by the following lemma.

Lemma 2.3 (Stability of $\mathcal{I}_h$ on rational expressions). *Given elementwise polynomial functions $q_h, r_h \in C(\overline{\Omega})$, i.e., $q_h|_K, r_h|_K \in P_r(K)^m$ for all $K \in \mathcal{T}_h$, and such that $0 < c_\ell \leq |q_h| \leq c_\ell^{-1}$ we have for $k \in \{0,1\}$ and $1 \leq p \leq \infty$ that*

$$\left\| \mathcal{I}_h \left(\frac{r_h}{|q_h|} \right) \right\|_{W^{k,p}} \leq c \left\| \frac{r_h}{|q_h|} \right\|_{W^{k,p}}.$$

Proof. We note that for $K \in \mathcal{T}_h$ the set of pairs

$$\mathcal{K} = \left\{ (r_h, q_h) \in (P_r(K)^m)^2 : \|r_h\|_{L^p(K)} = 1, \, c_\ell \leq |q_h| \leq c_\ell^{-1} \right\}$$

is compact so that the continuous function

$$F : \mathcal{K} \to \mathbb{R}, \quad (r_h, q_h) \mapsto \frac{\|\mathcal{I}_h(r_h/|q_h|)\|_{L^p(K)}}{\|r_h/|q_h|\|_{L^p(K)}}$$

attains its maximum which implies the L^p stability result. With this, we also have the stability in $W^{1,p}$ norms, as, e.g., using the inverse estimate (10),

$$\|\nabla \mathcal{I}_h(r_h/|q_h|)\|_{L^p(K)} \leq ch^{-1} \|r_h/|q_h| - \alpha_K\|_{L^p(K)} \leq c\|\nabla(r_h/|q_h|)\|_{L^p(K)},$$

where α_K is the mean of $r_h/|q_h|$ on K. $\qquad\square$

3. FULLY DISCRETE SCHEME

In this section we devise the fully discrete time-stepping scheme and state some elementary properties about the discrete projection operator.

3.1. **Finite element discretization.**
The orthogonality condition included in the time-stepping scheme needs to be suitably discretized in a fully discrete scheme. Following [2, 13] we impose it at the nodes of the triangulation and define for $\widehat{u} \in C(\overline{\Omega}, \mathbb{R}^m)$, with $|\widehat{u}|^2 = 1$, a discrete tangent space via

$$T_h(\widehat{u}) = \left\{ \phi_h \in V_h : \mathcal{I}_h(\phi_h \cdot \widehat{u}) = 0 \right\}.$$

The scheme (6) thus computes the sequence $(u_h^n)_{n=0,\ldots,N}$ by determining $d_t u_h^n \in T_h(\widehat{u}_h^n)$ with $\widehat{u}_h^n = N(u_h^{n-1})$ that fulfills

$$(d_t u_h^n, \phi_h) + (\nabla u_h^n, \nabla \phi_h) = 0$$

for all $\phi_h \in T_h(\widehat{u}_h^n)$ with $u_h^n = u_h^{n-1} + \tau d_t u_h^n$.

We note that the scheme is unconditionally well defined and stable in the sense that solutions satisfy energy estimates, e.g., choosing $\phi_h = d_t u_h^n$ and using the binomial formula

$$(\nabla u_h^n, \nabla d_t u_h^n) = \frac{d_t}{2} \|\nabla u_h^n\|^2 + \frac{\tau}{2} \|\nabla d_t u_h^n\|^2,$$

we deduce that for $N' = 1, 2, \ldots, N$ we have

$$\frac{1}{2} \|\nabla u_h^{N'}\|^2 + \tau \sum_{n=1}^{N'} \|d_t u_h^n\|^2 \leq \frac{1}{2} \|\nabla u_h^0\|^2.$$

Moreover, we have the controlled violation of the constraint, i.e., arguing as in the derivation of (4) one finds that

$$(15) \qquad |u_h^n(z)|^2 - |u_h^0(z)|^2 = \tau^2 \sum_{j=1}^{n} |d_t u_h^j(z)|^2.$$

Noting that the iteration satisfies an energy decay property the term on the right-hand side is of order $\mathcal{O}(\tau)$ after summation over the nodes $z \in \mathcal{N}_h$.

3.2. **Discrete projection.** To quantify consistency properties of the fully discrete method we will often make use of the operator P_h defined for $\widehat{u} \in C(\overline{\Omega}; \mathbb{R}^m)$ and $v_h \in V_h$ by

$$P_h := \mathcal{I}_h P, \quad v_h \mapsto \mathcal{I}_h(P(\widehat{u})v_h).$$

Although it will not be used below, we note that it defines a projection onto $T_h(\widehat{u})$ with respect to the inner product

$$(v, w)_h = \int_\Omega \mathcal{I}_h(v \cdot w) \, \mathrm{d}x = \sum_{z \in \mathcal{N}_h} \beta_z v(z) \cdot w(z)$$

for $v, w \in C(\overline{\Omega}, \mathbb{R}^m)$ with $\beta_z = (1, \varphi_z)$ for all $z \in \mathcal{N}_h$. Note that this is in general not an inner product for higher-order methods.

Lemma 3.1 (Discrete projection). *Let $\widehat{u} \in C(\overline{\Omega})$ with $|\widehat{u}|^2 = 1$. The operator $P_h = \mathcal{I}_h P$ is self-adjoint with respect to $(\cdot, \cdot)_h$ i.e., for any $v_h, w_h \in V_h$ we have*

$$(P_h(\widehat{u})v_h, w_h)_h = (v_h, P_h(\widehat{u})w_h)_h.$$

Moreover, we have $P_h(\widehat{u})v_h \in T_h(\widehat{u})$ for every $v_h \in V_h$ and

$$(v_h - P_h(\widehat{u})v_h, w_h)_h = 0$$

for all $w_h \in T_h(\widehat{u})$, i.e., P_h is an orthogonal projection onto $T_h(\widehat{u})$ with respect to $(\cdot, \cdot)_h$.

Proof. For every $z \in \mathcal{N}_h$ we have that $P(\widehat{u}(z)) = I - \widehat{u}(z)\widehat{u}(z)^\mathsf{T}$ is symmetric and hence

$$(P(\widehat{u})v_h \cdot w_h)(z) = (v_h \cdot P(\widehat{u})w_h)(z),$$

so that a summation over $z \in \mathcal{N}_h$ yields the self-adjointness. Moreover, we find that $(P(\widehat{u})v_h)(z) \cdot \widehat{u}(z) = 0$ so that $P_h(\widehat{u})v_h \in T_h(\widehat{u})$. This and the self-adjointness imply the asserted orthogonality relation. $\qquad\square$

Remark 3.2. *The linearization of the length constraint at the nodes follows the aproaches from [4, 15]. An averaged version of the related orthogonality has been considered in [1] by defining*

$$T_h^{\mathrm{avg}}(u) = \big\{ \phi_h \in V_h : \Pi_h(u \cdot \phi_h) = 0 \big\},$$

with the (scalar) L^2 projection Π_h onto a finite element space. Defining P_h^{avg} as the L^2 projection onto T_h^{avg} leads to various stability estimates that require subtle arguments. The nodal variant considered here leads to simpler proofs of the estimates and allows for a straightforward numerical realization.

3.3. **Further properties of** P_h. We next establish stability and approximation properties of the discrete projection operator $P_h = \mathcal{I}_h P$. Similar properties were shown in [1, Section 5] for the averaged discrete tangential projection P_h^{avg}. Although the results are similar, the proofs given here are immediate consequences of nodal interpolation estimates.

Lemma 3.3 (Approximation). *For $u_h \in V_h$ with $1/2 \leq |u_h| \leq 2$ define $\widehat{u}_h = N(u_h)$. For $k \in \{0,1\}$ and $v_h \in V_h$ we have*

$$\|(P_h(\widehat{u}_h) - P(\widehat{u}_h))v_h\|_{H^k} \leq ch^{2-k}\|v_h\|_{W^{1,p}}\|\nabla u_h\|_{L^\infty}^2.$$

Proof. We deduce the estimates from corresponding elementwise estimates. Since $v_h|_K$ is linear for every $K \in \mathcal{T}_h$ we have

$$\|(P_h(\widehat{u}_h) - P(\widehat{u}_h))v_h\|_{H^k} \leq ch^{2-k}\|\mathrm{D}_h^2(P(\widehat{u}_h)v_h)\|$$
$$\leq ch^{2-k}\|v_h\|_{H^1}(\|\mathrm{D}_h^2\widehat{u}_h\|_{L^\infty} + \|\nabla\widehat{u}_h\|_{L^\infty}^2).$$

Incorporating Lemma 2.1 yields the estimate. $\qquad\square$

For the continuous projection operator we have the local Lipschitz estimates from [1, Lemma 4.1], i.e., for $k \in \{0,1\}$ and $u, \widetilde{u} \in W^{k,\infty}(\Omega)$ with $|u|, |\widetilde{u}| \leq 1$ in Ω and all $v \in W^{1,\infty}(\Omega)$ we have

$$(16) \qquad \|(P(u) - P(\widetilde{u}))v\|_{H^k} \leq c\|v\|_{W^{k,\infty}}\|\widetilde{u}\|_{W^{k,\infty}}\|u - \widetilde{u}\|_{H^k}.$$

The estimate follows from the identity

$$(17) \qquad\qquad -(P(u) - P(\widetilde{u})) = ee^{\mathsf{T}} + e\widetilde{u}^{\mathsf{T}} + \widetilde{u}e^{\mathsf{T}}$$

with $e = u - \widetilde{u}$. The discrete projection P_h satisfies similar estimates.

Lemma 3.4 (Discrete local Lipschitz estimate). *Let $u_{*,h}, u_h \in V_h$ such that $1/2 \leq |u_{*,h}|, |u_h| \leq 2$ and define $\widehat{u}_{*,h} = N(u_{*,h})$ and $\widehat{u}_h = N(u_h)$. Then, for all $v_h \in V_h$ and $k \in \{0,1\}$ we have that*

$$\|(P_h(\widehat{u}_{*,h}) - P_h(\widehat{u}_h))v_h\|_{H^k} \leq c\|v_h\|_{W^{k,\infty}}\|u_{*,h}\|_{W^{k,\infty}}\|u_{*,h} - u_h\|_{H^k}.$$

and

$$\|(P_h(\widehat{u}_{*,h}) - P_h(\widehat{u}_h))v_h\|_{L^1} \leq c\|v_h\|\|u_{*,h} - u_h\|.$$

Proof. Noting that, e.g., $P_h(\widetilde{u}_{*,h})v_h = \mathcal{I}_h(P(\mathcal{I}_h\widetilde{u}_{*,h})v_h)$, we deduce from Lemma 2.3 that

$$\|(P_h(\widehat{u}_{*,h}) - P_h(\widehat{u}_h))v_h\|_{W^{k,p}} = \|\mathcal{I}_h[(P(\mathcal{I}_h\widehat{u}_{*,h}) - P(\mathcal{I}_h\widehat{u}_h))v_h]\|_{W^{k,p}}$$
$$\leq c\|(P(\mathcal{I}_h\widehat{u}_{*,h}) - P(\mathcal{I}_h\widehat{u}_h))v_h\|_{W^{k,p}}.$$

With (16) we thus find that

$$\|(P_h(\widehat{u}_{*,h}) - P_h(\widehat{u}_h))v_h\|_{H^k} \leq c\|v_h\|_{W^{k,\infty}}\|\mathcal{I}_h\widehat{u}_{*,h}\|_{W^{k,\infty}}\|\mathcal{I}_h(\widehat{u}_{*,h} - \widehat{u}_h)\|_{H^k}.$$

Using Lemmas 2.3 and 2.2 we verify the first estimate. The estimate in L^1 is obtained similarly using a Hölder inequality, the bound (17), and Lemmas 2.3 and 2.2. $\qquad\square$

Our third result is the $W^{1,p}$ stability of P_h.

Lemma 3.5 (Stability). *For $u_h \in V_h$ with $1/2 \le |u_h| \le 2$ let $\widehat{u}_h = N(u_h)$. Then for $p \in \{2, \infty\}$ we have for every $v_h \in V_h$*

$$\|P_h(\widehat{u}_h)v_h\|_{W^{1,p}} \le c\|v_h\|_{W^{1,p}}\|u_h\|_{W^{1,\infty}}^2.$$

Proof. We note that $P_h(\widetilde{u}_h)v_h = \mathcal{I}_h(P(\mathcal{I}_h\widetilde{u}_h)v_h)$ and Lemma 2.3 verify that

$$\|P_h(\widehat{u}_h)v_h\|_{W^{1,p}} \le c\|P(\mathcal{I}_h\widehat{u}_h)v_h\|_{W^{1,p}} \le c\|\mathcal{I}_h\widehat{u}_h\|_{W^{1,\infty}}^2\|v_h\|_{W^{1,p}}.$$

The application of Lemmas 2.3 and 2.2 proves the result. □

4. Consistency estimates

We derive in this section consistency estimates for the fully discrete scheme under suitable regularity conditions adapting the approach from [1].

4.1. **Consistency bound.** Letting u be a regular solution of (1) and using the mean-preserving Ritz projection R_h and the normalization operator N we define for $n = 0, 1, \ldots, N$

$$u_*^n = u(t_n), \quad u_{*,h}^n = R_h u_*^n, \quad \widehat{u}_{*,h}^n = N(u_{*,h}^{n-1}).$$

For $u \in C^0(0, T, W^{2,\infty}(\Omega))$, using (14), we estimate

$$\|\,|u_{*,h}^n| - 1\|_{L^\infty} \le \|u_{*,h}^n - u_*^n\|_{L^\infty} \le ch\|u\|_{C^0(I_n, W^{2,\infty})}.$$

With this we deduce the uniform upper and lower bounds

$$(18) \qquad\qquad 1/2 \le |u_{*,h}^n(x)| \le 2,$$

for $n = 0, 1, \ldots, N$ with h sufficiently small. This implies that $\widehat{u}_{*,h}^n$ is well defined. Both upper and lower bounds will be used repeatedly. We define the full discretization defect $d_h^n \in T_h(\widehat{u}_{*,h}^n)$ via

$$(19) \qquad\qquad (d_h^n, \phi_h) = (d_t u_{*,h}^n, \phi_h) + (\nabla u_{*,h}^n, \nabla \phi_h)$$

for all $\phi_h \in T_h(\widehat{u}_{*,h}^n)$.

Lemma 4.1 (Consistency). *Assume that the solution u of (1) satisfies (7). We then have*

$$\|d_h^n\| \le c(h + \tau).$$

Proof. Letting $D_h^n = d_t u_{*,h}^n - \Delta u_*^n$ the definition of $u_{*,h}^n$ shows

$$(d_h^n, \phi_h) = (D_h^n, \phi_h)$$

for all $\phi_h \in T_h(\widehat{u}_{*,h}^n)$. We abbreviate $D^n = \partial_t u(t_n) - \Delta u(t_n)$ and use that $P(u_*^n)D^n = 0$ with the symmetric matrix $P(u_*^n)$ to infer that

$$\begin{aligned}(d_h^n, \phi_h) = {}&(D_h^n, (P_h(\widehat{u}_{*,h}^n) - P(\widehat{u}_{*,h}^n))\phi_h) \\ &+ (D_h^n - D^n, P(\widehat{u}_{*,h}^n)\phi_h) + (D^n, (P(\widehat{u}_{*,h}^n) - P(u_*^n))\phi_h).\end{aligned}$$

Choosing $\phi_h = d_h^n$ leads to

$$\begin{aligned}\|d_h^n\|^2 \le {}&\|D_h^n\|\|(P_h(\widehat{u}_{*,h}^n) - P(\widehat{u}_{*,h}^n))d_h^n\| \\ &+ \|D_h^n - D^n\|\|d_h^n\| + \|D^n\|_{L^\infty}\|P(\widehat{u}_{*,h}^n) - P(u_*^n)\|\|d_h^n\|.\end{aligned}$$

Using (8) and $W^{1,\infty}$ stability for R_h, we have

$$(20) \qquad \|D_h^n\|_{L^\infty} \le c(\|\partial_t u\|_{C^0(I_n, W^{1,\infty})} + \|\Delta u\|_{C^0(I_n, L^\infty)}).$$

With Lemma 3.3 and the inverse estimate (10), we find that

$$\|(P_h(\widehat{u}_{*,h}^n) - P(\widehat{u}_{*,h}^n)d_h\| \le ch\|d_h^n\|\|\nabla u_*^n\|_{L^\infty}^2 \le ch\|d_h^n\|\|u\|_{C^0(I_n, W^{1,\infty})}^2.$$

We next note that (9) implies

$$\|D_h^n - D^n\| = \|d_t u_{*,h}^n - \partial_t u(t_n)\| \le \tau\|\partial_t^2 u\|_{C^0(I_n, L^2)}.$$

Furthermore, we have

$$\|D^n\|_{L^\infty} \le \|\partial_t u\|_{C^0(I_n, L^\infty)} + \|u\|_{C^0(I_n, W^{2,\infty})}.$$

Finally, we use that $|u_*^{n-1}| = 1$ to deduce with (17) that

$$\begin{aligned} \|P(\widehat{u}_{*,h}^n) - P(u_*^n)\| &\le c\big(\|N(u_{*,h}^{n-1}) - N(u_*^{n-1})\| + \|u_*^{n-1} - u_*^n\|\big) \\ &\le c\big(\|u_{*,h}^{n-1} - u_*^{n-1}\| + \|u_*^{n-1} - u_*^n\|\big) \\ &\le c\big(h\|u\|_{C^0(I_n, H^2)} + \tau\|\partial_t u\|_{C^0(I_n, L^2)}\big). \end{aligned}$$

A combination of the estimates proves the result. $\qquad\square$

4.2. **Residual estimate.** The residual measures the violation of the numerical scheme by the Ritz projections of the exact solution relative to the orthogonality constraint defined by the numerical solution, i.e., we define $r_h^n \in T_h(\widehat{u}_h^n)$ via

$$(21) \qquad (r_h^n, \phi_h) = (d_t u_{*,h}^n, \phi_h) + (\nabla u_{*,h}^n, \nabla \phi_h)$$

for all $\phi_h \in T_h(\widehat{u}_h^n)$. Note that the defect d_h^n defined in (19) belongs to the space $T_h(\widehat{u}_{*,h}^n)$. The following lemma controls the difference.

Lemma 4.2 (Residual). *Assume that the solution u of (1) satisfies (7) and that $1/2 \le |u_h^{n-1}| \le 2$. We then have that*

$$\|r_h^n\| \le c\big(\|d_h^n\| + \|u_{*,h}^{n-1} - u_h^{n-1}\|\big).$$

Proof. Using the definition of $u_{*,h}^n$, noting $P_h(\widehat{u}_h^n)\phi_h = \phi_h$, incorporating the definition of d_h^n, and abbreviating $D_h^n = d_t u_{*,h}^n - \Delta u_*^n$, we have

$$\begin{aligned} (r_h^n, \phi_h) &= (D_h^n, P_h(u_{*,h}^n)\phi_h) + (D_h^n, (P_h(\widehat{u}_h^n) - P_h(\widehat{u}_{*,h}^n))\phi_h) \\ &= (d_h^n, P_h(u_{*,h}^n)\phi_h) + (D_h^n, (P_h(\widehat{u}_h^n) - P_h(\widehat{u}_{*,h}^n))\phi_h). \end{aligned}$$

Hence, with $\phi_h = r_h^n$ we deduce with Lemmas 3.4 and 3.5 that

$$\|r_h^n\|^2 \le c\big(\|d_h^n\| + \|D_h^n\|_{L^\infty}\|u_h^{n-1} - u_{*,h}^{n-1}\|\big)\|r_h^n\|.$$

Incorporating (20) implies the result. $\qquad\square$

4.3. **Test function correction.** By subtracting (21) from (6), the residuals r_h^n gives rise to the error equation

$$(22) \qquad (d_t e_h^n, \phi_h) + (\nabla e_h^n, \nabla \phi_h) = -(r_h, \phi_h)$$

with $e_h^n = u_h^n - u_{*,h}^n$ and for all $\phi_h \in T_h(\widehat{u}_h^n)$. Since $d_t e_h^n$ is in general not an admissible test function, we follow [1] and use $P_h(\widehat{u}_h^n) d_t e_h^n$. The following lemma controls the corresponding correction error.

Lemma 4.3 (Projected test function). *Assume that the solution u of* (1) *satisfies* (7) *and that* $1/2 \le |u_h^{n-1}| \le 2$. *There exist functions* $s_h^n, q_h^n \in V_h$ *such that*

$$(I - P_h(\widehat{u}_h^n)) d_t e_h^n = s_h^n + q_h^n$$

and, for $k = 0, 1$,

$$\|s_h^n\|_{H^1} \le c(h + \tau), \quad \|q_h^n\|_{H^k} \le c\|e_h^{n-1}\|_{H^k}.$$

Proof. Since $(I - P_h(\widehat{u}_h^n)) d_t u_h^n = 0$ we have that

$$(I - P_h(\widehat{u}_h^n)) d_t e_h^n = -(I - P_h(\widehat{u}_{*,h}^n)) d_t u_{*,h}^n - (P_h(\widehat{u}_{*,h}^n) - P_h(\widehat{u}_h^n)) d_t u_{*,h}^n$$
$$=: s_h^n + q_h^n.$$

(i) To estimate s_h^n we note that $(I - P(u_*^n)) \partial_t u(t_n) = 0$ and write

$$s_h^n = (\partial_t u(t_n) - d_t u_{*,h}^n) + (P(u_*^n) \partial_t u(t_n) - P_h(\widehat{u}_{*,h}^n) d_t u_{*,h}^n) =: \alpha + \beta.$$

We have

$$\|\alpha\|_{H^1} \le \|d_t u_{*,h}^n - d_t u_*^n\|_{H^1} + \|d_t u_*^n - \partial_t u(t_n)\|_{H^1}$$
$$\le ch\|\partial_t u\|_{C^0(I_n, H^2)} + c\tau\|\partial_t^2 u\|_{C^0(I_n, H^1)}.$$

To estimate for β we write

$$\beta = (P_h(\widehat{u}_{*,h}^n) - P(\widehat{u}_{*,h}^n)) d_t u_{*,h}^n + P(\widehat{u}_{*,h}^n)(d_t u_{*,h}^n - \partial_t u(t_n))$$
$$+ (P(\widehat{u}_{*,h}^n) - P(u_*^n)) \partial_t u(t_n) =: \beta_1 + \beta_2 + \beta_3.$$

Using Lemma 3.3 and the H^1- and $W^{1,\infty}$ stability of R_h leads to

$$\|\beta_1\|_{H^1} \le ch\|d_t u_*^n\|_{H^1}\|\nabla u_{*,h}^{n-1}\|_{L^\infty}^2$$
$$\le ch\|\partial_t u\|_{C^0(I_n, H^1)}\|u\|_{C^0(I_n, W^{1,\infty})}^2.$$

Using an H^1-bound for P, Lemma 2.1, and H^1-stability of R_h shows that

$$\|\beta_2\|_{H^1} \le c\|\widehat{u}_{*,h}^n\|_{W^{1,\infty}}^2\|d_t u_{*,h}^n - \partial_t u(t_n)\|_{H^1}$$
$$\le c\|u_{*,h}^{n-1}\|_{W^{1,\infty}}^2(\|d_t u_{*,h}^n - d_t u_*^n\|_{H^1} + \|d_t u_*^n - \partial_t u(t_n)\|_{H^1})$$
$$\le c\|u\|_{C^0(I_n; W^{1,\infty})}^2(h\|\partial_t u\|_{C^0(I_n, H^2)} + \tau\|\partial_t^2 u\|_{C^0(I_n, H^1)}),$$

Finally, using (16) and Lemma 2.2 we verify that

$$\|\beta_3\|_{H^1} \le c\|u_*^n\|_{W^{1,\infty}}\|\partial_t u(t_n)\|_{W^{1,\infty}}\|\widehat{u}_{*,h}^n - u_*^n\|_{H^1}$$
$$\le c(\|u_{*,h}^{n-1} - u_*^{n-1}\|_{H^1} + \|u_*^{n-1} - u_*^n\|_{H^1})$$
$$\le c(h\|u\|_{C^0(I_n, H^2)} + \tau\|\partial_t u\|_{C^0(I_n, H^1)}).$$

This implies the estimate for s_h^n.

(ii) To bound q_h^n we use Lemma 3.4 to verify that

$$\|q_h^n\|_{H^k} \le c\|d_t u_{*,h}^n\|_{W^{1,\infty}}\|u_{*,h}^{n-1}\|_{W^{1,\infty}}\|u_{*,h}^{n-1} - u_h^{n-1}\|_{H^k}.$$

The $W^{1,\infty}$ stability of R_h implies that $\|u_{*,h}^{n-1}\|_{W^{1,\infty}} \le c\|u\|_{C^0(I_n,W^{1,\infty})}$ and, incorporating (8), imply

$$\|d_t u_{*,h}^n\|_{W^{1,\infty}} \le c\|d_t u_*^{n-1}\|_{W^{1,\infty}} \le c\|\partial_t u\|_{C^0(I_n,W^{1,\infty})}.$$

A combination of the estimates proves the estimates for $\|q_h^n\|_{H^k}$. $\qquad\square$

Remark 4.4. *In the application of Lemmas 4.1, 4.2, and 4.3 only weighted sums of the squares of $\|d_h^n\|$ and $\|s_h^n\|_{H^1}$ are needed, cf. Lemma 5.1. Therefore, the conditions $u \in C^1([0,T], H^1(\Omega) \cap W^{1,\infty}(\Omega))$ and $u \in C^2([0,T], H^1(\Omega))$ can be replaced by the weaker requirements $u \in H^1([0,T], H^1(\Omega) \cap W^{1,\infty}(\Omega))$ and $u \in H^2([0,T], H^1(\Omega))$, respectively.*

5. Error analysis

The following lemma provides a discrete error estimate for the difference between the numerical solutions and the Ritz projections of a regular solution. It results from a stability argument for the error equation (22).

Lemma 5.1 (Error equation stability). *Let $(u_h^n)_{n=0,...,N}$ solve (6), and define $u_{*,h}^n = R_h(u(t_n)))$ for a solution u of (1) satisfying (7). Then for $h, \tau > 0$ sufficiently small, the discrete error $e_h^n = u_h^n - u_{*,h}^n$ satisfies*

$$(23) \qquad \max_{n=0,...,N} \|e_h^n\|_{H^1}^2 \le c_{\mathrm{stab}} B_{h,\tau}^2,$$

where $B_{h,\tau}$ is defined via

$$B_{h,\tau}^2 = \|e_h^0\|_{H^1}^2 + \tau \sum_{n=1}^N \left(\|d_h^n\|^2 + \|s_h^n\|_{H^1}^2\right),$$

provided that $B_{h,\tau}^2 \le c_B^2 h$ with c_B sufficiently small.

If $\|e_h^0\|_{H^1} \le ch$ then Lemmas 4.1 and 4.3 imply that $B_{h,\tau} \le c(h + \tau)$ so that the condition of Lemma 5.1 is satisfied if $\tau \le c_{\mathrm{m}} h^{1/2}$ with $c_{\mathrm{m}} > 0$ sufficiently small. The error estimates for the Ritz projections imply that

$$\max_{n=0,...,N} \|u(t_n) - u_h^n\|_{H^1} \le c(h + \tau).$$

This implies the result of Theorem 1.1. Moreover, the inverse estimate (11) shows $1/2 \le |u_h^{n-1}| \le 2$ and Lemma 2.2 implies that for the normalized approximations $N(u_h^n)$ we have

$$\|u(t_n) - N(u_h^n)\|_{H^1} = \|N(u(t_n)) - N(u_h^n)\|_{H^1} \le c\|u(t_n) - u_h^n\|_{H^1},$$

so that these satisfy the same approximation properties. We note that in view of a sharper inverse estimate the step-size condition can be weakened to $\tau \le c_{\mathrm{m}}(1 + |\log h|)^{-1}$ if $n_\Omega = 2$ and $\tau \le c_{\mathrm{m}}$ if $n_\Omega = 1$.

Proof. (i) To ensure the stability of the normalization the uniform bound $1/2 \leq |u_h^{n-1}| \leq 2$ is needed. We argue by induction and assume that (23) holds with N replaced by $N' - 1$. For $N' - 1 = 0$ this is satisfied since by definition of $B_{h,\tau}$ we have $\|e_h^0\|_{H^1} \leq B_{h,\tau}$. Then, the inverse estimate (11) and the assumption on $B_{h,\tau}$ with c_B small enough imply that

$$\|e_h^{n-1}\|_{L^\infty}^2 \leq c_{\mathrm{inv}}^2 h^{-1} \|e_h^{n-1}\|_{H^1}^2 \leq c_{\mathrm{inv}}^2 h^{-1} c_{\mathrm{stab}} B_{h,\tau}^2 \leq c_{\mathrm{inv}}^2 c_{\mathrm{stab}} c_B^2 \leq \frac{1}{16},$$

for all $n \leq N'$. We then deduce with (14) that

$$\|\,|u_h^{n-1}| - 1\|_{L^\infty} \leq \|u_h^{n-1} - u_*^{n-1}\|_{L^\infty} \leq \|e_h^{n-1}\|_{L^\infty} + \|u_{*,h}^{n-1} - u_*^{n-1}\|_{L^\infty} \leq \frac{1}{2}$$

for h sufficiently small.

(ii) For $n \leq N'$ we test the error equation (22) by $\phi_h = P_h(\widehat{u}_h^n) d_t e_h^n \in T_h(\widehat{u}_h^n)$ and use Lemma 4.3, which shows $\phi_h = d_t e_h^n - s_h^n - q_h^n$. This leads to

$$\|d_t e_h^n\|^2 + (\nabla e_h^n, \nabla d_t e_h^n) = (d_t e_h^n, s_h^n + q_h^n) + (\nabla e_h^n, \nabla(s_h^n + q_h^n))$$
$$- (r_h^n, d_t e_h^n - s_h^n - q_h^n).$$

With a binomial formula and Hölder and Young inequalites we deduce that

$$(24) \quad \begin{aligned} &\|d_t e_h^n\|^2 + \frac{d_t}{2}\|\nabla e_h^n\|^2 + \frac{\tau}{2}\|\nabla d_t e_h^n\|^2 \\ &\leq \frac{1}{2}\|d_t e_h^n\|^2 + 2\|s_h^n + q_h^n\|^2 + 2\|r_h^n\|^2 + \frac{1}{2}\|\nabla e_h^n\|^2 + \frac{1}{2}\|\nabla(s_h^n + q_h^n)\|^2. \end{aligned}$$

To obtain the full H^1-norm of e_h^n on the left-hand side we note that

$$\frac{d_t}{2}\|e_h^n\|^2 + \frac{\tau}{2}\|d_t e_h^n\|^2 = (e_h^n, d_t e_h^n) \leq \frac{1}{2}\|e_h^n\|^2 + \frac{1}{2}\|d_t e_h^n\|^2.$$

Hence, by adding $(1/2)\|e_h^n\|^2$ to both sides of (24) we find

$$\frac{d_t}{2}\|e_h^n\|_{H^1}^2 \leq 2\|s_h^n + q_h^n\|^2 + 2\|r_h^n\|^2 + \frac{1}{2}\|e_h^n\|_{H^1}^2 + \frac{1}{2}\|\nabla(s_h^n + q_h^n)\|^2.$$

Multiplication by 2τ and summation over $n = 1, 2, \ldots, n'$ with $n' \leq N'$ show that

$$\|e_h^{n'}\|_{H^1}^2 \leq \|e_h^0\|_{H^1}^2 + \tau \sum_{n=1}^{n'} \|e_h^n\|_{H^1}^2 + 4\tau \sum_{n=1}^{n'} \left(\|s_h^n + q_h^n\|_{H^1}^2 + \|r_h^n\|^2\right).$$

Absorbing $\|e_h^{n'}\|_{H^1}^2$ for τ sufficiently small and incorporating Lemmas 4.2 and 4.3 leads to

$$\|e_h^{n'}\|_{H^1}^2 \leq c_1 \tau \sum_{n=1}^{n'-1} \|e_h^n\|_{H^1}^2 + c_2 B_{h,\tau}^2$$

for all $n' = 1, 2, \ldots, N'$. A discrete Gronwall inequality proves

$$\max_{n'=1,\ldots,N'} \|e_h^{n'}\|_{H^1}^2 \leq c_{\mathrm{stab}} B_{h,\tau}^2,$$

with $c_{\text{stab}} = \exp(c_1 \tau N) \leq \exp(c_1 T)$ independently of N'. Hence, (23) holds with N' and this completes the induction argument. $\qquad\square$

REFERENCES

[1]   G. Akrivis, M. Feischl, B. Kovács, and C. Lubich. Higher-order linearly implicit full discretization of the Landau-Lifshitz-Gilbert equation. *Math. Comp.*, 90(329):995–1038, 2021. DOI: `10.1090/mcom/3597`.

[2]   F. Alouges. A new algorithm for computing liquid crystal stable configurations: the harmonic mapping case. *SIAM J. Numer. Anal.*, 34(5):1708–1726, 1997. DOI: `10.1137/S0036142994264249`.

[3]   F. Alouges. A new finite element scheme for Landau-Lifchitz equations. *Discrete Contin. Dyn. Syst. Ser. S*, 1(2):187–196, 2008. DOI: `10.3934/dcdss.2008.1.187`.

[4]   F. Alouges and P. Jaisson. Convergence of a finite element discretization for the Landau-Lifshitz equations in micromagnetism. *Math. Models Methods Appl. Sci.*, 16(2):299–316, 2006. DOI: `10.1142/S0218202506001169`.

[5]   F. Alouges, E. Kritsikis, J. Steiner, and J.-C. Toussaint. A convergent and precise finite element scheme for Landau-Lifschitz-Gilbert equation. *Numer. Math.*, 128(3):407–430, 2014. DOI: `10.1007/s00211-014-0615-3`.

[6]   R. An. Optimal error estimates of linearized Crank-Nicolson Galerkin method for Landau-Lifshitz equation. *J. Sci. Comput.*, 69(1):1–27, 2016. DOI: `10.1007/s10915-016-0181-1`.

[7]   R. An, H. Gao, and W. Sun. Optimal error analysis of Euler and Crank-Nicolson projection finite difference schemes for Landau-Lifshitz equation. *SIAM J. Numer. Anal.*, 59(3):1639–1662, 2021. DOI: `10.1137/20M1335431`.

[8]   R. An and W. Sun. Analysis of backward Euler projection FEM for the Landau–Lifshitz equation. *IMA Journal of Numerical Analysis*, 2021. DOI: `10.1093/imanum/drab038`. DOI:10.1093/imanum/drab038.

[9]   Ľ. Baňas, A. Prohl, and R. Schätzle. Finite element approximations of harmonic map heat flows and wave maps into spheres of nonconstant radii. *Numer. Math.*, 115(3):395–432, 2010. DOI: `10.1007/s00211-009-0282-y`.

[10]   J. W. Barrett, S. Bartels, X. Feng, and A. Prohl. A convergent and constraint-preserving finite element method for the p-harmonic flow into spheres. *SIAM J. Numer. Anal.*, 45(3):905–927, 2007. DOI: `10.1137/050639429`.

[11]   S. Bartels. *Numerical methods for nonlinear partial differential equations*, volume 47 of *Springer Series in Computational Mathematics*. Springer, Cham, 2015, pages x+393. DOI: `10.1007/978-3-319-13797-1`.

[12]   S. Bartels. Projection-free approximation of geometrically constrained partial differential equations. *Math. Comp.*, 85(299):1033–1049, 2016. DOI: `10.1090/mcom/3008`.

[13]   S. Bartels. Stability and convergence of finite-element approximation schemes for harmonic maps. *SIAM J. Numer. Anal.*, 43(1):220–238, 2005. DOI: `10.1137/040606594`.

[14]   S. Bartels, C. Lubich, and A. Prohl. Convergent discretization of heat and wave map flows to spheres using approximate discrete Lagrange multipliers. *Math. Comp.*, 78(267):1269–1292, 2009. DOI: `10.1090/S0025-5718-09-02221-2`.

[15]   S. Bartels and A. Prohl. Constraint preserving implicit finite element discretization of harmonic map flow into spheres. *Math. Comp.*, 76(260):1847–1859, 2007. DOI: `10.1090/S0025-5718-07-02026-1`.

[16] S. Bartels and A. Prohl. Convergence of an implicit finite element method for the Landau-Lifshitz-Gilbert equation. *SIAM J. Numer. Anal.*, 44(4):1405–1419, 2006. DOI: `10.1137/050631070`.

[17] S. Bartels and P. Reiter. Stability of a simple scheme for the approximation of elastic knots and self-avoiding inextensible curves. *Math. Comp.*, 90(330):1499–1526, 2021. DOI: `10.1090/mcom/3633`.

[18] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008, pages xviii+397. DOI: `10.1007/978-0-387-75934-0`.

[19] K.-C. Chang, W. Y. Ding, and R. Ye. Finite-time blow-up of the heat flow of harmonic maps from surfaces. *J. Differential Geom.*, 36(2):507–515, 1992.

[20] Y. M. Chen and M. Struwe. Existence and partial regularity results for the heat flow for harmonic maps. *Math. Z.*, 201(1):83–103, 1989. DOI: `10.1007/BF01161997`.

[21] I. Cimrák. Error estimates for a semi-implicit numerical scheme solving the Landau-Lifshitz equation with an exchange field. *IMA J. Numer. Anal.*, 25(3):611–634, 2005. DOI: `10.1093/imanum/dri011`.

[22] J. Eells Jr. and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964. DOI: `10.2307/2373037`.

[23] H. Gao. Optimal error estimates of a linearized backward Euler FEM for the Landau-Lifshitz equation. *SIAM J. Numer. Anal.*, 52(5):2574–2593, 2014. DOI: `10.1137/130936476`.

[24] X. Gui, B. Li, and J. Wang. Convergence of renormalized finite element methods for heat flow of harmonic maps. *SIAM J. Numer. Anal.*, 60(1):312–338, 2022. DOI: `10.1137/21M1402212`.

[25] J. Kraus, C.-M. Pfeiler, D. Praetorius, M. Ruggeri, and B. Stiftner. Iterative solution and preconditioning for the tangent plane scheme in computational micromagnetics. *J. Comput. Phys.*, 398:108866, 27, 2019. DOI: `10.1016/j.jcp.2019.108866`.

[26] N. J. Mauser, C.-M. Pfeiler, D. Praetorius, and M. Ruggeri. Unconditional well-posedness and IMEX improvement of a family of predictor-corrector methods in micromagnetics. *Appl. Numer. Math.*, 180:33–54, 2022. DOI: `10.1016/j.apnum.2022.05.008`.

[27] D. Praetorius, M. Ruggeri, and B. Stiftner. Convergence of an implicit-explicit midpoint scheme for computational micromagnetics. *Comput. Math. Appl.*, 75(5):1719–1738, 2018. DOI: `10.1016/j.camwa.2017.11.028`.

[28] A. Prohl. *Computational micromagnetism*. Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 2001, pages xviii+304. DOI: `10.1007/978-3-663-09498-2`.

[29] M. Struwe. *Variational methods*, volume 34. Springer-Verlag, Berlin, second edition, 1996, pages xvi+272. DOI: `10.1007/978-3-662-03212-1`.

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