Explicit and efficient error estimation for convex minimization problems

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April 22, 2022

Abstract

We combine a systematic approach for deriving general a posteriori error estimates for convex minimization problems based on convex duality relations with a recently derived generalized Marini formula. The a posteriori error estimates are essentially constant-free and apply to a large class of variational problems including the p–Dirichlet problem, as well as degenerate minimization, obstacle and image de-noising problems. In addition, these a posteriori error estimates are based on a comparison to a given non-conforming finite element solution. For the p–Dirichlet problem, these a posteriori error bounds are equivalent to residual type a posteriori error bounds and, hence, reliable and efficient.

Keywords: Convex minimization, finite elements, non-conforming methods, a posteriori error estimates, adaptive mesh refinement, p–Dirichlet problem, optimal design problem

AMS MSC (2020): 49M29, 65K15, 65N15, 65N50

1. Introduction

1.1 Available estimates

Various computable a posteriori error estimates have recently been derived for convex minimization problems such as the p–Dirichlet problem or degenerate minimization problems, cf. [21, 12, 33, 32, 9]. These a posteriori error estimates are typically defined for a particular finite element method and an appropriate discretization of the problem including a suitable choice of quadrature. Further error sources, e.g., resulting from a stopping criterion of an iterative solution procedure are often not considered. In this article, we systematically derive a general a posteriori error estimate which applies to a large class of conforming and non-conforming numerical methods for non-linear and non-differentiable problems, and provides a computable error bound that is independent of particular discretizations or iteration errors. One important application of general estimates is the concept of model hierarchies, e.g., determining the error of the numerical solution of a linearized model in solving a more complex non-linear problem. Our estimates avoid the use of discrete and continuous Euler–Lagrange equations and only resort to first-order relations. This approach combines a concept used in [42] with representations of a discrete dual solution obtained via post-processing non-conforming approximations derived in, e.g., [39, 7].

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1.2 Error bounds via convex duality

Given a proper, convex and lower semi-continuous functional $\phi : \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ and a (Lebesgue–)measurable functional $\psi : \Omega\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ such that for almost every $x \in \Omega$, where $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, is a bounded polyhedral Lipschitz domain, the functional $\psi(x,\cdot) : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ is proper, convex and lower semi-continuous, we consider the minimization of the functional $I : W^{1,p}_D(\Omega) \to \mathbb{R} \cup \{+\infty\}$, $p \in (1,\infty)$, for every $v \in W^{1,p}_D(\Omega)$ defined by

$$I(v) := \int_\Omega \phi(\nabla v)\,\mathrm{d}x + \int_\Omega \psi(\cdot, v)\,\mathrm{d}x, \tag{1.1}$$

where $W^{1,p}_D(\Omega)$ denotes a suitable Sobolev space with a homogeneous Dirichlet boundary condition on a non-empty boundary part $\Gamma_D \subseteq \partial\Omega$ and may be replaced by $BV(\Omega)$, i.e., the space of functions of bounded variation. Given a possibly non-unique minimizer $u \in W^{1,p}_D(\Omega)$ and an arbitrary conforming approximation $\tilde{u}_h \in W^{1,p}_D(\Omega)$, which may result from a post-processing of a non-conforming, discontinuous approximation, the convexity properties of (1.1), measured by an error functional $\rho_I^2 : W^{1,p}_D(\Omega) \times W^{1,p}_D(\Omega) \to \mathbb{R}_{\geq 0}$, lead to the error estimate

$$\rho_I^2(\tilde{u}_h, u) \leq I(\tilde{u}_h) - I(u). \tag{1.2}$$

Here, $\rho_I^2 : W^{1,p}_D(\Omega)\times W^{1,p}_D(\Omega)\to\mathbb{R}_{\geq 0}$ has the interpretation of a distance measure and is, e.g., (if existent) a lower bound for the second variation of the functional $I : W^{1,p}_D(\Omega)\to\mathbb{R}\cup\{+\infty\}$. Also for degenerate problems a meaningful distance measure can be defined making use of a co-coercivity property. To get a computable upper bound for the approximation error (1.2), we resort to the (Fenchel) dual problem to the minimization of (1.1) which, if, e.g., $\phi\in C^0(\mathbb{R}^d)$ and $\psi : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping[1], cf. [24, p. 113 ff.], is given by the maximization of the functional $D : W^{p'}_N(\mathrm{div};\Omega) \to \mathbb{R} \cup \{-\infty\}$, for every $y \in W^{p'}_N(\mathrm{div};\Omega)$ defined by

$$D(y) := -\int_\Omega \phi^*(y)\,\mathrm{d}x - \int_\Omega \psi^*(\cdot, \mathrm{div}(y))\,\mathrm{d}x, \tag{1.3}$$

where $W^{p'}_N(\mathrm{div};\Omega)$ consists of all vector fields in $L^{p'}(\Omega;\mathbb{R}^d)$ whose distributional divergence exists in $L^{p'}(\Omega)$ and whose normal component vanishes on $\Gamma_N := \partial\Omega\setminus\Gamma_D$. Apart from that, the functionals $\phi^* : \mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ and $\psi^* : \Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ denote the Fenchel conjugates to $\phi : \mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ and $\psi : \Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ (with respect to the second argument), resp. A weak duality relation implies $I(u) \geq D(\tilde{z}_h)$ for all $\tilde{z}_h \in W^{p'}_N(\mathrm{div};\Omega)$, cf. [24, Proposition 1.1]. In particular, for every conforming approximations $\tilde{u}_h \in W^{1,p}_D(\Omega)$ and $\tilde{z}_h \in W^{p'}_N(\mathrm{div};\Omega)$, which both may result from a post-processing of a non-conforming, discontinuous approximation, we obtain from an integration-by-parts, the general primal-dual a posteriori error estimate

$$\begin{aligned}\rho_I^2(\tilde{u}_h, u) &\leq \int_\Omega \phi(\nabla\tilde{u}_h) - \nabla\tilde{u}_h\cdot\tilde{z}_h + \phi^*(\tilde{z}_h)\,\mathrm{d}x \\ &\quad+ \int_\Omega \psi(\cdot,\tilde{u}_h) - \tilde{u}_h\,\mathrm{div}(\tilde{z}_h) + \psi^*(\cdot,\mathrm{div}(\tilde{z}_h))\,\mathrm{d}x \\ &=: \eta_h^2(\tilde{u}_h, \tilde{z}_h).\end{aligned} \tag{1.4}$$

The bound (1.4) and variants of it are well-known in the literature, cf. [44, 43, 45, 30, 42, 6, 9, 7]. The practical realizations of these bounds require the construction of an appropriate – ideally optimal – $\tilde{z}_h \in W^{p'}_N(\mathrm{div};\Omega)$. Apparently, the estimate (1.4) can only be efficient, i.e., provide an optimal upper bound, if a strong duality relation applies, i.e., if we have that

$$\inf_{v\in W^{1,p}_D(\Omega)} I(v) = \sup_{y\in W^{p'}_N(\mathrm{div};\Omega)} D(y).$$

[1] A mapping $f : \Omega\times\mathbb{R}^k \to \mathbb{R}^l$, $k, l \in \mathbb{N}$, is said to be a *Carathéodory mapping*, if $\psi(x,\cdot) \in C^0(\mathbb{R}^k;\mathbb{R}^l)$ for almost every $x \in \Omega$ and $\psi(\cdot, a) : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}^l$ is (Lebesgue–)measurable for all $a \in \mathbb{R}^k$.

1.3 Practical realization

We assume here that for all $t \in \mathbb{R}$, $\psi(\cdot,t) = \psi_h(\cdot,t)$ is element-wise constant with respect to a triangulation $\mathcal{T}_h$, $h > 0$, of Ω and choose an admissible vector field $\tilde{z}_h \in W^{p'}_N(\mathrm{div};\Omega)$ in the Raviart–Thomas finite element space $\mathcal{RT}^0_N(\mathcal{T}_h) \subseteq W^{p'}_N(\mathrm{div};\Omega)$, cf. [41], i.e., the space of element-wise affine vector fields that have continuous constant normal components on element sides which vanish on Γ_N. Assume that $\tilde{u}_h \in W^{1,p}_D(\Omega)$ belongs to $\mathcal{S}^1_D(\mathcal{T}_h) \subseteq W^{1,p}_D(\Omega)$, i.e., the space of globally continuous, element-wise affine functions that vanish on Γ_D. Then, the primal-dual a posteriori error estimator defined in (1.4) can be re-written as

$$\begin{aligned} \eta_h^2(\tilde{u}_h,\tilde{z}_h) = &\int_\Omega \phi(\nabla\tilde{u}_h) - \nabla\tilde{u}_h\cdot\Pi_h\tilde{z}_h + \phi^*(\Pi_h\tilde{z}_h)\,\mathrm{d}x \\ &+ \int_\Omega \psi_h(\cdot,\Pi_h\tilde{u}_h) - \Pi_h\tilde{u}_h\,\mathrm{div}(\tilde{z}_h) + \psi_h^*(\cdot,\mathrm{div}(\tilde{z}_h))\,\mathrm{d}x \\ &+ \int_\Omega \psi_h(\cdot,\tilde{u}_h) - \psi_h(\cdot,\Pi_h\tilde{u}_h)\,\mathrm{d}x + \int_\Omega \phi^*(\tilde{z}_h) - \phi^*(\Pi_h\tilde{z}_h)\,\mathrm{d}x, \end{aligned} \tag{1.5}$$

where $\Pi_h : L^1(\Omega;\mathbb{R}^l) \to \mathcal{L}^0(\mathcal{T}_h)^l$, $l \in \mathbb{N}$, denotes the L^2–projection operator onto element-wise constant functions and vector fields, resp. Then, the integrands of the first two integrals on the right-hand side in (1.5) are element-wise constant and, by the Fenchel–Young inequality, non-negative. In addition, by Jensen's inequality, it holds $\int_T \psi_h(\cdot,\tilde{u}_h) - \psi_h(\cdot,\Pi_h\tilde{u}_h)\,\mathrm{d}x \geq 0$ and $\int_T \phi^*(\tilde{z}_h) - \phi^*(\Pi_h\tilde{z}_h)\,\mathrm{d}x \geq 0$ for all $T \in \mathcal{T}_h$. More generally, it can be bounded reliably using a trapezoidal quadrature rule leading to a fully practical contribution.

1.4 Non-conforming representation

A quasi-optimal discrete vector field $\tilde{z}_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ now is found via post-processing a non-conforming, discontinuous Crouzeix–Raviart approximation of the primal problem, i.e., the minimization of $I_h^{cr} : \mathcal{S}^{1,cr}_D(\mathcal{T}_h) \to \mathbb{R}\cup\{+\infty\}$, for every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h) := \int_\Omega \phi(\nabla_h v_h)\,\mathrm{d}x + \int_\Omega \psi_h(\cdot,\Pi_h v_h)\,\mathrm{d}x. \tag{1.6}$$

where $\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ denotes the Crouzeix–Raviart finite element space, i.e, the space of element-wise affine functions that are continuous at the midpoints of element sides and vanish in midpoints (barycenters) of element sides belonging to Γ_D, and where $\nabla_h : \mathcal{S}^{1,cr}_D(\mathcal{T}_h) \to \mathcal{L}^0(\mathcal{T}_h)^d$ denotes the element-wise application of the gradient operator. In [18, 7], it has been shown that a discrete (Fenchel) dual problem is given via the maximization of $D_h^{rt} : \mathcal{RT}^0_N(\mathcal{T}_h) \to \mathbb{R}\cup\{-\infty\}$, for every $y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ defined by

$$D_h^{rt}(y_h) := -\int_\Omega \phi^*(\Pi_h y_h)\,\mathrm{d}x - \int_\Omega \psi_h^*(\cdot,\mathrm{div}(y_h))\,\mathrm{d}x. \tag{1.7}$$

Then, a maximizer $z_h^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_h)$ of (1.7) represents a quasi-optimal choice for $\tilde{z}_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ in the a posteriori error estimate (1.5). However, owing to typical constraints such as, e.g., $-\mathrm{div}(z_h^{rt}) = f_h$ in $\mathcal{L}^0(\mathcal{T}_h)$, this often requires solving a (potentially non-linear) saddle point problem. The latter can, fortunately, be avoided via post-processing the minimizer of (1.6). More precisely, if $\phi \in C^1(\mathbb{R}^d)$ and $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for almost every $x \in \Omega$, then for a minimizer $u_h^{cr} \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ of (1.6), we may represent $z_h^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_h)$ via the reconstruction formula

$$z_h^{rt} = D\phi(\nabla_h u_h^{cr}) + D\psi_h(\cdot,\Pi_h u_h^{cr})d^{-1}\big(\mathrm{id}_{\mathbb{R}^d} - \Pi_h\mathrm{id}_{\mathbb{R}^d}\big) \quad \text{in } \mathcal{RT}^0_N(\mathcal{T}_h), \tag{1.8}$$

derived, e.g., in [7, Proposition 3.1]. Even if $\phi : \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ and $\psi_h : \Omega\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ are non-differentiable, it is sometimes possible to derive reconstruction formulas similar to (1.8) for quasi-optimal discrete vector fields $z_h^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_h)$, e.g., resorting to regularization arguments or given discrete Lagrange multipliers.

For the non-linear Dirichlet problem, i.e., if $\phi \in C^1(\mathbb{R}^d)$ and $\psi_h(x,t) := -f_h(x)t$ for almost every $x\in\Omega$ and all $t\in\mathbb{R}$, where $f_h := \Pi_h f \in \mathcal{L}^0(\mathcal{T}_h)$ and $f \in L^{p'}(\Omega)$, $p\in[1,\infty]$, (1.5) implies

$$\begin{aligned}\rho_I^2(\tilde{u}_h,u) &\le \eta_h^2(\tilde{u}_h,z_h^{rt})\\ &\le \int_\Omega \big(D\phi(\nabla\tilde{u}_h) - D\phi(\nabla_h u_h^{cr})\big)\cdot(\nabla\tilde{u}_h - \nabla_h u_h^{cr})\,\mathrm{d}x\\ &\quad + \int_\Omega \big(D\phi^*(z_h^{rt}) - \phi^*(\Pi_h z_h^{rt})\big)\cdot(z_h^{rt} - \Pi_h z_h^{rt})\,\mathrm{d}x,\end{aligned}\tag{1.9}$$

where the first integral on the right-hand side has the interpretation of a residual, while the second integral contains data approximation errors. The estimate (1.9) is less accurate than the estimate (1.4) but turns out to be particularly useful for establishing efficiency properties. In fact, for the p–Dirichlet problem, i.e., if $\phi := \frac{1}{p}|\cdot|^p \in C^1(\mathbb{R}^d)$, $p\in(1,\infty)$, resorting to (1.9), we will find that the primal-dual a posteriori error estimator $\eta_h^2(u_h^c, z_h^{rt})$, where $u_h^c \in \mathcal{S}_D^1(\mathcal{T}_h)$ denotes the unique minimizer of $I_h^c := I|_{\mathcal{S}_D^1(\mathcal{T}_h)} : \mathcal{S}_D^1(\mathcal{T}_h) \to \mathbb{R}$, is globally equivalent to the classical residual type a posteriori error estimator $\eta_{res,h}^2(u_h^c)$, cf. [21], and, therefore, reliable, efficient and equivalent to the error quantity $\rho_I^2(u_h^c,u)$ (if suitably chosen). More generally, using the triangle inequality in (1.9), we see that the primal-dual a posteriori error estimator is estimated by approximation errors of conforming and non-conforming approximations.

1.5 New contributions

The bound (1.4) and variants thereof are well-known in literature, cf. [44, 43, 45, 42, 9, 7]. Foremost, in [42], S. I. Repin proposed general a posteriori error estimates based on estimating the approximation error by the primal-dual gap. In particular, he pointed out that these error bounds require a quasi-optimal dual vector field $\tilde{z}_h \in W_N^{p'}(\mathrm{div};\Omega)$ to be practicable. In the continuous case and if, e.g., $\phi\in C^1(\mathbb{R}^d)$ with $|D\phi(t)|\le c_0|t|^{p-1} + c_1$ for all $t\in\mathbb{R}^d$, which is already well-understood, the reconstruction of such a quasi-optimal vector field is challenging as, e.g., a maximizer $z\in W_N^{p'}(\mathrm{div};\Omega)$ of (1.3) needs to satisfy the optimality relation $z = D\phi(\nabla u)$ in $L^{p'}(\Omega;\mathbb{R}^d)$ and, thus, depends on a minimizer $u \in W_D^{1,p}(\Omega)$ of (1.1). For a discrete reconstruction, however, a discrete analogue of such an optimality relation has yet been unavailable. In [7], if, e.g., $\phi \in C^1(\mathbb{R}^d)$ and $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$ – or already L. D. Marini in [39] in the linear case – via (1.8), the desired discrete analogue, commonly referred to as generalized Marini formula, has recently been provided. We merge these results and extend them to non-differentiable convex minimization problems, cf. [8]. The reconstruction formula (1.8) was also referred to in [33, 32] in the case of the non-linear Dirichlet problem. However, the different a posteriori error estimates derived therein rely less on convex duality arguments such as, e.g., (1.4) or (1.9) do, but more on the reconstruction of an $\mathcal{S}_D^3(\mathcal{T}_h)$–conformal[2] companion, which is computationally cheap but rather indirect. Our approach is direct, general and incurs an effort for the computation of the right-hand side in (1.4) comparable to the effort of the computation of the left-hand side in (1.4), i.e., to the computation of the primal approximation $\tilde{u}_h \in \mathcal{S}_D^1(\mathcal{T}_h)$. In the case of the Poisson problem, in [14, 13], D. Braess and J. Schöberl equally resorted to convex duality arguments and the explicit reconstruction of a quasi-optimal vector field $\tilde{z}_h \in W_N^{p'}(\mathrm{div};\Omega)$. However, their reconstruction technique, also called equilibration, cf. [40, 1, 38, 26, 3, 46], is based on local corrections on each patch, which equally is computationally cheap. As a whole, we propose the combination of primal-dual a posteriori error estimates together with the reconstruction of quasi-optimal vector fields $\tilde{z}_h \in W_N^{p'}(\mathrm{div};\Omega)$ based on reconstruction formulas like (1.8) as a broadly applicable and usually computationally cheap alternative to residual type a posteriori error estimators. Apart from that, numerical experiments, cf. Section 6, justify the choice $\tilde{u}_h = I_h u_h^{cr} \in \mathcal{S}_D^1(\mathcal{T}_h)$, where $I_h : \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathcal{S}_D^1(\mathcal{T}_h)$ is a suitable quasi-interpolation operator, as conformal approximation of $u \in W_D^{1,p}(\Omega)$. This, in turn, reduces the computational effort to the same level as, e.g., for residual type error estimators.

[2]Here, $\mathcal{S}_D^3(\mathcal{T}_h)$ denotes space of globally continuous and element-wise cubic functions that vanish on Γ_D.

1.6 Outline

This article is organized as follows: In Section 2, we introduce the employed notation, define the relevant finite element spaces and give a brief review of continuous and discrete convex minimization problems. In particular, we prove discrete convex optimality relations under minimal regularity assumptions (cf. Proposition 2.1). In Section 3, we discuss a general a posteriori error estimate and potential error sources that may need to be taken into account. In Section 4, the general a posteriori error estimate is refined using particular convex duality relations and a post-processing of a discrete convex minimization problem. In Section 5, we apply these results to well-known convex minimization problems including the p–Dirichlet problem and an optimal design problem, a prototypical example from topology optimization. For the p–Dirichlet problem, we establish a reliability and efficiency result (cf. Theorem 5.2). In Section 6, we confirm our theoretical findings via numerical experiments. In Appendix A.1, we collect definitions and results from convex analysis needed in the paper. Error estimates for the node-averaging operator in terms of shifted N–functions are proved in Appendix A.2.

2. PRELIMINARIES

Throughout the entire article, if not otherwise specified, we denote by $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, a bounded polyhedral Lipschitz domain, whose topological boundary is disjointly divided into a closed Dirichlet part Γ_D and a Neumann part Γ_N, i.e., $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\emptyset = \Gamma_D \cap \Gamma_N$.

2.1 Standard function spaces

For $p \in [1,\infty]$ and $l \in \mathbb{N}$, we employ the standard notations[3]

$$W^{1,p}_D(\Omega;\mathbb{R}^l) := \big\{ v \in L^p(\Omega;\mathbb{R}^l) \mid \nabla v \in L^p(\Omega;\mathbb{R}^{l\times d}),\ \mathrm{tr}(v) = 0 \text{ in } L^p(\Gamma_D;\mathbb{R}^l) \big\},$$
$$W^p_N(\mathrm{div};\Omega) := \big\{ y \in L^p(\Omega;\mathbb{R}^d) \mid \mathrm{div}(y) \in L^p(\Omega),\ \mathrm{tr}(y)\cdot n = 0 \text{ in } W^{-\frac{1}{p},p}(\Gamma_N) \big\},$$

$W^{1,p}(\Omega;\mathbb{R}^l) := W^{1,p}_D(\Omega;\mathbb{R}^l)$ if $\Gamma_D = \emptyset$, and $W^p(\mathrm{div};\Omega) := W^p_N(\mathrm{div};\Omega)$ if $\Gamma_N = \emptyset$, where we denote by $\mathrm{tr}: W^{1,p}(\Omega;\mathbb{R}^l) \to L^p(\partial\Omega;\mathbb{R}^l)$ and by $\mathrm{tr}(\cdot)\cdot n : W^p(\mathrm{div};\Omega) \to W^{-\frac{1}{p},p}(\partial\Omega)$, the trace and normal trace operator, resp. In particular, we predominantly omit $\mathrm{tr}(\cdot)$ in this context. In addition, we resort to the abbreviations $L^p(\Omega) := L^p(\Omega;\mathbb{R}^1)$, $W^{1,p}(\Omega) := W^{1,p}(\Omega;\mathbb{R}^1)$ and $W^{1,p}_D(\Omega) := W^{1,p}_D(\Omega;\mathbb{R}^1)$.

2.2 Triangulations and standard finite element spaces

In what follows, we will always denote by $\mathcal{T}_h$, $h > 0$, a sequence of regular, i.e., uniformly shape regular and conforming, triangulations of $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, cf. [25]. The sets $\mathcal{S}_h$ and $\mathcal{N}_h$ contain the sides and vertices, resp., of the elements of $\mathcal{T}_h$. In the context of locally refined meshes, we employ the average mesh-size $h := (|\Omega|/\mathrm{card}(\mathcal{N}_h))^{\frac{1}{d}} > 0$. In addition, we define $h_T := \mathrm{diam}(T)$ for all $T \in \mathcal{T}_h$ and $h_S := \mathrm{diam}(S)$ for all $S \in \mathcal{S}_h$. For $k \in \mathbb{N}\cup\{0\}$ and $T \in \mathcal{T}_h$, let $\mathcal{P}_k(T)$ denote the set of polynomials of maximal degree k on T. Then, for $k \in \mathbb{N}\cup\{0\}$ and $l \in \mathbb{N}$, the sets of continuous and element-wise polynomial functions or vector fields, resp., are defined by

$$\mathcal{S}^k(\mathcal{T}_h)^l := \big\{ v_h \in C^0(\overline{\Omega};\mathbb{R}^l) \mid v_h|_T \in \mathcal{P}_k(T)^l \text{ for all } T \in \mathcal{T}_h \big\},$$
$$\mathcal{L}^k(\mathcal{T}_h)^l := \big\{ v_h \in L^\infty(\Omega;\mathbb{R}^l) \mid v_h|_T \in \mathcal{P}_k(T)^l \text{ for all } T \in \mathcal{T}_h \big\}.$$

The element-wise mesh-size function $h_{\mathcal{T}} \in \mathcal{L}^0(\mathcal{T}_h)$ is defined by $h_{\mathcal{T}}|_T := h_T$ for all $T \in \mathcal{T}_h$. For every $T \in \mathcal{T}_h$ and $S \in \mathcal{S}_h$, we denote by $x_T := \frac{1}{d+1}\sum_{z\in\mathcal{N}_h\cap T} z$ and $x_S := \frac{1}{d}\sum_{z\in\mathcal{N}_h\cap S} z$, the midpoints (barycenters) of T and S, resp. The L^2–projection operator onto element-wise constant functions or vector fields, resp., is denoted by

$$\Pi_h : L^1(\Omega;\mathbb{R}^l) \to \mathcal{L}^0(\mathcal{T}_h)^l.$$

[3] Here, $W^{-\frac{1}{p},p}(\Gamma_N) := (W^{1-\frac{1}{p'},p'}(\Gamma_N))^*$ and $W^{-\frac{1}{p},p}(\partial\Omega) := (W^{1-\frac{1}{p'},p'}(\partial\Omega))^*$.

For every $v_h \in \mathcal{L}^1(\mathcal{T}_h)^l$, it holds $\Pi_h v_h|_T = v_h(x_T)$ for every $T \in \mathcal{T}_h$. For $p \in [1,\infty]$, there exists a constant $c_\Pi > 0$ such that for all $v \in L^p(\Omega;\mathbb{R}^l)$ and $T \in \mathcal{T}_h$, cf. [25, Thm. 18.16], it holds

(L0.1) $\|\Pi_h v\|_{L^p(T;\mathbb{R}^l)} \leq \|v\|_{L^p(T;\mathbb{R}^l)}$,

(L0.2) $\|v - \Pi_h v\|_{L^p(T;\mathbb{R}^l)} \leq c_\Pi h_T \|\nabla v\|_{L^p(T;\mathbb{R}^{l\times d})}$ if $v \in W^{1,p}(\Omega;\mathbb{R}^l)$.

The node-averaging operator $\mathcal{J}_h^{av} : \mathcal{L}^k(\mathcal{T}_h)^l \to \mathcal{S}^k_D(\mathcal{T}_h)^l$, where $\mathcal{S}^k_D(\mathcal{T}_h)^l := \mathcal{S}^k(\mathcal{T}_h)^l \cap W^{1,1}_D(\Omega)^l$, denoting for $z \in \mathcal{D}_h^k$, where $\mathcal{D}_h^k$ denotes the set of degrees of freedom associated with $\mathcal{S}^k(\mathcal{T}_h)^l$, by $\mathcal{T}_h(z) := \{T \in \mathcal{T}_h \mid z \in T\}$ the set of elements sharing z, for all $v_h \in \mathcal{L}^k(\mathcal{T}_h)^l$, is defined by

$$\mathcal{J}_h^{av} v_h := \sum_{z\in\mathcal{D}_h^k} \langle v_h\rangle_z \varphi_z, \qquad \langle v_h\rangle_z := \begin{cases} \frac{1}{\operatorname{card}(\mathcal{T}_h(z))}\sum_{T\in\mathcal{T}_h(z)} (v_h|_T)(z) & \text{if } z \in \Omega\cup\Gamma_N,\\ 0 & \text{if } z\in\Gamma_D,\end{cases}$$

where we denote by $(\varphi_z)_{z\in\mathcal{D}_h^k} \subseteq \mathcal{S}^k(\mathcal{T}_h)$, the nodal basis of $\mathcal{S}^k(\mathcal{T}_h)$. For $p\in[1,\infty]$, there exists a constant $c_{av} > 0$ such that for all $v_h \in \mathcal{L}^k(\mathcal{T}_h)^l$, $T\in\mathcal{T}_h$, $m\in\{0,\dots,k+1\}$, cf. [25, Lem. 22.12],[4]

(AV.1) $\|\nabla_h^m(v_h - \mathcal{J}_h^{av} v_h)\|_{L^p(T;\mathbb{R}^{l\times d^m})} \leq c_{av}\sum_{S\in\mathcal{S}_h(T)} \big\|h_S^{1/p-m}[\![v_h]\!]_S\big\|_{L^p(S;\mathbb{R}^l)}$,

(AV.2) $\|\mathcal{J}_h^{av} v_h\|_{L^p(T;\mathbb{R}^l)} \leq c_{av}\|v_h\|_{L^p(\omega_T;\mathbb{R}^l)}$,

where $\mathcal{S}_h(T) := \{S \in \mathcal{S}_h \mid S\cap \operatorname{int}(\omega_T) \neq \emptyset\}$ and $\omega_T := \bigcup\{T' \in \mathcal{T}_h \mid T'\cap T \neq\emptyset\}$ for all $T\in\mathcal{T}_h$ and $\nabla_h^m : \mathcal{L}^k(\mathcal{T}_h)^l \to \mathcal{L}^{k-1}(\mathcal{T}_h)^{l\times d^m}$, for every $v_h \in \mathcal{L}^k(\mathcal{T}_h)^l$ defined by $(\nabla_h^m v_h)|_T := \nabla^m(v_h|_T)$ for all $T\in\mathcal{T}_h$, denotes the element-wise m–th gradient operator.

2.3 *Crouzeix–Raviart finite elements*

A particular instance of a larger class of non-conforming finite element spaces, introduced in [20], is the Crouzeix–Raviart finite element space, which consists of element-wise affine functions that are continuous at the midpoints (barycenters) of inner element sides, i.e.,

$$\mathcal{S}^{1,cr}(\mathcal{T}_h) := \left\{ v_h \in \mathcal{L}^1(\mathcal{T}_h) \,\Big|\, \int_S [\![v_h]\!]_S \,\mathrm{d}s = 0 \text{ for all } S\in\mathcal{S}_h\setminus\partial\Omega\right\}.$$

Crouzeix–Raviart finite element functions that vanish at the midpoints of boundary element sides that correspond to the Dirichlet boundary Γ_D are contained in the space

$$\mathcal{S}^{1,cr}_D(\mathcal{T}_h) := \big\{ v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h) \mid v_h(x_S) = 0 \text{ for all } S\in\mathcal{S}_h\cap\Gamma_D\big\}.$$

In particular, we have that $\mathcal{S}^{1,cr}_D(\mathcal{T}_h) = \mathcal{S}^{1,cr}(\mathcal{T}_h)$ if $\Gamma_D = \emptyset$. A basis of $\mathcal{S}^{1,cr}(\mathcal{T}_h)$ is given by functions $\varphi_S \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$, $S\in\mathcal{S}_h$, satisfying the Kronecker property $\varphi_S(x_{S'}) = \delta_{S,S'}$ for all $S, S' \in \mathcal{S}_h$. A basis of $\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ is given by $(\varphi_S)_{S\in\mathcal{S}_h; S\not\subseteq\Gamma_D}$. Since for every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, it holds $v_h = v_h(x_S) + \nabla_h v_h(\mathrm{id}_{\mathbb{R}^d} - x_S)$ in $T_+\cup T_-$ for all $T_+, T_- \in \mathcal{T}_h$ with $T_+\cap T_- = S \in \mathcal{S}_h$, we have that $[\![v_h]\!]_S = [\![\nabla_h v_h]\!]_S(\mathrm{id}_{\mathbb{R}^d} - x_S)$ in S for all $S\in\mathcal{S}_h$. As an immediate consequence, also resorting to the discrete trace inequality[5], for fixed $p\in[1,\infty]$ and every $S\in\mathcal{S}_h$, it holds

$$\begin{aligned}\big\|h_S^{1/p}[\![v_h]\!]_S\big\|_{L^p(S)} &\leq \big\|h_S^{1/p+1}[\![\nabla_h v_h]\!]_S\big\|_{L^p(S;\mathbb{R}^d)}\\ &\leq c_{\mathrm{tr}}\sum_{T\in\mathcal{T}_h; S\subseteq\partial T}\big\|h_T\nabla_h v_h\big\|_{L^p(T;\mathbb{R}^d)}.\end{aligned} \tag{2.1}$$

A combination of (2.1) and *(AV.1)* implies that for $p\in[1,\infty]$, there exists a constant $c_{av} > 0$ such that for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, $T\in\mathcal{T}_h$ and $m\in\{0,1,2\}$, we have that

(AV.3) $\|\nabla_h^m(v_h - \mathcal{J}_h^{av} v_h)\|_{L^p(T;\mathbb{R}^{d^m})} \leq c_{av}\sum_{S\in\mathcal{S}_h(T)}\big\|h_S^{1/p+1-m}[\![\nabla_h v_h]\!]_S\big\|_{L^p(S;\mathbb{R}^d)}$,

(AV.4) $\|\nabla_h^m(v_h - \mathcal{J}_h^{av} v_h)\|_{L^p(T;\mathbb{R}^{d^m})} \leq c_{av}\big\|h_{\mathcal{T}}^{1-m}\nabla_h v_h\big\|_{L^p(\omega_T;\mathbb{R}^d)}$.

[4] Here, for every $S\in\mathcal{S}_h\setminus\partial\Omega$, $[\![v_h]\!]_S := v_h|_{T_+} - v_h|_{T_-}$ on S, where $T_+, T_- \in \mathcal{T}_h$ satisfy $\partial T_+\cap\partial T_- = S$, and for every $S\in\mathcal{S}_h\cap\partial\Omega$, $[\![v_h]\!]_S := v_h|_T$ on S, where $T\in\mathcal{T}_h$ satisfies $S\subseteq\partial T$.

[5] Appealing to [25, Lemma 12.8], for $p\in[1,\infty]$ and $k\in\mathbb{N}\cup\{0\}$, there exists a constant $c_{\mathrm{tr}} > 0$ such that for every $v_h\in\mathcal{L}^0(\mathcal{T}_h)$, it holds $h_T^{1/p}\|v_h\|_{L^p(S)} \leq c_{\mathrm{tr}}\|v_h\|_{L^p(T)}$ for all $T\in\mathcal{T}_h$ and $S\in\mathcal{S}_h$ with $S\subseteq\partial T$.

2.4 Raviart–Thomas finite elements

The lowest order Raviart–Thomas finite element space, introduced in [41], consists of element-wise affine vector fields that have continuous constant normal components on inner elements sides, i.e.,[6]

$$\begin{aligned}\mathcal{RT}^0(\mathcal{T}_h) := \big\{ y_h \in \mathcal{L}^1(\mathcal{T}_h)^d \mid y_h|_T \cdot n_T = \text{const in } \partial T \text{ for all } T \in \mathcal{T}_h, \\ [\![y_h \cdot n]\!]_S = 0 \text{ on } S \text{ for all } S \in \mathcal{S}_h \setminus \partial\Omega \big\}.\end{aligned}$$

Raviart–Thomas finite element functions that possess vanishing normal components on the Neumann boundary Γ_N are contained in the space

$$\mathcal{RT}^0_N(\mathcal{T}_h) := \big\{ y_h \in \mathcal{RT}^0(\mathcal{T}_h) \mid y_h \cdot n = 0 \text{ on } \Gamma_N \big\}.$$

In particular, we have that $\mathcal{RT}^0_N(\mathcal{T}_h) = \mathcal{RT}^0(\mathcal{T}_h)$ if $\Gamma_N = \emptyset$. A basis of $\mathcal{RT}^0(\mathcal{T}_h)$ is given by vector fields $\psi_S \in \mathcal{RT}^0(\mathcal{T}_h)$, $S \in \mathcal{S}_h$, satisfying the Kronecker property $\psi_S|_{S'} \cdot n_{S'} = \delta_{S,S'}$ on S' for all $S' \in \mathcal{S}_h$, where n_S for all $S \in \mathcal{S}_h$ is the unit normal vector on S that points from T_- to T_+ if $T_+ \cap T_- = S \in \mathcal{S}_h$. A basis of $\mathcal{RT}^0_N(\mathcal{T}_h)$ is given by $\psi_S \in \mathcal{RT}^0_N(\mathcal{T}_h)$, $S \in \mathcal{S}_h \setminus \Gamma_N$.

2.5 Integration-by-parts formula with respect to $\mathcal{S}^{1,cr}(\mathcal{T}_h)$ and $\mathcal{RT}^0(\mathcal{T}_h)$

An element-wise integration-by-parts implies that for all $v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}^0(\mathcal{T}_h)$, we have the integration-by-parts formula

$$\int_\Omega \nabla_h v_h \cdot \Pi_h y_h \,\mathrm{d}x + \int_\Omega \Pi_h v_h \,\mathrm{div}(y_h)\,\mathrm{d}x = \int_{\partial\Omega} v_h \, y_h \cdot n \,\mathrm{d}s. \tag{2.2}$$

Here, we have exploited that $y_h \in \mathcal{RT}^0(\mathcal{T}_h)$ has continuous constant normal components on inner element sides, i.e., $[\![y_h \cdot n]\!]_S = 0$ on S for every $S \in \mathcal{S}_h \setminus \partial\Omega$, and that the jumps of $v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ across inner element sides have vanishing integral mean, i.e., $\int_S [\![v_h]\!]_S \,\mathrm{d}s = 0$ for every $S \in \mathcal{S}_h \setminus \partial\Omega$. In particular, for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$, (2.2) reads as

$$\int_\Omega \nabla_h v_h \cdot \Pi_h y_h \,\mathrm{d}x = -\int_\Omega \Pi_h v_h \,\mathrm{div}(y_h)\,\mathrm{d}x. \tag{2.3}$$

In [18, 7], the integration-by-parts formula (2.3) formed a cornerstone in the derivation of discrete convex duality relations and, as such, also plays a central role in the hereinafter analysis.

2.6 Convex minimization problems

Let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous functional and let $\psi : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be (Lebesgue–)measurable such that for almost every $x \in \Omega$, the function $\psi(x,\cdot) : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. Then, for given $p \in (1,\infty)$, we examine the convex minimization problem that seeks for a function $u \in W^{1,p}_D(\Omega)$ that is minimal for $I : W^{1,p}_D(\Omega) \to \mathbb{R} \cup \{+\infty\}$, for every $v \in W^{1,p}_D(\Omega)$ defined by

$$I(v) := \int_\Omega \phi(\nabla v)\,\mathrm{d}x + \int_\Omega \psi(\cdot, v)\,\mathrm{d}x. \tag{2.4}$$

We will always assume that $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ and $\psi : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ are such that (2.4) is proper, convex, weakly coercive, and lower semi-continuous, so that the direct method in the calculus of variations implies the existence of a minimizer $u \in W^{1,p}_D(\Omega)$ of (2.4). A (Fenchel) dual problem to (2.4) is given by the maximization of $D : L^{p'}(\Omega;\mathbb{R}^d) \to \mathbb{R} \cup \{-\infty\}$, for every $y \in L^{p'}(\Omega;\mathbb{R}^d)$ defined by

$$D(y) := -\int_\Omega \phi^*(y)\,\mathrm{d}x - F^*(\mathrm{div}(y)), \tag{2.5}$$

[6] Here, for every $S \in \mathcal{S}_h \setminus \partial\Omega$, $[\![y_h \cdot n]\!]_S := y_h|_{T_+} \cdot n_{T_+} + y_h|_{T_-} \cdot n_{T_-}$ on S, where $T_+, T_- \in \mathcal{T}_h$ satisfy $\partial T_+ \cap \partial T_- = S$, and for every $T \in \mathcal{T}_h$, $n_T : \partial T \to \mathbb{S}^{d-1}$ denotes the outward unit normal vector field to T, and for every $S \in \mathcal{S}_h \cap \partial\Omega$, $[\![y_h \cdot n]\!]_S := y_h|_T \cdot n$ on S, where $T \in \mathcal{T}_h$ satisfies $S \subseteq \partial T$ and $n : \partial\Omega \to \mathbb{S}^{d-1}$ denotes the outward unit normal vector field to Ω.

where $\operatorname{div}: L^{p'}(\Omega;\mathbb{R}^d)\to (W^{1,p}_D(\Omega))^*$ for every $y\in L^{p'}(\Omega;\mathbb{R}^d)$ and $v\in W^{1,p}_D(\Omega)$ is defined by $\langle \operatorname{div}(y),v\rangle_{W^{1,p}_D(\Omega)} := -\int_\Omega y\cdot\nabla v\,\mathrm{d}x$, and $F^*: L^{p'}(\Omega)\to\mathbb{R}\cup\{\pm\infty\}$ denotes the Fenchel conjugate to $F: L^p(\Omega)\to\mathbb{R}\cup\{+\infty\}$, defined by $F(v):=\int_\Omega\psi(\cdot,v)\,\mathrm{d}x$ for every $v\in L^p(\Omega)$. Note that for every $y\in W^{p'}_N(\operatorname{div};\Omega)$, we have the explicit representation

$$D(y) = -\int_\Omega \phi^*(y)\,\mathrm{d}x - \int_\Omega \psi^*(\cdot,\operatorname{div}(y))\,\mathrm{d}x.$$

In general, cf. [24, Proposition 1.1], we have the weak duality relation

$$I(u) = \inf_{v\in W^{1,p}_D(\Omega)} I(v) \geq \sup_{y\in L^{p'}(\Omega;\mathbb{R}^d)} D(y). \tag{2.6}$$

If, for instance, $\phi\in C^0(\mathbb{R}^d)$ and $\psi:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory mapping, then, in [24, p. 113 ff.], it is shown that (2.5) admits at least one maximizer $z\in W^{p'}_N(\operatorname{div};\Omega)$, i.e., (2.5) can be restricted to the maximization in $W^{p'}_N(\operatorname{div};\Omega)$ and strong duality applies, i.e., we have that

$$I(u) = D(z). \tag{2.7}$$

In addition, cf. [24, Proposition 5.1], we then have the optimality relations

$$\begin{aligned} z\cdot\nabla u &= \phi^*(z)+\phi(\nabla u) &&\text{a.e. in }\Omega,\\ \operatorname{div}(z)\,u &= \psi^*(\cdot,\operatorname{div}(z))+\psi(\cdot,u) &&\text{a.e. in }\Omega. \end{aligned} \tag{2.8}$$

If $\phi\in C^1(\mathbb{R}^d)$ and there exist $c_0,c_1\geq 0$ such that $|D\phi(t)|\leq c_0|t|^{p-1}+c_1$ for all $t\in\mathbb{R}^d$, then, by the Fenchel–Young identity (cf. (A.3)), $(2.8)_1$ is equivalent to

$$z = D\phi(\nabla u) \quad \text{in } L^{p'}(\Omega;\mathbb{R}^d). \tag{2.9}$$

Similarly, if $\psi(x,\cdot)\in C^1(\mathbb{R})$ for almost every $x\in\Omega$ and there exist $c_0\geq 0$ and $c_1\in L^{p'}(\Omega)$ such that $|D\psi(x,t)|\leq c_0|t|^{p-1}+c_1(x)$ for almost every $x\in\Omega$ and all $t\in\mathbb{R}$, then $(2.8)_2$ is equivalent to

$$\operatorname{div}(z) = D\psi(\cdot,u) \quad \text{in } L^{p'}(\Omega). \tag{2.10}$$

In addition, for the remainder of this article, we further assume that (2.4) is co-coercive at a minimizer $u\in W^{1,p}_D(\Omega)$, i.e., there exists a functional $\rho_I^2: W^{1,p}_D(\Omega)\times W^{1,p}_D(\Omega)\to\mathbb{R}_{\geq 0}$ such that for every $v\in W^{1,p}_D(\Omega)$, it holds

$$\rho_I^2(v,u)\leq I(v)-I(u). \tag{2.11}$$

Note that, in general, $\rho_I^2: W^{1,p}_D(\Omega)\times W^{1,p}_D(\Omega)\to\mathbb{R}_{\geq 0}$ does not need be definite and, hence, $u\in W^{1,p}_D(\Omega)$ does not need be unique. If, however, $\rho_I^2: W^{1,p}_D(\Omega)\times W^{1,p}_D(\Omega)\to\mathbb{R}_{\geq 0}$ is definite, then $u\in W^{1,p}_D(\Omega)$ is unique.

2.7 Discrete convex minimization problem

Let $\psi_h:\Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ denote a suitable approximation of $\psi:\Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ such that $\psi_h(\cdot,t)\in\mathcal{L}^0(\mathcal{T}_h)$ for all $t\in\mathbb{R}$ and for almost every $x\in\Omega$, $\psi_h(x,\cdot):\Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is a proper, convex and lower semi-continuous functional. Then, for given $p\in(1,\infty)$, we examine the (discrete) convex minimization problem that seeks for a function $u_h^{cr}\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ that is minimal for $I_h^{cr}:\mathcal{S}^{1,cr}_D(\mathcal{T}_h)\to\mathbb{R}\cup\{+\infty\}$, for every $v_h\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h) := \int_\Omega \phi(\nabla_h v_h)\,\mathrm{d}x + \int_\Omega \psi_h(\cdot,\Pi_h v_h)\,\mathrm{d}x. \tag{2.12}$$

Once again, we always assume that $\phi:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ and $\psi_h:\Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ are such that (2.12) admits a minimizer $u_h^{cr}\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$. The replacement of $\psi:\Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ by the approximation $\psi_h:\Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ and inserting the L^2–projection operator

$\Pi_h : L^1(\Omega) \to \mathcal{L}^0(\mathcal{T}_h)$ in (2.12) is crucial for the derivation of discrete convex duality relations, as it leads to

$$\inf_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} I^{cr}_h(v_h) = \inf_{\overline{v}_h \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))} \overline{I}^{cr}_h(\overline{v}_h),$$

cf. [18, 7], where $\overline{I}^{cr}_h : \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)) \to \mathbb{R} \cup \{+\infty\}$ for every $\overline{v}_h \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))$ is defined by

$$\overline{I}^{cr}_h(\overline{v}_h) := \inf_{\substack{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h) \\ \Pi_h v_h = \overline{v}_h \text{ in } \mathcal{L}^0(\mathcal{T}_h)}} \int_\Omega \phi(\nabla_h v_h)\,\mathrm{d}x + \int_\Omega \psi_h(\cdot, \overline{v}_h)\,\mathrm{d}x, \tag{2.13}$$

i.e., the minimization of (2.12) is equivalently expressible through the minimization of (2.13). This motivates to examine (2.13) for its (Fenchel) dual problem via the Lagrange functional $\overline{L}_h : \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)) \times \mathcal{R}T^0_N(\mathcal{T}_h) \to \mathbb{R} \cup \{\pm\infty\}$, for every $(\overline{v}_h, y_h)^\top \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)) \times \mathcal{R}T^0_N(\mathcal{T}_h)$ defined by

$$\overline{L}_h(\overline{v}_h, y_h) := -\int_\Omega \operatorname{div}(y_h)\,\overline{v}_h\,\mathrm{d}x - \int_\Omega \phi^*(\Pi_h y_h)\,\mathrm{d}x + \int_\Omega \psi_h(\cdot, \overline{v}_h)\,\mathrm{d}x. \tag{2.14}$$

On the basis of the Lagrange functional (2.14), in [7, 10], it has been established that a (Fenchel) dual problem to the minimization of (2.12) and (2.13), resp., is given by the maximization of $D^{rt}_h : \mathcal{R}T^0_N(\mathcal{T}_h) \to \mathbb{R} \cup \{-\infty\}$, for every $y_h \in \mathcal{R}T^0_N(\mathcal{T}_h)$ defined by

$$D^{rt}_h(y_h) := -\int_\Omega \phi^*(\Pi_h y_h)\,\mathrm{d}x - \int_\Omega \psi^*_h(\cdot, \operatorname{div}(y_h))\,\mathrm{d}x. \tag{2.15}$$

Appealing to [7, Proposition 3.1] or [10, Corollary 3.6], the discrete weak duality relation

$$\begin{aligned}
\inf_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} I^{cr}_h(v_h) &= \inf_{\overline{v}_h \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))} \overline{I}^{cr}_h(\overline{v}_h) \\
&\geq \inf_{\overline{v}_h \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))} \sup_{y_h \in \mathcal{R}T^0_N(\mathcal{T}_h)} \overline{L}_h(\overline{v}_h, y_h) \\
&\geq \sup_{y_h \in \mathcal{R}T^0_N(\mathcal{T}_h)} \inf_{\overline{v}_h \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))} \overline{L}_h(\overline{v}_h, y_h) \\
&\geq \sup_{y_h \in \mathcal{R}T^0_N(\mathcal{T}_h)} D^{rt}_h(y_h).
\end{aligned}$$

holds. If, in addition, $\phi \in C^1(\mathbb{R}^d)$ and $\psi_h(x, \cdot) \in C^1(\mathbb{R})$ for almost every $x \in \Omega$, then a minimizer $u^{cr}_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ of (2.12) and a maximizer $z^{rt}_h \in \mathcal{R}T^0_N(\mathcal{T}_h)$ of (2.15), cf. [10, Corollary 3.7], are related by

$$\begin{aligned}
\Pi_h z^{rt}_h &= D\phi(\nabla_h u^{cr}_h) && \text{in } \mathcal{L}^0(\mathcal{T}_h)^d, \\
\operatorname{div}(z^{rt}_h) &= D\psi_h(\cdot, \Pi_h u^{cr}_h) && \text{in } \mathcal{L}^0(\mathcal{T}_h).
\end{aligned} \tag{2.16}$$

Apart from that, note that by the Fenchel–Young identity (cf. (A.3)), (2.16) is equivalent to

$$\begin{aligned}
\Pi_h z^{rt}_h \cdot \nabla_h u^{cr}_h &= \phi^*(\Pi_h z^{rt}_h) + \phi(\nabla_h u^{cr}_h) && \text{a.e. in } \Omega, \\
\operatorname{div}(z^{rt}_h)\,\Pi_h u^{cr}_h &= \psi^*_h(\cdot, \operatorname{div}(z^{rt}_h)) + \psi_h(\cdot, \Pi_h u^{cr}_h) && \text{a.e. in } \Omega.
\end{aligned} \tag{2.17}$$

Eventually, in this case, we have the discrete reconstruction formula

$$z^{rt}_h = D\phi(\nabla_h u^{cr}_h) + D\psi_h(\cdot, \Pi_h u^{cr}_h) d^{-1}\big(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}\big) \quad \text{in } \mathcal{R}T^0_N(\mathcal{T}_h), \tag{2.18}$$

and discrete strong duality relation applies, i.e.,

$$I^{cr}_h(u^{cr}_h) = D^{rt}_h(z^{rt}_h). \tag{2.19}$$

More generally, without additional regularity assumptions on $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ and $\psi_h : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, the following discrete convex optimality relations apply:

Proposition 2.1. *Assume that $u_h^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_h)$ satisfy*

$$\int_\Omega \phi(\nabla_h u_h^{cr})\,\mathrm{d}x = \sup_{y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)} \int_\Omega \nabla_h u_h^{cr} \cdot \Pi_h y_h\,\mathrm{d}x - \int_\Omega \phi^*(\Pi_h y_h)\,\mathrm{d}x, \tag{2.20}$$

$$\int_\Omega \psi_h^*(\cdot, \mathrm{div}(z_h^{rt}))\,\mathrm{d}x = \sup_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} \int_\Omega \mathrm{div}(z_h^{rt})\,\Pi_h v_h\,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x, \tag{2.21}$$

hold. Then, the following statements are equivalent:
(i) (2.19) *holds.*
(ii) $(\Pi_h u_h^{cr}, z_h^{rt})^\top \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)) \times \mathcal{RT}^0_N(\mathcal{T}_h)$ *is a saddle point of* (2.14).
Moreover, if either of the cases (i) or (ii) applies, then the optimality relations (2.17) *hold.*

Proof. Using the assumptions (2.20) and (2.21), and the integration-by-parts formula (2.3), a direct calculation shows that

$$I_h^{cr}(u_h^{cr}) = \sup_{y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)} \overline{L}_h(\Pi_h u_h^{cr}, y_h), \qquad D_h^{rt}(z_h^{rt}) = \inf_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} \overline{L}_h(\Pi_h v_h, z_h^{rt}). \tag{2.22}$$

ad (i) ⇒ (ii). Combining (2.22) and (2.19), we find that

$$\begin{aligned} I_h^{cr}(u_h^{cr}) &= \sup_{y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)} \overline{L}_h(\Pi_h u_h^{cr}, y_h) \geq \overline{L}_h(\Pi_h u_h^{cr}, z_h^{rt}) \\ &\geq \inf_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} \overline{L}_h(\Pi_h v_h, z_h^{rt}) = D_h^{rt}(z_h^{rt}) = I_h^{cr}(u_h^{cr}). \end{aligned} \tag{2.23}$$

As a result of (2.23), $(\Pi_h u_h^{cr}, z_h^{rt})^\top \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)) \times \mathcal{RT}^0_N(\mathcal{T}_h)$ is a saddle point of (2.14), i.e.,

$$\max_{y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)} \overline{L}_h(\Pi_h u_h^{cr}, y_h) = \overline{L}_h(\Pi_h u_h^{cr}, z_h^{rt}) = \min_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} \overline{L}_h(\Pi_h v_h, z_h^{rt}). \tag{2.24}$$

ad (ii) ⇒ (i). If $(\Pi_h u_h^{cr}, z_h^{rt})^\top \in \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)) \times \mathcal{RT}^0_N(\mathcal{T}_h)$ is a saddle point of (2.14), i.e., (2.24) applies, then the infimum and supremum in (2.22) become a minimum and maximum, resp., so that from (2.24), it immediately follows that (2.19) applies.

Optimality relations. From (2.24) we deduce that $0 \in (\partial_1 \overline{L}_h)(\Pi_h u_h^{cr}, z_h^{rt})$, where the sub-differential ∂_1 is taken in $\Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))$ equipped with $(\cdot,\cdot)_{L^2(\Omega)}$, and $0 \in (\partial_2 \overline{L}_h)(\Pi_h u_h^{cr}, z_h^{rt})$, where the sub-differential ∂_2 is taken in $\mathcal{RT}^0_N(\mathcal{T}_h)$ equipped with $(\cdot,\cdot)_{L^2(\Omega;\mathbb{R}^d)}$. Then, by the Fenchel–Young identity (cf. (A.3)), $0 \in (\partial_1 \overline{L}_h)(\Pi_h u_h^{cr}, z_h^{rt})$ is equivalent to

$$\int_\Omega \Pi_h z_h^{rt} \cdot \nabla_h u_h^{cr}\,\mathrm{d}x = \int_\Omega \phi^*(\Pi_h z_h^{rt}) + \phi(\nabla_h u_h^{cr})\,\mathrm{d}x, \tag{2.25}$$

while $0 \in (\partial_2 \overline{L}_h)(\Pi_h u_h^{cr}, z_h^{rt})$ is equivalent to

$$\int_\Omega \mathrm{div}(z_h^{rt})\,\Pi_h u_h^{cr}\,\mathrm{d}x = \int_\Omega \psi_h^*(\cdot, \mathrm{div}(z_h^{rt})) + \psi_h(\cdot, \Pi_h u_h^{cr})\,\mathrm{d}x. \tag{2.26}$$

Eventually, since, by the Fenchel–Young inequality (cf. (A.2)), we have that $\Pi_h z_h^{rt} \cdot \nabla_h u_h^{cr} \leq \phi^*(\Pi_h z_h^{rt}) + \phi(\nabla_h u_h^{cr})$ and $\mathrm{div}(z_h^{rt})\Pi_h u_h^{cr} \leq \psi_h^*(\cdot, \mathrm{div}(z_h^{rt})) + \psi_h(\cdot, \Pi_h u_h^{cr})$ almost everywhere in Ω, from (2.25) and (2.26), we conclude that (2.17) hold. □

Remark 2.2 (Sufficient conditions for (2.20) and (2.21))**.** *(i) For* (2.20) *it is, e.g., sufficient that $\phi \in C^1(\mathbb{R}^d)$ (cf. [7, Proposition 2.2]) or that there exist regularizations $(\phi_\varepsilon)_{\varepsilon>0} \subseteq C^1(\mathbb{R}^d)$ such that $\phi_\varepsilon \to \phi$ and $\phi_\varepsilon^* \to \phi^*$ locally uniformly on their domains (cf. [7, Remark 2.3]), e.g., for $\phi := |\cdot| \in C^0(\mathbb{R}^d)$, one can employ $(\phi_\varepsilon)_{\varepsilon>0} \subseteq C^1(\mathbb{R}^d)$, defined by $\phi_\varepsilon(t) := \min\{|t| + \frac{\varepsilon}{2}, \frac{|t|^2}{2\varepsilon}\}$ for every $t \in \mathbb{R}^d$ and $\varepsilon > 0$, for which we have that $\phi_\varepsilon^*(t) = \varepsilon\frac{t^2}{2}$ if $|t| \leq 1$ and $\phi_\varepsilon^*(t) = \infty$ else for every $\varepsilon > 0$.*
(ii) For (2.21) *it is, e.g., sufficient that $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for almost every $x \in \Omega$ or that $\mathcal{L}^0(\mathcal{T}_h) = \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))$, e.g., if $\Gamma_D \neq \partial\Omega$ (cf. [10, Corollary 3.2]).*

3. General a posteriori error estimation

In this section, we derive general a posteriori error estimates for convex, possibly non-differentiable, minimization problems such as in Subsection 2.6. These error estimates are well-known in the literature, cf. [44, 43, 45, 30, 42, 6, 9, 7], and form the fundament for any a posteriori error analysis on the basis of convex duality relations. As its proof is proportionate simple, for the benefit of the reader, we want to briefly reproduce it here. Moreover, similar to [29], we want to point out potential error sources that may need to be taken into account and discuss various practical aspects of the concept.

Proposition 3.1. *Let $p\in[1,\infty)$ and $\psi(x,t)=\psi_h(x,t)$ for almost every $x\in\Omega$ and all $t\in\mathbb{R}$. Then, for every $\tilde{u}_h\in W^{1,p}_D(\Omega)$ and $\tilde{z}_h\in W^{p'}_N(\operatorname{div};\Omega)$, we have that*

$$\begin{aligned}\rho_I^2(\tilde{u}_h,u)\leq \int_\Omega \phi(\nabla\tilde{u}_h)-\tilde{z}_h\cdot\nabla\tilde{u}_h+\phi^*(\tilde{z}_h)\,\mathrm{d}x\\ +\int_\Omega \psi_h(\cdot,\tilde{u}_h)-\operatorname{div}(\tilde{z}_h)\,\tilde{u}_h+\psi_h^*(\cdot,\operatorname{div}(\tilde{z}_h))\,\mathrm{d}x=:\eta_h^2(\tilde{u}_h,\tilde{z}_h).\end{aligned}\tag{3.1}$$

Proof. Let $\tilde{u}_h\in W^{1,p}_D(\Omega)$ and $\tilde{z}_h\in W^{p'}_N(\operatorname{div};\Omega)$ be fixed, but arbitrary. Then, referring to the co-coercivity of $I:W^{1,p}_D(\Omega)\to\mathbb{R}\cup\{+\infty\}$ (cf. (2.11)) and the weak duality principle (cf. (2.6)), we find that

$$\begin{aligned}\rho_I^2(\tilde{u}_h,u)&\leq I(\tilde{u}_h)-I(u)\leq I(\tilde{u}_h)-D(\tilde{z}_h)\\ &=\int_\Omega\phi(\nabla\tilde{u}_h)\,\mathrm{d}x+\int_\Omega\psi_h(\cdot,\tilde{u}_h)\,\mathrm{d}x+\int_\Omega\phi^*(\tilde{z}_h)\,\mathrm{d}x+\int_\Omega\psi_h^*(\cdot,\operatorname{div}(\tilde{z}_h))\,\mathrm{d}x.\end{aligned}$$

Eventually, using the integration-by-parts formula (2.3), we conclude the assertion. ☐

Remark 3.2. *(i) The primal-dual a posteriori error estimator $\eta_h^2(\tilde{u}_h,\tilde{z}_h)$ for $\tilde{u}_h\in W^{1,p}_D(\Omega)$ and $\tilde{z}_h\in W^{p'}_N(\operatorname{div};\Omega)$ yields a reliable upper bound for the approximation error $\rho_I^2(\tilde{u}_h,u)$, i.e., we do not have to compute any exact/discrete solution of the primal or dual problem, resp. However, note that for non-admissible $\tilde{u}_h\in W^{1,p}_D(\Omega)$ and/or $\tilde{z}_h\in W^{p'}_N(\operatorname{div};\Omega)$, the critical case $\eta_h^2(\tilde{u}_h,\tilde{z}_h)=+\infty$ might occur.*

(ii) The a posteriori error estimate (3.1) is entirely constant-free, making the error estimator $\eta_h^2:W^{1,p}_D(\Omega)\times W^{p'}_N(\operatorname{div};\Omega)\to\mathbb{R}\cup\{+\infty\}$ attractive compared to classical residual type error estimators, which usually depend on constants that are difficult to bound accurately.

(iii) The (local) refinement indicators $(\eta_{h,T}^2(\tilde{u}_h,\tilde{z}_h))_{T\in\mathcal{T}_h}$, for every $T\in\mathcal{T}_h$ defined by

$$\begin{aligned}\eta_{h,T}^2(\tilde{u}_h,\tilde{z}_h):=\int_T\phi(\nabla\tilde{u}_h)-\tilde{z}_h\cdot\nabla\tilde{u}_h+\phi^*(\tilde{z}_h)\,\mathrm{d}x\\ +\int_T\psi_h(\cdot,\tilde{u}_h)-\operatorname{div}(\tilde{z}_h)\,\tilde{u}_h+\psi_h^*(\cdot,\operatorname{div}(\tilde{z}_h))\,\mathrm{d}x,\end{aligned}\tag{3.2}$$

are non-negative by the Fenchel–Young inequality (cf. (A.2)).

(iv) If we choose $\tilde{u}_h\in\mathcal{S}^1_D(\mathcal{T}_h)\subseteq W^{1,p}_D(\Omega)$ and $\tilde{z}_h\in\mathcal{RT}^0_N(\mathcal{T}_h)\subseteq W^{p'}_N(\operatorname{div};\Omega)$ in Proposition 3.1, then, exploiting that $\nabla\tilde{u}_h\in\mathcal{L}^0(\mathcal{T}_h)^d$ and $\operatorname{div}(\tilde{z}_h)\in\mathcal{L}^0(\mathcal{T}_h)$, for every $T\in\mathcal{T}_h$, it holds

$$\begin{aligned}\eta_{h,T}^2(\tilde{u}_h,\tilde{z}_h)&=\int_T\phi(\nabla\tilde{u}_h)-\Pi_h\tilde{z}_h\cdot\nabla\tilde{u}_h+\phi^*(\Pi_h\tilde{z}_h)\,\mathrm{d}x\\ &\quad+\int_T\psi_h(\cdot,\Pi_h\tilde{u}_h)-\operatorname{div}(\tilde{z}_h)\,\Pi_h\tilde{u}_h+\psi_h^*(\cdot,\operatorname{div}(\tilde{z}_h))\,\mathrm{d}x\\ &\quad+\int_T\psi_h(\cdot,\tilde{u}_h)-\psi_h(\cdot,\Pi_h\tilde{u}_h)\,\mathrm{d}x+\int_T\phi^*(\tilde{z}_h)-\phi^*(\Pi_h\tilde{z}_h)\,\mathrm{d}x\\ &=:\eta_{A,T}^2(\tilde{u}_h,\tilde{z}_h)+\eta_{B,T}^2(\tilde{u}_h,\tilde{z}_h)+\eta_{C,T}^2(\tilde{u}_h)+\eta_{D,T}^2(\tilde{z}_h).\end{aligned}\tag{3.3}$$

Then, $\eta_{A,h}^2(\tilde{u}_h,\tilde{z}_h)$, $\eta_{B,h}^2(\tilde{u}_h,\tilde{z}_h)$, $\eta_{C,h}^2(\tilde{u}_h)$ and $\eta_{D,h}^2(\tilde{z}_h)$ are defined by summation of the corresponding element-wise quantities.

(iv.a) *The representation* (3.3) *for* $\tilde{u}_h \in \mathcal{S}^1_D(\mathcal{T}_h)$ *and* $\tilde{z}_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ *has the particular advantage that the integrands of the first two integrals on the right-hand side, i.e., of* $\eta^2_{A,h}(\tilde{u}_h,\tilde{z}_h)$ *and* $\eta^2_{B,h}(\tilde{u}_h,\tilde{z}_h)$*, are element-wise constant, i.e., we have that*

$$\phi(\nabla\tilde{u}_h) - \Pi_h\tilde{z}_h\cdot\nabla\tilde{u}_h + \phi^*(\Pi_h\tilde{z}_h) \in \mathcal{L}^0(\mathcal{T}_h),$$
$$\psi_h(\cdot,\Pi_h\tilde{u}_h) - \operatorname{div}(\tilde{z}_h)\,\Pi_h\tilde{u}_h + \psi_h^*(\cdot,\operatorname{div}(\tilde{z}_h)) \in \mathcal{L}^0(\mathcal{T}_h),$$

which settles the question of a suitable choice of a quadrature for these integrals, i.e., they do not produce any further quadrature errors.

(iv.b) *Noting that both* $(x\mapsto\psi_h(x,\tilde{u}_h(x))),(x\mapsto\phi^*(\tilde{z}_h(x))):T\to\mathbb{R}\cup\{+\infty\}$ *define convex functions for all* $T\in\mathcal{T}_h$, $\tilde{u}_h\in\mathcal{S}^1_D(\mathcal{T}_h)$ *and* $\tilde{z}_h\in\mathcal{RT}^0_N(\mathcal{T}_h)$*, where we used that* $\psi_h(\cdot,t)\in\mathcal{L}^0(\mathcal{T}_h)$ *for all* $t\in\mathbb{R}$*, similar to [9, Remark 4.8], we propose a trapezoidal quadrature that leads to a reliable upper bound, i.e., for every* $T\in\mathcal{T}_h$*, we have that*

$$\begin{aligned}\eta^2_{C,T}(\tilde{u}_h) &\le \widehat{\eta}^2_{C,T}(\tilde{u}_h) := \int_T \widehat{I}_h\big[\psi_h(\cdot,\tilde{u}_h)\big] - \psi_h(\cdot,\Pi_h\tilde{u}_h)\,\mathrm{d}x,\\ \eta^2_{D,T}(\tilde{z}_h) &\le \widehat{\eta}^2_{D,T}(\tilde{z}_h) := \int_T \widehat{I}_h\big[\phi^*(\tilde{z}_h)\big] - \phi^*(\Pi_h\tilde{z}_h)\,\mathrm{d}x,\end{aligned}\tag{3.4}$$

where we denote by $\widehat{I}_h : C^0(\mathcal{T}_h)\to\mathcal{L}^1(\mathcal{T}_h)$[7] *the element-wise nodal interpolation operator, for every* $v_h\in C^0(\mathcal{T}_h)$ *defined by* $\widehat{I}_h v_h|_T := \sum_{z\in\mathcal{N}_h\cap T}(v_h|_T)(z)\varphi_z$ *for all* $T\in\mathcal{T}_h$*. More precisely, we propose the trapezoidal a posteriori error estimator* $\widehat{\eta}^2_h:\mathcal{S}^1_D(\mathcal{T}_h)\times\mathcal{RT}^0_N(\mathcal{T}_h)\to\mathbb{R}\cup\{+\infty\}$*, for every* $\tilde{u}_h\in\mathcal{S}^1_D(\mathcal{T}_h)$ *and* $\tilde{z}_h\in\mathcal{RT}^0_N(\mathcal{T}_h)$ *defined by*

$$\widehat{\eta}^2_h(\tilde{u}_h,\tilde{z}_h) := \eta^2_{A,h}(\tilde{u}_h,\tilde{z}_h) + \eta^2_{B,h}(\tilde{u}_h,\tilde{z}_h) + \widehat{\eta}^2_{C,h}(\tilde{u}_h) + \widehat{\eta}^2_{D,h}(\tilde{z}_h),\tag{3.5}$$

i.e., $\widehat{\eta}^2_h(\tilde{u}_h,\tilde{z}_h) = \sum_{T\in\mathcal{T}_h}\widehat{\eta}^2_{h,T}(\tilde{u}_h,\tilde{z}_h)$*, where for every* $T\in\mathcal{T}_h$

$$\widehat{\eta}^2_{h,T}(\tilde{u}_h,\tilde{z}_h) := \eta^2_{A,T}(\tilde{u}_h,\tilde{z}_h) + \eta^2_{B,T}(\tilde{u}_h,\tilde{z}_h) + \widehat{\eta}^2_{C,T}(\tilde{u}_h) + \widehat{\eta}^2_{D,T}(\tilde{z}_h).\tag{3.6}$$

Then, for every $\tilde{u}_h\in\mathcal{S}^1_D(\mathcal{T}_h)$ *and* $\tilde{z}_h\in\mathcal{RT}^0_N(\mathcal{T}_h)$*, appealing to* (3.4) *and (iii), we have that*

$$\widehat{\eta}^2_{h,T}(\tilde{u}_h,\tilde{z}_h) \ge \eta^2_{h,T}(\tilde{u}_h,\tilde{z}_h) \ge 0 \quad \text{for all } T\in\mathcal{T}_h.\tag{3.7}$$

(v) *The assumption that* $\psi(x,t)=\psi_h(x,t)$ *for almost every* $x\in\Omega$ *and all* $t\in\mathbb{R}$ *can be avoided by considering* $I_h : W^{1,p}_D(\Omega)\to\mathbb{R}\cup\{+\infty\}$*, for every* $v\in W^{1,p}_D(\Omega)$ *defined by*

$$I_h(v) := \int_\Omega \phi(\nabla v)\,\mathrm{d}x + \int_\Omega \psi_h(\cdot,v)\,\mathrm{d}x,$$

and noting that for every $v\in W^{1,p}_D(\Omega)$*, it holds* $I(v)-I_h(v) = \int_\Omega \psi(\cdot,v)-\psi_h(\cdot,v)\,\mathrm{d}x$*. In combination with a priori bounds for a minimizing function* $u\in W^{1,p}_D(\Omega)$ *of the original functional* $I:W^{1,p}_D(\Omega)\to\mathbb{R}\cup\{+\infty\}$*, this approximation error leads to a computable bound that can be included in the error analysis:*

(v.a) *If* $\psi(x,t):=-f(x)t$ *and* $\psi_h(x,t):=-f_h(x)t$ *for almost every* $x\in\Omega$ *and all* $t\in\mathbb{R}$*, where* $f\in L^{p'}(\Omega)$*,* $p\in[1,\infty]$*, and* $f_h:=\Pi_h f\in\mathcal{L}^0(\mathcal{T}_h)$*, with an element-wise application of Poincaré's inequality, we find that*

$$I(u)-I_h(u) \le c_{\mathrm{P}}\|h_{\mathcal{T}}(f-f_h)\|_{L^{p'}(\Omega)}\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}.$$

(v.b) *If* $\psi(x,t):=\frac{\alpha}{2}(t-g(x))^2$ *and* $\psi_h(x,t):=\frac{\alpha}{2}(t-g_h(x))^2$ *for almost every* $x\in\Omega$ *and all* $t\in\mathbb{R}$*, where* $g\in L^\infty(\Omega)$ *and* $g_h:=\Pi_h g\in\mathcal{L}^0(\mathcal{T}_h)$*, then we find that*

$$I(u)-I_h(u) \le \alpha\big(\|g-g_h\|^2_{L^2(\Omega)} + \|u-\Pi_h u\|^2_{L^2(\Omega)}\big).$$

[7] Here, $C^0(\mathcal{T}_h) := \{v_h\in L^\infty(\Omega)\mid v_h|_T\in C^0(T)\text{ for all }T\in\mathcal{T}_h\}$.

4. A POSTERIORI ERROR ESTIMATION BASED ON POST-PROCESSING NON-CONFORMING APPROXIMATIONS

In this section, we refine the a posteriori error estimate (3.1) resorting to a post-processing of the non-conforming, discontinuous approximation (2.12) of the primal problem (2.4) and discrete convex optimality relations derived in [7, 10] or Proposition 2.1.

Proposition 4.1. *Let $\psi(x,t) := \psi_h(x,t)$ for almost every $x \in \Omega$ and all $t \in \mathbb{R}$. Moreover, let $u_h^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_h)$ be such that (2.19), (2.20) and (2.21) are satisfied. Then, for every $\tilde{u}_h \in \mathcal{S}^1_D(\mathcal{T}_h)$, we have that*

$$\begin{aligned}\eta^2_{A,h}(\tilde{u}_h, z_h^{rt}) &= \int_\Omega \phi(\nabla \tilde{u}_h) - \Pi_h z_h^{rt} \cdot (\nabla \tilde{u}_h - \nabla_h u_h^{cr}) - \phi(\nabla_h u_h^{cr})\,\mathrm{d}x\\ \eta^2_{B,h}(\tilde{u}_h, z_h^{rt}) &= \int_\Omega \psi_h(\cdot, \Pi_h \tilde{u}_h) - \operatorname{div}(z_h^{rt})\,(\Pi_h \tilde{u}_h - \Pi_h u_h^{cr}) - \psi_h(\cdot, \Pi_h u_h^{cr})\,\mathrm{d}x.\end{aligned} \tag{4.1}$$

Proof. Appealing to Proposition 2.1, the optimality relations (2.17) apply. Thus, choosing $\tilde{z}_h = z_h^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_h)$, we immediately conclude the assertion. □

If $\phi \in C^1(\mathbb{R}^d)$ or $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for almost every $x \in \Omega$, then (3.1) can be refined.

Corollary 4.2. *Let the assumptions of Proposition 4.1 be satisfied. Then, the following statements apply:*

(i) If $\phi \in C^1(\mathbb{R}^d)$, then for every $\tilde{u}_h \in \mathcal{S}^1_D(\mathcal{T}_h)$, we have that

$$\eta^2_{A,h}(\tilde{u}_h, z_h^{rt}) \le \int_\Omega \big(D\phi(\nabla \tilde{u}_h) - D\phi(\nabla_h u_h^{cr})\big) \cdot (\nabla \tilde{u}_h - \nabla_h u_h^{cr})\,\mathrm{d}x. \tag{4.2}$$

(ii) If $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for almost every $x \in \Omega$, then for every $\tilde{u}_h \in \mathcal{S}^1_D(\mathcal{T}_h)$, we have that

$$\eta^2_{B,h}(\tilde{u}_h, z_h^{rt}) \le \int_\Omega \big(D\psi_h(\cdot, \Pi_h \tilde{u}_h) - D\psi_h(\cdot, \Pi_h u_h^{cr})\big) \cdot (\Pi_h \tilde{u}_h - \Pi_h u_h^{cr})\,\mathrm{d}x, \tag{4.3}$$

and

$$\eta^2_{C,h}(\tilde{u}_h) \le \int_\Omega \big(D\psi_h(\cdot, \tilde{u}_h) - D\psi_h(\cdot, \Pi_h \tilde{u}_h)\big) \cdot (\tilde{u}_h - \Pi_h \tilde{u}_h)\,\mathrm{d}x. \tag{4.4}$$

(iii) If $\phi^* \in C^1(\mathbb{R}^d)$, then for every $\tilde{z}_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$, we have that

$$\eta^2_{D,h}(\tilde{z}_h) \le \int_\Omega \big(D\phi^*(\tilde{z}_h) - D\phi^*(\Pi_h \tilde{z}_h)\big) \cdot (\tilde{z}_h - \Pi_h \tilde{z}_h)\,\mathrm{d}x. \tag{4.5}$$

Proof. Let $\tilde{u}_h \in \mathcal{S}^1_D(\mathcal{T}_h)$ be fixed, but arbitrary.

ad (i) If $\phi \in C^1(\mathbb{R}^d)$, then, owing to the convexity of $\phi : \mathbb{R}^d \to \mathbb{R}$, it holds

$$\phi(\nabla_h u_h^{cr}) \ge \phi(\nabla \tilde{u}_h) + D\phi(\nabla \tilde{u}_h) \cdot (\nabla_h u_h^{cr} - \nabla \tilde{u}_h) \quad \text{ in } \Omega. \tag{4.6}$$

Then, the optimality relation $\Pi_h z_h^{rt} = D\phi(\nabla_h u_h^{cr})$ in $\mathcal{L}^0(\mathcal{T}_h)^d$ (cf. $(2.22)_1$) and (4.6) yield the assertion.

ad (ii) If $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for almost every $x \in \Omega$, then, owing to the convexity of $\psi_h(x,\cdot) : \mathbb{R} \to \mathbb{R}$ for almost every $x \in \Omega$, it holds

$$\begin{aligned}\psi_h(\cdot, \Pi_h u_h^{cr}) &\ge \psi_h(\cdot, \Pi_h \tilde{u}_h) + D\psi_h(\cdot, \Pi_h \tilde{u}_h)\,(\Pi_h u_h^{cr} - \Pi_h \tilde{u}_h) &&\text{in } \Omega,\\ \psi_h(\cdot, \Pi_h \tilde{u}_h) &\ge \psi_h(\cdot, \tilde{u}_h) + D\psi_h(\cdot, \tilde{u}_h)\,(\Pi_h \tilde{u}_h - \tilde{u}_h) &&\text{in } \Omega.\end{aligned} \tag{4.7}$$

Then, the optimality relation $\operatorname{div}(z_h^{rt}) = D\psi_h(\cdot, \Pi_h u_h^{cr})$ in $\mathcal{L}^0(\mathcal{T}_h)$ (cf. $(2.22)_2$), (4.7) and $\tilde{u}_h - \Pi_h \tilde{u}_h \perp D\psi_h(\cdot, \Pi_h \tilde{u}_h)$ in $L^2(\Omega)$ yield the assertion.

ad (iii) If $\phi^* \in C^1(\mathbb{R}^d)$, then, due to the convexity of $\phi^* : \mathbb{R}^d \to \mathbb{R}$, it holds

$$\phi^*(\Pi_h \tilde{z}_h) \geq \phi^*(\tilde{z}_h) + D\phi^*(\tilde{z}_h) \cdot (\Pi_h \tilde{z}_h - \tilde{z}_h) \quad \text{ in } \Omega. \tag{4.8}$$

Then, (4.8) and $\tilde{z}_h - \Pi_h \tilde{z}_h \perp D\phi^*(\Pi_h \tilde{z}_h)$ in $L^2(\Omega;\mathbb{R}^d)$ yield the assertion. ☐

Remark 4.3 (Improved estimates for strongly convex functionals)**.** *If $\phi \in C^1(\mathbb{R}^d)$ is strongly convex, i.e., there exists a possibly vanishing bi-variate functional $\rho_\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ such that for every $a, b \in \mathbb{R}^d$, we have that $\phi(b) \geq \phi(a) + D\phi(a)\cdot(a-b) + \rho_\phi(a-b, a-b)$, then applying Corollary 4.2, (3.1) can be improved since we can subtract the error quantity*

$$|\tilde{u}_h - u_h^{cr}|_\phi^2 := \int_\Omega \rho_\phi(\nabla \tilde{u}_h - \nabla_h u_h^{cr}, \nabla \tilde{u}_h - \nabla_h u_h^{cr})\,\mathrm{d}x$$

on the right-hand side or, equivalently, add it to the left-hand side, i.e., in the inequality (3.1), the error quantity $\rho_I^2(\tilde{u}_h, u)$ can be replaced by the quantity $\tilde{\rho}_I^2(\tilde{u}_h, u) := \rho_I^2(\tilde{u}_h, u) + |\tilde{u}_h - u_h^{cr}|_\phi^2$. The same equally applies to the functional $\psi_h : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$.

Remark 4.4 (Iteration errors)**.** *If the Crouzeix–Raviart approximation $u_h^{cr,it} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ of the primal problem is obtained by an iterative scheme, then there exists a $r_h^{cr,it} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ such that $u_h^{cr,it} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ is minimal for the functional $I_h^{cr,it} : \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathbb{R} \cup \{\infty\}$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ defined by*

$$I_h^{cr,it}(v_h) := \int_\Omega \phi(\nabla_h v_h)\,\mathrm{d}x + \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x - \int_\Omega r_h^{cr,it}\, v_h\,\mathrm{d}x.$$

Then, the corresponding representation of $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ requires the contribution $r_h^{it} := \Pi_h r_h^{cr,it} \in \mathcal{L}^0(\mathcal{T}_h)$. The difference $r_h^{it} - r_h^{cr,it} \in \mathcal{L}^1(\mathcal{T}_h)$ can be controlled as the approximation error in Remark 3.2 (iv). The function $r_h^{it} \in \mathcal{L}^0(\mathcal{T}_h)$ is explicitly available via post-processing the discrete Euler–Lagrange equation

$$\int_\Omega D\phi(\nabla_h u_h^{cr,it}) \cdot \nabla_h v_h\,\mathrm{d}x + \int_\Omega D\psi_h(\cdot, \Pi_h u_h^{cr,it})\, \Pi_h v_h\,\mathrm{d}x = \int_\Omega r_h^{cr,it}\, v_h\,\mathrm{d}x$$

for all $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, assuming that $\phi \in C^1(\mathbb{R}^d)$ and $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for almost every $x \in \Omega$.

Remark 4.5 (Data errors)**.** *If $\phi \in C^1(\mathbb{R}^d)$ such that $D\phi^* \in C^{0,\alpha}(\mathbb{R}^d;\mathbb{R}^d)$ for some $\alpha \in (0,1]$ and $\psi_h(x,t) := -f_h(x)t$ for almost every $x \in \Omega$ and all $t \in \mathbb{R}$, where $f \in L^{p'}(\Omega)$, $p \in (1,\infty)$, and $f_h = \Pi_h f \in \mathcal{L}^0(\mathcal{T}_h)$, then, using the reconstruction formula (2.18), we observe that*

$$\int_\Omega \big(D\phi^*(z_h^{rt}) - D\phi^*(\Pi_h z_h^{rt})\big) \cdot (z_h^{rt} - \Pi_h z_h^{rt})\,\mathrm{d}x \leq c_{\alpha,\phi^*} \int_\Omega \big|f_h d^{-1}\big(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}\big)\big|^{1+\alpha}\,\mathrm{d}x,$$

while, owing to $\mathrm{div}(z_h^{rt}) = -f_h$ in $\mathcal{L}^0(\mathcal{T}_h)$, it holds

$$\begin{aligned}\int_\Omega \psi_h(\cdot, \Pi_h \tilde{u}_h) - \mathrm{div}(z_h^{rt})\,(\Pi_h \tilde{u}_h - \Pi_h u_h^{cr}) + \psi_h(\cdot, \Pi_h u_h^{cr})\,\mathrm{d}x &= 0,\\ \int_\Omega \psi_h(\cdot, \tilde{u}_h) - \psi_h(\cdot, \Pi_h \tilde{u}_h)\,\mathrm{d}x &= 0.\end{aligned}$$

As a result, resorting to Proposition 3.1, Proposition 4.1 and Corollary 4.2, we conclude that

$$\begin{aligned}\rho_I^2(\tilde{u}_h, u) \leq{}& \int_\Omega \big(D\phi(\nabla \tilde{u}_h) - D\phi(\nabla_h u_h^{cr})\big) \cdot (\nabla \tilde{u}_h - \nabla_h u_h^{cr})\,\mathrm{d}x\\ &+ c_{\alpha,\phi^*} \int_\Omega \big|f_h d^{-1}\big(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}\big)\big|^{1+\alpha}\,\mathrm{d}x,\end{aligned}$$

where the first integral on the right-hand side has the interpretation of a residual, while the second integral contains data approximation errors.

5. APPLICATION: NON-LINEAR DIRICHLET PROBLEMS

In this section, we apply the general theory built previously to well-known non-linear Dirichlet problems including the p–Dirichlet problem and an optimal design problem. For the p–Dirichlet problem, we establish a reliability and efficiency result (cf. Theorem 5.2).

5.1 Continuous non-linear Dirichlet problem

Given a right-hand side $f\in L^{p'}(\Omega)$, $p\in(1,\infty)$, and a proper, convex and lower semi-continuous functional $\phi:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$, we examine the non-linear Dirichlet problem defined by the minimization of $I:W^{1,p}_D(\Omega)\to\mathbb{R}\cup\{+\infty\}$, for every $v\in W^{1,p}_D(\Omega)$ defined by

$$I(v):=\int_\Omega \phi(\nabla v)\,\mathrm{d}x-\int_\Omega f\,v\,\mathrm{d}x, \tag{5.1}$$

i.e., $\psi(t,x):=-f(x)t$ for almost every $x\in\Omega$ and $t\in\mathbb{R}$. Proceeding as, e.g., in [24, p. 113 ff.], one finds that the (Fenchel) dual problem determines a vector field $z\in W^{p'}_N(\mathrm{div};\Omega)$ that is maximal for $D:W^{p'}_N(\mathrm{div};\Omega)\to\mathbb{R}\cup\{-\infty\}$, for every $y\in W^{p'}_N(\mathrm{div};\Omega)$ defined by

$$D(y):=-\int_\Omega \phi^*(y)\,\mathrm{d}x-I_{\{-f\}}(\mathrm{div}(y)), \tag{5.2}$$

where $I_{\{-f\}}:L^{p'}(\Omega)\to\mathbb{R}\cup\{+\infty\}$ is defined by $I_{\{-f\}}(v):=0$ if $v=-f$ and $I_{\{-f\}}(v):=+\infty$ else. If $\phi\in C^0(\mathbb{R}^d)$, then [24, Proposition 5.1] establishes the existence of a maximizer $z\in W^{p'}_N(\mathrm{div};\Omega)$ of (5.6) and that strong duality, i.e., $D(z)=\inf_{v\in W^{1,p}_D(\Omega)}I(v)$, applies. If, in addition, a minimizer $u\in W^{1,p}_D(\Omega)$ of (5.1) exists, e.g., if there exist $c_0>0$ and $c_1\ge 0$ such that $\phi(t)\ge c_0|t|^p-c_1$ for all $t\in\mathbb{R}^d$, then [24, Proposition 5.1] yields the optimality relations

$$\mathrm{div}(z)=-f\quad\text{ in } L^{p'}(\Omega),\qquad z\cdot\nabla u=\phi^*(z)+\phi(\nabla u)\quad\text{ a.e. in }\Omega. \tag{5.3}$$

If $\phi\in C^1(\mathbb{R}^d)$ and there exist $c_0,c_1\ge 0$ such that $|D\phi(t)|\le c_0|t|^{p-1}+c_1$ for all $t\in\mathbb{R}^d$, then, by the Fenchel–Young identity (cf. (A.3)), the optimality relations (5.3) are equivalent to

$$\mathrm{div}(z)=-f\quad\text{ in } L^{p'}(\Omega),\qquad z=D\phi(\nabla u)\quad\text{ in } L^{p'}(\Omega;\mathbb{R}^d). \tag{5.4}$$

5.2 Discrete non-linear Dirichlet problem

Given a right-hand side $f\in L^{p'}(\Omega)$, $p\in(1,\infty)$, and a proper, convex and lower semi-continuous functional $\phi:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$, with $f_h:=\Pi_h f\in\mathcal{L}^0(\mathcal{T}_h)$, the discrete non-linear Dirichlet problem determines a Crouzeix–Raviart function $u_h^{cr}\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ that is minimal for $I_h^{cr}:\mathcal{S}^{1,cr}_D(\mathcal{T}_h)\to\mathbb{R}\cup\{+\infty\}$, for every $v_h\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h):=\int_\Omega \phi(\nabla_h v_h)\,\mathrm{d}x-\int_\Omega f_h\,\Pi_h v_h\,\mathrm{d}x, \tag{5.5}$$

Appealing to Subsection 2.7, the corresponding dual problem to (5.5) determines a Raviart–Thomas vector field $z_h^{rt}\in\mathcal{RT}^0_N(\mathcal{T}_h)$ that is maximal for $D_h^{rt}:\mathcal{RT}^0_N(\mathcal{T}_h)\to\mathbb{R}\cup\{-\infty\}$, for every $y_h\in\mathcal{RT}^0_N(\mathcal{T}_h)$ defined by

$$D_h^{rt}(y_h):=-\int_\Omega \phi^*(\Pi_h y_h)\,\mathrm{d}x-I_{\{-f_h\}}(\mathrm{div}(y_h)). \tag{5.6}$$

If a minimum $u_h^{cr}\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ of (5.5) and a maximum $z_h^{rt}\in\mathcal{RT}^0_N(\mathcal{T}_h)$ of (5.6) are given and discrete strong duality, i.e., $I_h^{cr}(u_h^{cr})=D_h^{rt}(z_h^{rt})$, and (2.20) apply, then Proposition 2.1 yields

$$\mathrm{div}(z_h^{rt})=-f_h\quad\text{ in }\mathcal{L}^0(\mathcal{T}_h),\qquad \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr}=\phi^*(\Pi_h z_h^{rt})+\phi(\nabla_h u_h^{cr})\quad\text{ in }\mathcal{L}^0(\mathcal{T}_h).$$

If a minimizer $u_h^{cr}\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ of (5.5) exists, e.g., if $\phi(t)\ge c_0|t|^p-c_1$ for all $t\in\mathbb{R}^d$ for some $c_0>0$ and $c_1\ge 0$, and if $\phi\in C^1(\mathbb{R}^d)$, then a maximizer $z_h^{rt}\in\mathcal{RT}^0_N(\mathcal{T}_h)$ of (5.6) is given by

$$z_h^{rt}:=D\phi(\nabla_h u_h^{cr})-f_h d^{-1}\big(\mathrm{id}_{\mathbb{R}^d}-\Pi_h\mathrm{id}_{\mathbb{R}^d}\big)\quad\text{ in }\mathcal{RT}^0_N(\mathcal{T}_h), \tag{5.7}$$

i.e., $\Pi_h z_h^{rt}=D\phi(\nabla_h u_h^{cr})$ in $\mathcal{L}^0(\mathcal{T}_h)^d$, and discrete strong duality applies.

Proposition 5.1 (Error estimate via comparison to a non-conforming approximation)**.** *Let $f = f_h \in \mathcal{L}^0(\mathcal{T}_h)$. Moreover, let $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ be such that (2.19), (2.20) and (2.21) are satisfied. Then, for every $\tilde{u}_h \in \mathcal{S}_D^1(\mathcal{T}_h)$, we have that*

$$\begin{aligned}\rho_I^2(\tilde{u}_h, u) &\leq \eta_h^2(\tilde{u}_h, z_h^{rt}) \\ &= \int_\Omega \phi(\nabla \tilde{u}_h) - \Pi_h z_h^{rt} \cdot \big(\nabla \tilde{u}_h - \nabla_h u_h^{cr}\big) - \phi(\nabla_h u_h^{cr})\,\mathrm{d}x \\ &\quad + \int_\Omega \phi^*(z_h^{rt}) - \phi^*(\Pi_h z_h^{rt})\,\mathrm{d}x,\end{aligned} \tag{5.8}$$

where, again, $\rho_I^2 : W_D^{1,p}(\Omega) \times W_D^{1,p}(\Omega) \to \mathbb{R}_{\geq 0}$ denotes a measure for the co-coercivity of $I : W_D^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ at $u \in W_D^{1,p}(\Omega)$, i.e., $\rho_I^2(v, u) \leq I(v) - I(u)$ for all $v \in W_D^{1,p}(\Omega)$.

Proof. Follows from Proposition 3.1, Proposition 4.1 and $\eta_{B,h}^2(\tilde{u}_h, z_h^{rt}) = \eta_{C,h}^2(\tilde{u}_h) = 0$. □

5.3 p–Dirichlet problem

In the particular case $\phi := \frac{1}{p}|\cdot|^p \in C^1(\mathbb{R}^d)$, $p \in (1,\infty)$, the non-linear Dirichlet problem (5.1) reduces to the well-known p–Dirichlet problem. An important property of the p–Dirichlet problem is that its defining functional (5.1) is not only co-coercive at a minimizer $u \in W_D^{1,p}(\Omega)$, but even strongly convex. More precisely, there exists a metric $\rho_I^2 : W_D^{1,p}(\Omega) \times W_D^{1,p}(\Omega) \to \mathbb{R}_{\geq 0}$ such that for every $v \in W_D^{1,p}(\Omega)$, it holds

$$c_\rho^{-1} \rho_I^2(v, u) \leq I(v) - I(u) \leq c_\rho \rho_I^2(v, u), \tag{5.9}$$

for some constant $c_\rho > 0$. But which is the right choice for $\rho_I^2 : W_D^{1,p}(\Omega) \times W_D^{1,p}(\Omega) \to \mathbb{R}_{\geq 0}$? The canonical choice $\rho_I^2(v, w) := \|\nabla v - \nabla w\|_{L^p(\Omega;\mathbb{R}^d)}^p$ for all $v, w \in W_D^{1,p}(\Omega)$ is not well-suited since (5.9), in general, does not hold and one obtains convergence rates that are sub-optimal for $\mathcal{S}_D^1(\mathcal{T}_h)$, cf. [5]. Instead, a so-called F–metric, for every $v, w \in W_D^{1,p}(\Omega)$ defined by

$$\rho_F^2(v, w) := \|F(\nabla v) - F(\nabla w)\|_{L^2(\Omega;\mathbb{R}^d)}^2,$$

where the vector-valued mapping $F : \mathbb{R}^d \to \mathbb{R}^d$ is defined by $F(a) := |a|^{\frac{p-2}{2}} a$ for all $a \in \mathbb{R}^d$, has been introduced and widely employed in the literature, cf. [5, 23, 34, 35, 36, 21, 12, 9]. The F–metric satisfies (5.9), cf. [21, Lemma 16], so that we set $\rho_I^2(v, w) := \rho_F^2(v, w)$ for all $v, w \in W_D^{1,p}(\Omega)$. Note that from (5.9) and the definiteness of $\rho_I^2 : W_D^{1,p}(\Omega) \times W_D^{1,p}(\Omega) \to \mathbb{R}_{\geq 0}$, it follows that the p–Dirichlet problem admits a unique minimizer.

This particular choice, in turn, also enables us to relate the primal-dual a posteriori error estimator to the residual type a posteriori error estimator in [21], i.e., if $u_h^c \in \mathcal{S}_D^1(\mathcal{T}_h)$ denotes the unique minimizer of $I_h^c := I|_{\mathcal{S}_D^1(\mathcal{T}_h)} : \mathcal{S}_D^1(\mathcal{T}_h) \to \mathbb{R}$, the quantity

$$\eta_{res,h}^2(u_h^c) := \sum_{T \in \mathcal{T}_h} \eta_{res,T}^2(u_h^c), \tag{5.10}$$

where for every $T \in \mathcal{T}_h$ and $S \in \mathcal{S}_h \setminus \partial\Omega$ with $S \subseteq \partial T$

$$\begin{aligned}\eta_{res,T}^2(u_h^c) &:= \eta_{E,T}^2(u_h^c) + \sum_{S \in \mathcal{S}_h \setminus \partial\Omega;\, S \subseteq \partial T} \eta_{J,S}^2(u_h^c), \\ \eta_{E,T}^2(u_h^c) &:= \int_T \big(|\nabla u_h^c|^{p-1} + h_T |f_h|\big)^{p'-2} h_T^2 |f_h|^2\,\mathrm{d}x, \\ \eta_{J,S}^2(u_h^c) &:= \int_S h_S |[\![F(\nabla u_h^c)]\!]_S|^2\,\mathrm{d}s.\end{aligned} \tag{5.11}$$

In [21, Lemma 8 & Corollary 11], it has been shown that the error estimator is reliable and efficient, i.e., there exist constants $c_{rel}, c_{\mathit{eff}} > 0$ such that[8]

$$c_{\mathit{eff}}\, \eta_{res,h}^2(u_h^c) \leq \rho_I^2(u, u_h^c) \leq c_{rel}\, \eta_{res,h}^2(u_h^c). \tag{5.12}$$

[8] Here, we assume that $f = f_h \in \mathcal{L}^0(\mathcal{T}_h)$. For the general case $f \in L^{p'}(\Omega)$, $p \in (1,\infty)$, oscillation terms, cf. Remark 5.3 (ii), need to be added in (5.12).

Generalizing the procedure in [15, 27, 28] and resorting to particular properties of the node-averaging operator $\mathcal{J}_h^{av}:\mathcal{S}^{1,cr}_D(\mathcal{T}_h)\to\mathcal{S}^1_D(\mathcal{T}_h)$, cf. Appendix A.2, we are able to establish the global equivalence of the primal-dual a posteriori error estimator, cf. (5.8) with $\tilde{u}_h=u_h^c \in\mathcal{S}^1_D(\mathcal{T}_h)$ and $\tilde{z}_h=z_h^{rt}\in\mathcal{RT}^0_N(\mathcal{T}_h)$ and the residual type a posteriori error estimator, cf. (5.10).

Theorem 5.2 (Equivalence to residual type a posteriori error estimator)**.** *Let $\phi:=\frac{1}{p}|\cdot|^p\in C^1(\mathbb{R}^d)$, $p\in(1,\infty)$, and $f=f_h\in\mathcal{L}^0(\mathcal{T}_h)$. Then, there exists a constant $c_{eq}>0$ such that*

$$c_{eq}^{-1}\,\eta_h^2(u_h^c,z_h^{rt})\le \eta^2_{res,h}(u_h^c)\le c_{eq}\,\eta^2_h(u_h^c,z_h^{rt}). \tag{5.13}$$

Remark 5.3. *(i) Theorem 5.2 can be extended to the case $\phi=\varphi\circ|\cdot|\in C^0(\mathbb{R}^d)\cap C^2(\mathbb{R}^d\setminus\{0\})$, where $\varphi:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ denotes an N–function (cf. Appendix A.2) satisfying the Δ_2–condition, the ∇_2–condition, $\varphi\in C^2(0,\infty)$ and $\varphi'(t)\sim t\varphi''(t)$ uniformly in $t\ge0$*[9]*, cf. [21, Assumption 1].*

(ii) If $f\in L^{p'}(\Omega)$, $p\in(1,\infty)$, in Theorem 5.2, then the equivalence (5.13) can be extended by adding the oscillation term $\mathrm{osc}(u_h^c):=\sum_{T\in\mathcal{T}_h}\mathrm{osc}(u_h^c,T)$, where for all $T\in\mathcal{T}_h$

$$\mathrm{osc}(u_h^c,T):=\int_T\big(|\nabla u_h^c|^{p-1}+h_T|f-\Pi_hf|\big)^{p'-2}h_T^2|f-\Pi_hf|^2\,\mathrm{d}x.$$

Proof. Introducing the $\mathcal{S}^1_D$–$\mathcal{S}^{1,cr}_D$–error $e_h:=u_h^c-u_h^{cr}\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, uniformly in $h>0$, we get

$$\begin{aligned}\int_\Omega\big(D\phi(\nabla u_h^c)-D\phi(\nabla_h u_h^{cr})\big)\cdot\nabla_h e_h\,\mathrm{d}x&=\int_\Omega D\phi(\nabla u_h^c)\cdot\nabla_h(e_h-\mathcal{J}_h^{av}e_h)\,\mathrm{d}x\\&\quad+\int_\Omega f_h\,(\mathcal{J}_h^{av}e_h-e_h)\,\mathrm{d}x\\&\quad+\int_\Omega\big(D\phi(\nabla u_h^c)-D\phi(\nabla u)\big)\cdot\nabla\mathcal{J}_h^{av}e_h\,\mathrm{d}x\\&=:I_h^1+I_h^2+I_h^3.\end{aligned}\tag{5.14}$$

Using that $[\![D\phi(\nabla u_h^c)n\cdot(e_h-\mathcal{J}_h^{av}e_h)]\!]_S=[\![D\phi(\nabla u_h^c)n]\!]_S\cdot\{e_h-\mathcal{J}_h^{av}e_h\}_S+\{D\phi(\nabla u_h^c)n\}_S\cdot[\![e_h-\mathcal{J}_h^{av}e_h]\!]_S$ on S, $\int_S[\![e_h-\mathcal{J}_h^{av}e_h]\!]_S\,\mathrm{d}s=0$ and $\{D\phi(\nabla u_h^c)\}_S=\mathrm{const}$ on S for all $S\in\mathcal{S}_h\setminus\partial\Omega$, an element-wise integration-by-parts, a discrete trace inequality [25, Lem. 12.8] and *(AV.4)*, we find that

$$\begin{aligned}I_h^1&=\sum_{S\in\mathcal{S}_h\setminus\partial\Omega}\int_S[\![D\phi(\nabla u_h^c)n]\!]_S\cdot\{e_h-\mathcal{J}_h^{av}e_h\}_S\,\mathrm{d}s\\&\le\sum_{S\in\mathcal{S}_h\setminus\partial\Omega}|[\![D\phi(\nabla u_h^c)n]\!]_S|\int_S|e_h-\mathcal{J}_h^{av}e_h|\,\mathrm{d}s\\&\le c_{\mathrm{tr}}\sum_{S\in\mathcal{S}_h\setminus\partial\Omega}|[\![D\phi(\nabla u_h^c)n]\!]_S|\sum_{T\in\mathcal{T}_h;S\subseteq\partial T}h_T^{-1}\int_T|e_h-\mathcal{J}_h^{av}e_h|\,\mathrm{d}s\\&\le\tilde{c}_{av}c_{\mathrm{tr}}\sum_{S\in\mathcal{S}_h\setminus\partial\Omega}\sum_{T\in\mathcal{T}_h;S\subseteq\partial T}\int_{\omega_T}|[\![D\phi(\nabla u_h^c)n]\!]_S||\nabla_h e_h|\,\mathrm{d}x.\end{aligned}\tag{5.15}$$

Then, proceeding as for [21, (3.8)–(3.10)], up to obvious adjustments, in particular, using for every $T\in\mathcal{T}_h$, in the patch ω_T, the ε–Young inequality (cf. (A.4)) for the shifted N–function $\varphi_{|\nabla u_h^c(T)|}:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$, cf. Appendix A.2 or [21, Remark 5], defined by

$$\varphi_{|\nabla u_h^c(T)|}(t):=\int_0^t(|\nabla u_h^c(T)|+s)^{p-2}s\,\mathrm{d}s\qquad\text{for all }t\ge0,$$

and $(\varphi_{|\nabla u_h^c(T)|})^*(|[\![D\phi(\nabla u_h^c)n]\!]_S|)\sim|[\![F(\nabla u_h^c)]\!]_S|^2$ on S for all $S\in\mathcal{S}_h\setminus\partial\Omega$ with $S\subseteq\partial T$ (cf. [21, Cor. 6]), where we for any $T\in\mathcal{T}_h$ write $\nabla u_h^c(T)$ to indicate that the shift on the whole

[9]Here, we employ the notation $f\sim g$ for two (Lebesgue–)measurable functions $f,g:\Omega\to\mathbb{R}$, if there exists a constant $c>0$ such that $c^{-1}f\le g\le cf$ almost everywhere in Ω.

patch ω_T depends on the value of ∇u_h^c on the triangle T and where $(\varphi_{|\nabla u_h^c(T)|})^* : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denotes the Fenchel conjugate to $\varphi_{|\nabla u_h^c(T)|} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, for any $\varepsilon > 0$, we conclude that

$$\begin{aligned} I_h^1 &\leq c_{av} c_{\mathrm{tr}} \sum_{S \in \mathcal{S}_h \setminus \partial\Omega} \sum_{T \in \mathcal{T}_h; S \subseteq \partial T} \int_{\omega_T} c_\varepsilon \, (\varphi_{|\nabla u_h^c(T)|})^*(|[\![D\phi(\nabla u_h^c) n]\!]_S|) + \varepsilon \, \varphi_{|\nabla u_h^c(T)|}(|\nabla_h e_h|) \, \mathrm{d}x \\ &\leq c_{av} c_{\mathrm{tr}} c_\varepsilon \sum_{S \in \mathcal{S}_h \setminus \partial\Omega} \eta_{J,S}^2(u_h^c) + \tilde{c}_{av} c_{\mathrm{tr}} \varepsilon \sum_{T \in \mathcal{T}_h} \int_{\omega_T} \varphi_{|\nabla u_h^c(T)|}(|\nabla_h e_h|) \, \mathrm{d}x. \end{aligned} \tag{5.16}$$

Using the ε–Young inequality (cf. (A.4)), $(|\nabla u_h^c|^{p-1} + h_T |f_h|)^{p'-2} h_T^2 |f_h|^2 \sim (\varphi_{|\nabla u_h^c|})^*(h_T |f_h|)$ in T for all $T \in \mathcal{T}_h$ (uniformly in $h > 0$, cf. [21, (2.6)]) and Corollary A.2, for any $\varepsilon > 0$, we get

$$\begin{aligned} I_h^2 &\leq \sum_{T \in \mathcal{T}_h} \int_T c_\varepsilon \, (\varphi_{|\nabla u_h^c|})^*(h_T |f_h|) + \varepsilon \, \varphi_{|\nabla u_h^c|}\big(h_T^{-1} |e_h - \mathcal{J}_h^{av} e_h|\big) \, \mathrm{d}x \\ &\leq c_\varepsilon \sum_{T \in \mathcal{T}_h} \eta_{E,T}^2(u_h^c) + \tilde{c}_{av} \varepsilon \sum_{T \in \mathcal{T}_h} \int_{\omega_T} \varphi_{|\nabla u_h^c(T)|}(|\nabla_h e_h|) \, \mathrm{d}x. \end{aligned} \tag{5.17}$$

The ε–Young inequality (cf. (A.4)), $(\varphi_{|\nabla u_h^c|})^*(|D\phi(\nabla u_h^c) - D\phi(\nabla u)|) \sim |F(\nabla u_h^c) - F(\nabla u)|^2$ in T for all $T \in \mathcal{T}_h$ (uniformly in $h > 0$, cf. [21, Cor. 6]), and Corollary A.2, for any $\varepsilon > 0$, yield

$$\begin{aligned} I_h^3 &\leq \sum_{T \in \mathcal{T}_h} \int_T c_\varepsilon \, (\varphi_{|\nabla u_h^c|})^*(|D\phi(\nabla u_h^c) - D\phi(\nabla u)|) + \varepsilon \, \varphi_{|\nabla u_h^c|}(|\nabla \mathcal{J}_h^{av} e_h|) \, \mathrm{d}x \\ &\leq c_\varepsilon \, \rho_I^2(u_h^c, u) + \tilde{c}_{av} \varepsilon \sum_{T \in \mathcal{T}_h} \int_{\omega_T} \varphi_{|\nabla u_h^c(T)|}(|\nabla_h e_h|) \, \mathrm{d}x. \end{aligned} \tag{5.18}$$

Proceeding as in [21, p. 9 & 10], we obtain a constant $c > 0$ such that

$$\sum_{T \in \mathcal{T}_h} \int_{\omega_T} \varphi_{|\nabla u_h^c(T)|}(|\nabla_h e_h|) \, \mathrm{d}x \leq c \sum_{T \in \mathcal{T}_h} \int_T \varphi_{|\nabla u_h^c|}(|\nabla_h e_h|) \, \mathrm{d}x + c \sum_{S \in \mathcal{S}_h \setminus \partial\Omega} \eta_{J,S}^2(u_h^c). \tag{5.19}$$

Then, combining (5.12) and (5.14)–(5.19), for any $\varepsilon > 0$, we conclude that

$$\int_\Omega \big(D\phi(\nabla u_h^c) - D\phi(\nabla_h u_h^{cr})\big) \cdot \nabla_h e_h \, \mathrm{d}x \leq c_\varepsilon \, \eta_{res,h}^2(u_h^c) + \varepsilon \sum_{T \in \mathcal{T}_h} \int_T \varphi_{|\nabla u_h^c|}(|\nabla_h e_h|) \, \mathrm{d}x.$$

Resorting the reconstruction formula (5.7) and [21, Lemma 3], we obtain a constant $c > 0$ such that for every $T \in \mathcal{T}_h$, we deduce that

$$\int_T \big(D\phi^*(z_h^{rt}) - D\phi^*(\Pi_h z_h^{rt})\big) \cdot (z_h^{rt} - \Pi_h z_h^{rt}) \, \mathrm{d}x \leq c \int_T (\varphi_{|\nabla_h u_h^{cr}|})^*(h_T |f_h|) \, \mathrm{d}x.$$

Then, a change of shift (cf. [21, Corollary 28]), for every $\varepsilon > 0$, provides a constant $c_\varepsilon > 0$ such that for every $T \in \mathcal{T}_h$, it holds

$$\int_T (\varphi_{|\nabla_h u_h^{cr}|})^*(h_T |f_h|) \, \mathrm{d}x \leq c_\varepsilon \, \eta_{E,T}^2(u_h^c) + \varepsilon \int_T \varphi_{|\nabla u_h^c|}(|\nabla_h e_h|) \, \mathrm{d}x.$$

Thanks to $\varphi_{|\nabla u_h^c|}(|\nabla_h e_h|) \sim (D\phi(\nabla u_h^c) - D\phi(\nabla_h u_h^{cr})) \cdot (\nabla u_h^c - \nabla_h u_h^{cr})$ in T for all $T \in \mathcal{T}_h$ (uniformly in $h > 0$), for $\varepsilon > 0$ sufficiently small, we obtain a constant $\tilde{c}_{eq} > 0$ such that

$$\begin{aligned} &\int_\Omega \big(D\phi(\nabla u_h^c) - D\phi(\nabla_h u_h^{cr})\big) \cdot (\nabla u_h^c - \nabla_h u_h^{cr}) \, \mathrm{d}x \\ &\quad + \int_\Omega \big(D\phi^*(z_h^{rt}) - D\phi^*(\Pi_h z_h^{rt})\big) \cdot (z_h^{rt} - \Pi_h z_h^{rt}) \, \mathrm{d}x \leq \tilde{c}_{eq} \, \eta_{res,h}^2(u_h^c). \end{aligned} \tag{5.20}$$

From (5.20), Proposition 5.1, Corollary 4.2 (i) & (iii) and (5.12) we, in turn, conclude that $\rho_I^2(u_h^c, u) \leq \eta_h^2(u_h^c, z_h^{rt}) \leq \tilde{c}_{eq} \, \eta_{res,h}^2(u_h^c) \leq \tilde{c}_{eq} \, c_{\mathit{eff}} \, \rho_I^2(u_h^c, u)$, which implies (5.13). □

Corollary 5.4 (Gobal reliability and efficiency)**.** *Let $\phi := \frac{1}{p}|\cdot|^p \in C^1(\mathbb{R}^d)$, $p \in (1,\infty)$, and $f = f_h \in \mathcal{L}^0(\mathcal{T}_h)$. Then, there exist constants $c_{rel}, c_{eff} > 0$ such that*

$$c_{eff}\, \rho_I^2(u_h^c, u) \leq \eta_h^2(u_h^c, z_h^{rt}) \leq c_{rel}\, \rho_I^2(u_h^c, u). \tag{5.21}$$

Remark 5.5. *(i) The extensions described in Remark 5.3 equally apply to Corollary 5.4.*

(ii) Since we have used global arguments, e.g., discrete and continuous Euler–Lagrange equations, cf. (5.14), and element-wise integration-by-parts, cf. (5.15), it remains unclear whether the primal-dual a posteriori error estimator $\eta_h^2(u_h^c, z_h^{rt})$ and the residual type a posteriori error estimator $\eta_{res,h}^2(u_h^c)$ are also locally equivalent, i.e., if there exists a constant $c_{eq}>0$, such that $c_{eq}^{-1}\, \eta_{h,T}^2(u_h^c, z_h^{rt}) \leq \eta_{res,T}^2(u_h^c) \leq c_{eq}\, \eta_{h,T}^2(u_h^c, z_h^{rt})$ for all $T \in \mathcal{T}_h$.

(iii) As, according to (ii), the local equivalence of the primal-dual a posteriori error estimator $\eta_h^2(u_h^c, z_h^{rt})$ and the residual type a posteriori error estimator $\eta_{res,h}^2(u_h^c)$ is still open, we cannot refer to [21] to infer the convergence of the adaptive algorithm, cf. Algorithm 6.1. In fact, in [21], it was decisively used that the residual type a posteriori error estimator $\eta_{res,h}^2(u_h^c)$ is locally efficient, i.e., there exists a constant $c_{eff} > 0$ such that for every $T \in \mathcal{T}_h$, it holds $c_{eff}\, \eta_{res,T}^2(u_h^c) \leq \|F(\nabla u_h^c) - F(\nabla u)\|_{L^2(T;\mathbb{R}^d)}^2$. This, in turn, suggests to use residual type a posteriori error estimators for adaptive mesh refinement and primal-dual a posteriori error estimators for error estimation.

5.4 A degenerate minimization problem: An optimal design problem

If $p = 2$ and $\phi := \psi \circ |\cdot| \in C^1(\mathbb{R}^d)$, where $\psi \in C^1(\mathbb{R}_{\geq 0})$ is prescribed by the initial value $\psi(0) = 0$ and the derivative $\psi' \in C^0(\mathbb{R}_{\geq 0})$, for all $t \geq 0$ defined by

$$\psi'(t) := \begin{cases} \mu_2 t & \text{for } t \in [0, t_1] \\ \mu_2 t_1 & \text{for } t \in [t_1, t_2] \\ \mu_1 t & \text{for } t \in [t_2, +\infty) \end{cases}, \tag{5.22}$$

$0 < t_1 < t_2$ and $0 < \mu_1 < \mu_2$ are given parameters such that $t_1\mu_2 = t_2\mu_1$, then the non-linear Dirichlet problem (5.1) reduces to the optimal design problem for maximal torsion stiffness of an infinite bar of a given geometry and unknown distribution of two materials of prescribed amounts, a classical example from topology optimization, cf. [19]. The optimal design problem is a degenerate convex minimization problem, i.e., in contrast to the p–Dirichlet problem, the defining functional (5.1) is not strongly convex, but only co-coercive since the energy density $\phi := \psi \circ |\cdot| \in C^1(\mathbb{R}^d)$, cf. [11, Proposition 4.2], for every $a, b \in \mathbb{R}^d$ satisfies

$$(2\mu_2)^{-1}|D\phi(a) - D\phi(b)|^2 \leq \phi(a) - \phi(b) - D\phi(a)\cdot(a-b). \tag{5.23}$$

If $u \in W_D^{1,2}(\Omega)$ is minimal for (5.1), from the finite-dimensional co-coercivity property (5.23), for every $v \in W_D^{1,2}(\Omega)$, we have the following infinite-dimensional co-coercivity property

$$(2\mu_2)^{-1}\|D\phi(\nabla v) - D\phi(\nabla u)\|_{L^2(\Omega;\mathbb{R}^d)}^2 \leq I(v) - I(u). \tag{5.24}$$

It is well-known that minimizers $u \in W_D^{1,2}(\Omega)$ of (5.1) are (possibly) non-unique, while from (5.24) we directly conclude that $D\phi(\nabla u) \in L^2(\Omega;\mathbb{R}^d)$ is unique. The co-coercivity property motivates to define a measure $\rho_I^2 : W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega) \to \mathbb{R}_{\geq 0}$ for the co-coercivity of (5.1) by $\rho_I^2(v,w) := \|D\phi(\nabla v) - D\phi(\nabla w)\|_{L^2(\Omega;\mathbb{R}^d)}^2$ for all $v, w \in W_D^{1,2}(\Omega)$. However, since explicit representations of minimizers $u \in W_D^{1,2}(\Omega)$ of (5.1) for simple data, e.g., $f = 1$, are rare, in our experiments, we consider $\rho_I^2 : W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega) \to \mathbb{R}_{\geq 0}$, for every $v, w \in W_D^{1,2}(\Omega)$ defined by

$$\rho_I^2(v,w) := I(v) - I(w), \tag{5.25}$$

which has the particular advantage that the exact value $I(u)$ can be approximated resorting to Aitken's δ^2–process, cf. [2].

6. NUMERICAL EXPERIMENTS

In this section, we verify our theoretical findings via numerical experiments. More precisely, we present numerical results for the approximation of the p–Dirichlet problem and an optimal design problem by deploying adaptive mesh refinements on the basis of the trapezoidal primal-dual a posteriori error estimators $(\widehat{\eta}^2_{h,T}(\tilde{u}_h, z_h^{rt}))_{T\in\mathcal{T}_h}$, cf. (3.6).

Before we present the computational experiments, we briefly outline the general details of our implementations. In general, we follow the adaptive algorithm, cf. [48, 47, 21, 17]:

Algorithm 6.1 (AFEM)**.** *Let $\varepsilon_{\mathrm{STOP}} > 0$, $\theta \in (0,1)$ and $\mathcal{T}_0$ a conforming initial triangulation of Ω. Then, for $k \geq 0$:*

('Solve') *Compute both a conforming approximation $\tilde{u}_k \in \mathcal{S}^1_D(\mathcal{T}_k)$ and a minimizer $u_k^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_k)$ of (2.12). Post-process $u_k^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_k)$ to obtain a maximizer $z_k^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_k)$ of (2.15).*

('Estimate') *Compute the refinement indicators $(\widehat{\eta}^2_{k,T}(\tilde{u}_k, z_k^{rt}))_{T\in\mathcal{T}_k}$. If $\widehat{\eta}^2_k(\tilde{u}_k, z_k^{rt}) \leq \varepsilon_{\mathrm{STOP}}$, then* STOP.

('Mark') *Choose a minimal (in terms of cardinality) subset $\mathcal{M}_k \subseteq \mathcal{T}_k$ such that*

$$\sum_{T\in\mathcal{M}_k} \widehat{\eta}^2_{k,T}(\tilde{u}_k, z_k^{rt}) \geq \theta^2 \sum_{T\in\mathcal{T}_k} \widehat{\eta}^2_{k,T}(\tilde{u}_k, z_k^{rt}).$$

('Refine') *Perform a (minimal) conforming refinement of $\mathcal{T}_k$ to obtain $\mathcal{T}_{k+1}$ such that each $T \in \mathcal{M}_k$ is refined in $\mathcal{T}_{k+1}$, i.e., each $T \in \mathcal{M}_k$ and each of its sides contains a node of $\mathcal{T}_{k+1}$ in its interior. Increase $k \to k+1$ and continue with ('Solve').*

Remark 6.2. *(i) The computation of a conforming approximation $\tilde{u}_k \in \mathcal{S}^1_D(\mathcal{T}_k)$ and a minimizer $u_k^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_k)$ of (2.12) in ('Solve') is adjusted to the respective problem.*

(ii) The reconstruction of a maximizer $z_k^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_k)$ of (2.15) in ('Solve') is based on explicit representation formulas and does not entail further computational costs.

(iii) If not otherwise specified, we employ the parameter $\theta = \frac{1}{2}$ in ('Estimate').

(iv) To find the minimal (in terms of cardinality) set $\mathcal{M}_k \subseteq \mathcal{T}_k$ in ('Mark'), we deploy the Dörfler marking strategy, cf. [22].

(v) The (minimal) conforming refinement of $\mathcal{T}_k$ with respect to $\mathcal{M}_k$ in ('Refine') is obtained deploying the red–green–blue–refinement algorithm.

All experiments were conducted using the finite element software package FEniCS, cf. [37]. All graphics are generated using the Matplotlib library, cf. [31].

6.1 p–Dirichlet problem

We examine the p–Dirichlet problem with prescribed in-homogeneous Dirichlet boundary data on an L–shaped domain. More precisely, we let $\Omega := (-1,1)^2 \setminus ([0,1]\times[-1,0])$, $\Gamma_D := \partial\Omega$, $\Gamma_N := \emptyset$, $\phi := \frac{1}{p}|\cdot|^p \in C^1(\mathbb{R}^2)$, $p \in (1,\infty)$, and prescribe in-homogeneous Dirichlet boundary data $u_D = u|_{\partial\Omega} \in L^p(\partial\Omega)$ through restriction of the unique exact solution $u \in W^{1,p}(\Omega)$, in polar coordinates, for every $(r,\theta)^\top \in (0,\infty)\times(0,2\pi)$ defined by

$$u(r,\theta) := r^\delta \sin(\delta\theta),$$

to the boundary $\partial\Omega$. The particular choice of $\delta > 0$ will be specified later in dependence of the choice of $p \in (1,\infty)$. Then, the corresponding non-smooth right-hand side $f \in L^{p'}(\Omega)$, in polar coordinates, is for every $(r,\theta)^\top \in (0,\infty)\times(0,2\pi)$ defined by

$$f(r,\theta) := -(2-p)\delta^{p-1}(1-\delta)r^{(\delta-1)(p-1)-1}\sin(\delta\theta).$$

For $p \in (1,\infty)$, we let $\delta = \frac{6}{5}(1-\frac{1}{p}) > 0$. Then, we have that $u \in W^{1,p}(\Omega)$, but $u \notin W^{2,p}(\Omega)$.

The initial triangulation $\mathcal{T}_0$ consists of 96 elements and 65 vertices. We use $f_k := \Pi_k f \in \mathcal{L}^0(\mathcal{T}_k)$ for all $k = 0, \dots, 19$. In what follows, for every $k = 0, \dots, 19$, we denote by $u_k^c \in \mathcal{S}_D^1(\mathcal{T}_k)$ the minimizer of $I_k^c := I|_{\mathcal{S}_D^1(\mathcal{T}_k)} : \mathcal{S}_D^1(\mathcal{T}_k) \to \mathbb{R}$ and by $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ the minimizer of (5.5). Both minimizers are computed using the Newton line search algorithm of PETSc, cf. [4], with an absolute tolerance of about $\tau_{abs} = 1\mathrm{e}{-8}$ and a relative tolerance of about $\tau_{rel} = 1\mathrm{e}{-10}$. The linear system emerging in each Newton step is solved using the generalized minimal residual method (GMRES). Globally convergent semi-implicit discretizations of the respective L^2–gradient flows yield comparable results, but terminate significantly slower. Then, for every $k = 0, \dots, 19$, via post-processing $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$, we obtain a maximizer $z_k^{rt} \in \mathcal{RT}^0(\mathcal{T}_k)$ of (5.6) by resorting to the reconstruction formula (5.7). In Figure 1, for every $k = 0, \dots, 19$ and $\tilde{u}_{h_k} := u_k^c \in \mathcal{S}_D^1(\mathcal{T}_k)$, the square root of the trapezoidal primal-dual a posteriori error estimator

$$\begin{aligned} \widehat{\eta}_k^2(\tilde{u}_{h_k}, z_k^{rt}) = &\int_\Omega \phi(\nabla \tilde{u}_{h_k}) - \Pi_{h_k} z_k^{rt} \cdot (\nabla \tilde{u}_{h_k} - \nabla_{h_k} u_k^{cr}) + \phi(\nabla_{h_k} \tilde{u}_{h_k})\,\mathrm{d}x \\ &+ \int_\Omega \widehat{I}_{h_k}\left[\phi^*(z_k^{rt})\right] - \phi^*(\Pi_{h_k} z_k^{rt})\,\mathrm{d}x, \end{aligned} \tag{6.1}$$

and square root of the error on the left-hand side of the estimate in Proposition 5.1, i.e.,

$$\rho_I^2(u, \tilde{u}_{h_k}) = \|F(\nabla u) - F(\nabla \tilde{u}_{h_k})\|^2_{L^2(\Omega;\mathbb{R}^2)}, \tag{6.2}$$

are plotted versus the number of degrees of freedom $N_k := \mathrm{card}(\mathcal{N}_{h_k})$ in a log log–plot. In it, one clearly observes that mesh adaptivity yields the quasi-optimal convergence rate $h_k \sim N_k^{-\frac{1}{2}}$. In particular, for every $k = 0, \dots, 19$, the trapezoidal primal-dual a posteriori error estimator $\widehat{\eta}_k^2(u_k^c, z_k^{rt})$ defines a reliable upper bound for the error quantity $\rho_I^2(u, u_k^c)$. Also note that data approximation terms such as, e.g., in Remark 3.2 (v.b) are disregarded in all experiments.

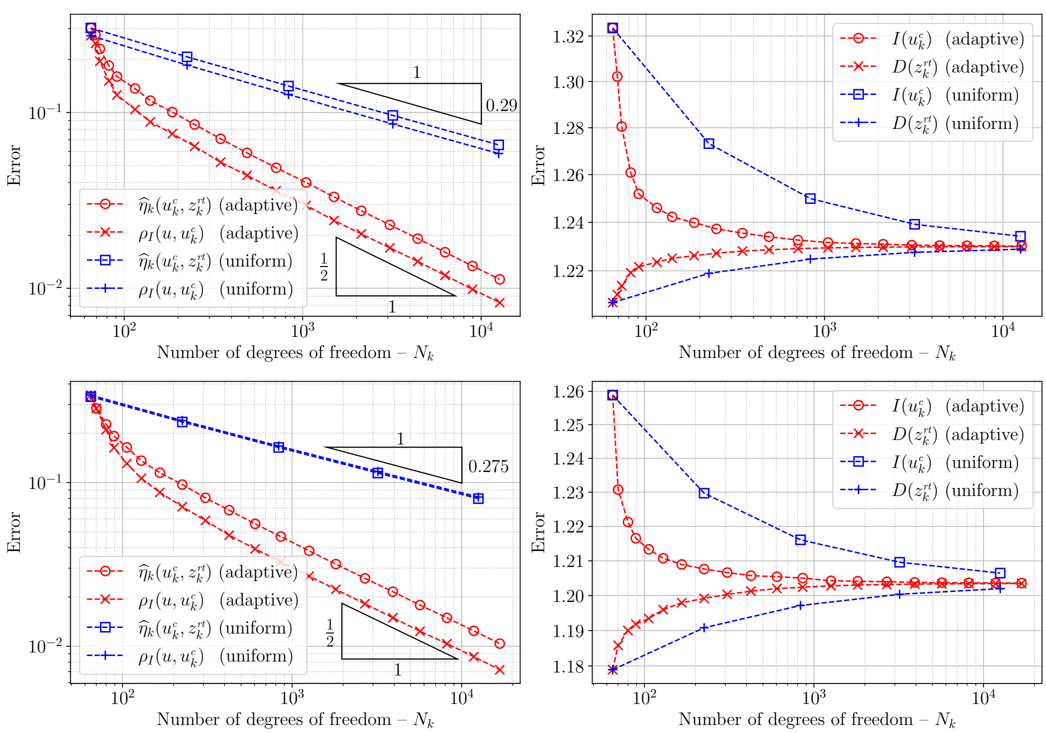

Figure 1: The trapezoidal primal-dual a posteriori error estimators $\widehat{\eta}_k^2(u_k^c, z_k^{rt})$, cf. (6.1), and the error quantities $\rho_I^2(u, u_k^c)$, cf. (6.2), (left) as well as the primal energies $I(u_k^c)$, cf. (5.1), and dual energies $D(z_k^{rt})$, (5.2), (right) for uniform and adaptive mesh refinement for $k = 0, \dots, 4$ and $k = 0, \dots, 19$, resp. Top: p–Dirichlet problem with $p = 1.6$. Bottom: p–Dirichlet problem with $p = 1.2$.

On the right-hand side of Figure 1, we displayed the energy curves for $I(u_k^c)$ and $D(z_k^{rt})$, $k=0,\ldots,19$, resp. The primal and dual energies converge to the optimal value $I(u)=D(z)$ and the primal-dual gap $I(u_k^c)-D(z_k^{rt})$, $k=0,\ldots,19$, converges to zero as $N_k\to\infty$, and even at a linear rate, when local mesh refinement is used.

In Figure 2, for every $k=0,\ldots,19$, we compare the trapezoidal primal-dual a posteriori error estimator $\widehat{\eta}_k^2(u_k^c,z_k^{rt})$ with the residual a posteriori error estimator $\eta^2_{res,k}(u_k^c):=\eta^2_{res,h_k}(u_k^c)$, for the p–Dirichlet problem with in-homogeneous Dirichlet boundary data on the L–shaped domain Ω for $p=1.6$ and $p=1.2$. In it, one observes that both estimators decay at the same quasi-optimal rate $\mathcal{O}(N_k^{-\frac{1}{2}})$. The experiments confirm that $\widehat{\eta}_k^2(u_k^c,z_k^{rt})$ and $\eta^2_{res,k}(u_k^c)$, up to an overestimation of $\eta^2_{res,k}(u_k^c)$, behave identically supporting the findings of Theorem 5.2.

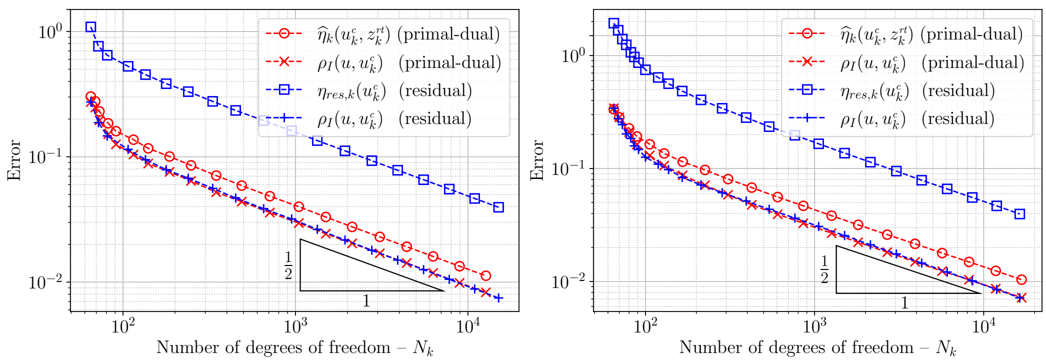

Figure 2: The trapezoidal primal-dual a posteriori error estimators $\widehat{\eta}_h^2(u_k^c,z_k^{rt})$, cf. (6.1), and residual type a posteriori error estimators $\eta^2_{res,k}(u_k^c)$, cf. (5.10), for $k=0,\ldots,19$. Left: p–Dirichlet problem with $p=1.6$. Right: p–Dirichlet problem with $p=1.2$.

In Figure 3, for every $k=0,\ldots,19$ and $\tilde{u}_{h_k}:=\mathcal{J}_{h_k}^{av}u_k^{cr}\in\mathcal{S}_D^1(\mathcal{T}_k)$, the square root of the primal-dual a posteriori error estimator $\widehat{\eta}_k^2(\mathcal{J}_{h_k}^{av}u_k^{cr},z_k^{rt})$, cf. (6.1), and of the error quantity $\rho_I^2(u,\mathcal{J}_{h_k}^{av}u_k^{cr})$, cf. (6.2), are plotted versus the number of degrees of freedom N_k in a loglog–plot. In it, one observes that mesh adaptivity yields the quasi-optimal convergence rate $h_k\sim N_k^{-\frac{1}{2}}$ and that for every $k=0,\ldots,19$, the trapezoidal primal-dual a posteriori error estimator $\widehat{\eta}_k^2(\mathcal{J}_{h_k}^{av}u_k^{cr},z_k^{rt})$ defines a reliable upper bound for the error quantity $\rho_I^2(u,\mathcal{J}_{h_k}^{av}u_k^{cr})$. Moreover, on the right-hand side of Figure 3, we displayed the energy curves for $I(\mathcal{J}_{h_k}^{av}u_k^{cr})$ and $D(z_k^{rt})$, $k=0,\ldots,19$, resp., whose distance likewise converges to zero as $N_k\to\infty$. The experiments justify to employ $\tilde{u}_{h_k}=\mathcal{J}_{h_k}^{av}u_k^{cr}\in\mathcal{S}_D^1(\mathcal{T}_k)$ instead of $\tilde{u}_{h_k}=u_k^c\in\mathcal{S}_D^1(\mathcal{T}_k)$ in Algorithm 6.1. Then, only one non-linear problem per iteration has to be solved in ('Solve').

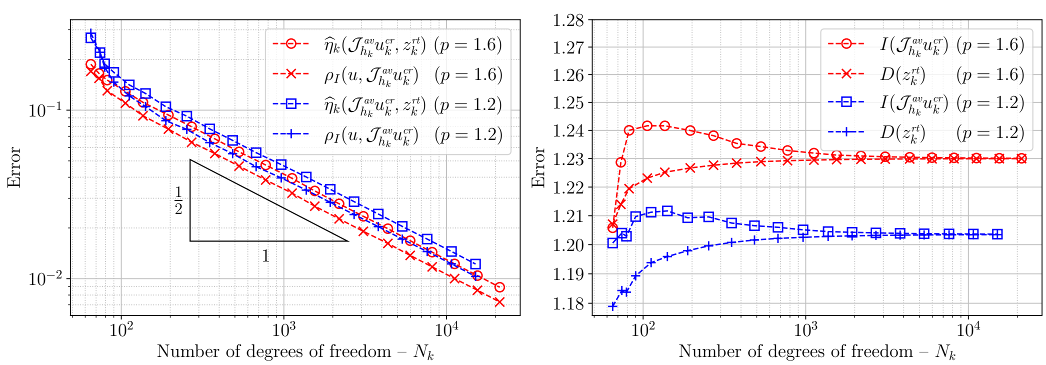

Figure 3: The trapezoidal primal-dual a posteriori error estimators $\widehat{\eta}_k^2(\mathcal{J}_{h_k}^{av}u_k^{cr},z_k^{rt})$, cf. (6.1), and the error quantities $\rho_I^2(u,\mathcal{J}_{h_k}^{av}u_k^{cr})$, cf. (6.2), (left) and the primal energies $I(\mathcal{J}_{h_k}^{av}u_k^{cr})$, cf. (5.1), and the dual energies $D(z_k^{rt})$, cf. (5.2), (right) for adaptive mesh refinement for $k=0,\ldots,19$, $p=1.6$ and $p=1.2$, resp.

6.2 Optimal design problem

We examine an optimal design problem with prescribed homogeneous Dirichlet boundary data on an L–shaped domain. More precisely, we let $\Omega := (-1,1)^2 \setminus ([0,1]\times[-1,0])$, $\Gamma_D := \partial\Omega$, $\Gamma_N := \emptyset$, $\phi := \psi \circ |\cdot| C^1(\mathbb{R}^2)$, where $\psi \in C^1(\mathbb{R}_{\geq 0})$ is defined by (5.22) for $\mu_1 = 1$, $\mu_2 = 2$, $t_1 = \sqrt{2\lambda\mu_1/\mu_2} = \sqrt{\lambda}$, $t_2 = \sqrt{2\lambda\mu_2/\mu_1} = 2\sqrt{\lambda}$, and $\lambda = 0.0145$ as in [11], and $f = 1$. We use the same initial triangulation $\mathcal{T}_0$ as in Subsection 6.1 and exploit that $f_k := \Pi_{h_k} f = f \in \mathcal{L}^0(\mathcal{T}_k)$ for all $k = 0,\ldots,19$. Apart from that, for every $k = 0,\ldots,19$, we again denote by $u_k^c \in \mathcal{S}^1_D(\mathcal{T}_k)$ the minimizer of $I_k^c := I|_{\mathcal{S}^1_D(\mathcal{T}_k)} : \mathcal{S}^1_D(\mathcal{T}_k) \to \mathbb{R}$ and by $u_k^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_k)$ the minimizer of (5.5). Both minimizers are computed resorting to a semi-implicit discretization of the respective L^2–gradient flows, cf. [7, Sec. 5], with stopping criterion $\varepsilon_{stop} = h^2/20$. Since these schemes are unconditionally strongly stable, cf. [7, Prop. 5.2], we employ the fixed step-size $\tau = 1$. In Figure 4, for every $k = 0,\ldots,19$, $\tilde{u}_{h_k} = u_k^c \in \mathcal{S}^{1,cr}_D(\mathcal{T}_k)$ and a maximizer of $z_k^{rt} \in \mathcal{RT}^0(\mathcal{T}_k)$ of (5.6) obtained using the reconstruction formula (5.7), the square root of the trapezoidal primal-dual a posteriori error estimator (6.1) and of $\rho_I^2(u,\tilde{u}_{h_k}) = I(\tilde{u}_{h_k}) - I(u)$, cf. (5.25), where the exact value $I(u) \approx -0.0745503$ is approximated using Aitken's δ^2–process, cf. [2], are plotted versus the number of degrees of freedom N_k in a log log–plot. In it, one observes that mesh adaptivity yields the quasi-optimal convergence rate $h_k \sim N_k^{-\frac{1}{2}}$. In particular, for every $k = 0,\ldots,19$, the trapezoidal primal-dual a posteriori error estimator $\widehat{\eta}_k^2(u_k^c, z_k^{rt})$ defines a reliable upper bound for the error quantity $\rho_I^2(u,u_k^c)$. On the right-hand side of Figure 4, we displayed the energy curves for $I(u_k^c)$ and $D(z_k^{rt})$, $k = 0,\ldots,19$, resp., whose distance converges to zero as $N_k \to \infty$. In agreement with experimental results in [16], our error estimator avoids a systematic reliability-efficiency gap that arises in residual-type estimates.

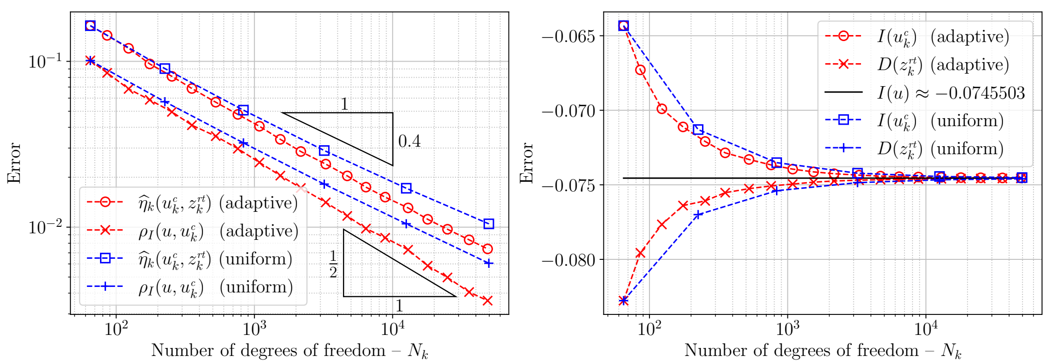

Figure 4: The trapezoidal primal-dual a posteriori error estimators $\widehat{\eta}_k^2(u_k^c, z_k^{rt})$, cf. (6.1), and the error quantities $\rho_I^2(u,u_k^c)$, cf. (6.2), (left) and the primal energy $I(u_k^c)$, cf. (5.1), and the dual energy $D(z_k^{rt})$, cf. (5.2), (right) for uniform and adaptive mesh refinement for $k = 0,\ldots,5$ and $k = 1,\ldots,19$, resp.

A. APPENDIX

A.1 Convex analysis

For a (real) Banach space X equipped with the norm $\|\cdot\|_X : X \to \mathbb{R}_{\geq 0}$, we denote its (continuous) dual space by X^* equipped with the dual norm $\|\cdot\|_{X^*} : X^* \to \mathbb{R}_{\geq 0}$, defined by $\|x^*\|_{X^*} := \sup_{\|x\|_X \leq 1} \langle x^*, x\rangle_X$ for every $x^* \in X^*$, where $\langle\cdot,\cdot\rangle_X : X^* \times X \to \mathbb{R}$, defined by $\langle x^*, x\rangle_X := x^*(x)$ for every $x^* \in X^*$ and $x \in X$, denotes the duality pairing. A functional $F : X \to \mathbb{R}\cup\{+\infty\}$ is called sub-differentiable in $x \in X$, if $F(x) < \infty$ and if there exists $x^* \in X^*$, called sub-gradient, such that for every $y \in X$, it holds

$$\langle x^*, y - x\rangle_X \leq F(y) - F(x). \tag{A.1}$$

The sub-differential $\partial F : X \to 2^{X^*}$ of a functional $F : X \to \mathbb{R}\cup\{+\infty\}$ for every $x \in X$, is defined by $(\partial F)(x) := \{x^* \in X^* \mid \text{(A.1) holds for } x^*\}$ if $F(x) < \infty$ and $(\partial F)(x) := \emptyset$ else.

For a functional $F : X \to \mathbb{R} \cup \{\pm\infty\}$, we denote its (Fenchel) conjugate functional by $F^* : X^* \to \mathbb{R} \cup \{\pm\infty\}$, which for every $x^* \in X^*$ is defined by $F^*(x^*) := \sup_{x \in X} \langle x^*, x\rangle_X - F(x)$. If $F : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semi-continuous functional, then its conjugate $F^* : X^* \to \mathbb{R} \cup \{+\infty\}$ equally is proper, convex and lower semi-continuous functional, cf. [24, p. 17]. Moreover, for every $x^* \in X^*$ and $x \in X$ such that $F^*(x^*) + F(x)$ is well-defined, i.e., the critical case $\infty - \infty$ does not occur, the Fenchel–Young inequality

$$\langle x^*, x\rangle_X \le F^*(x^*) + F(x) \tag{A.2}$$

applies. In particular, for every $x^* \in X^*$ and $x \in X$, it holds

$$x^* \in (\partial F)(x) \quad \Leftrightarrow \quad \langle x^*, x\rangle_X = F^*(x^*) + F(x). \tag{A.3}$$

A.2 Estimates for node-averaging operator in terms of (shifted) N–functions

In this subsection, we want to prove several estimates for the node-averaging operator $\mathcal{J}_h^{av} : \mathcal{L}^k(\mathcal{T}_h) \to \mathcal{S}_D^k(\mathcal{T}_h)$ in terms of (shifted) N–functions. A convex function $\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be an N–function if and only if $\varphi(0) = 0$, $\varphi(t) > 0$ for all $t > 0$, $\lim_{t\to 0} \varphi(t)/t = 0$, and $\lim_{t\to\infty} \varphi(t)/t = \infty$. As a consequence, there exists a right-derivative $\varphi' : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$, which is non-decreasing and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for all $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$. In addition, an N–function $\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ satisfies the Δ_2–condition (in short, $\varphi \in \Delta_2$), if and only if there exists a constant $c > 0$ such that $\varphi(2t) \le c\,\varphi(t)$ for all $t \ge 0$. We denote the smallest such constant by $\Delta_2(\varphi) > 0$. We say that an N–function $\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ satisfies the ∇_2–condition (in short, $\varphi \in \nabla_2$), if its Fenchel conjugate $\varphi^* : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is an N–function satisfying the Δ_2–condition. If $\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ satisfies the Δ_2– and the ∇_2–condition (in short, $\varphi \in \Delta_2 \cap \nabla_2$), then for $a \ge 0$, we define $\varphi_a' : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ by $\varphi_a'(t) := \varphi'(a+t)\frac{t}{a+t}$ for all $t \ge 0$. Furthermore, for $a \ge 0$, we define $\varphi_a : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, called shifted N–functions, by $\varphi_a(t) := \int_0^t \varphi_a'(s)\,\mathrm{d}s$ for all $t \ge 0$. It holds $c_\varphi := \sup_{a \ge 0} \Delta_2(\varphi_a) < \infty$, cf. [21, Lemma 22]. In particular, for every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$, not depending on $a \ge 0$, such that for every $t, s \ge 0$ and $a \ge 0$, it holds

$$s\,t \le c_\varepsilon\,(\varphi_a)^*(s) + \varepsilon\,\varphi_a(t). \tag{A.4}$$

Proposition A.1. *Let $\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be an N–function such that $\varphi \in \Delta_2 \cap \nabla_2$. Then, for every $v_h \in \mathcal{L}^k(\mathcal{T}_h)^l$, $k, l \in \mathbb{N}$, $m \in \{1, \ldots, k+1\}$, $a \ge 0$ and $T \in \mathcal{T}_h$, we have that*

$$\fint_T \varphi_a\big(h_T^m |\nabla_h^m (v_h - \mathcal{J}_h^{av} v_h)|\big)\,\mathrm{d}x \le c_{av} \sum_{S \in \mathcal{S}_h(T)} \fint_S \varphi_a(|[\![v_h]\!]_S|)\,\mathrm{d}s,$$

where $c_{av} > 0$ only depends on $k, l \in \mathbb{N}$, $c_\varphi > 0$ and a constant $c_{\mathcal{T}} > 0$ that depends on geometry of the triangulations $\mathcal{T}_h$, $h > 0$, but not on their maximal, minimal or mean mesh-sizes.

Proof. Appealing to [25, Lemma 22.12], there exists a constant $\overline{c}_{av} > 0$, such that for every $v_h \in \mathcal{L}^k(\mathcal{T}_h)^l$ and $T \in \mathcal{T}_h$, we have that

$$h_T^m \|\nabla_h^m (v_h - \mathcal{J}_h^{av} v_h)\|_{L^\infty(T;\mathbb{R}^{l\times d^m})} \le \overline{c}_{av} \sum_{S \in \mathcal{S}_h(T)} \|[\![v_h]\!]_S\|_{L^\infty(S;\mathbb{R}^l)}. \tag{A.5}$$

Hence, since $\|[\![v_h]\!]_S\|_{L^\infty(S;\mathbb{R}^l)} \le c_{\mathcal{T}} \fint_S |[\![v_h]\!]_S|\,\mathrm{d}s$ (cf. [25, Lemma 12.1]), also using the Δ_2–condition and convexity of $\varphi_a : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, $a \ge 0$, in particular, Jensen's inequality, and that $\sup_{h>0} \sup_{T \in \mathcal{T}_h} \mathrm{card}(\mathcal{S}_h(T)) \le c_{\mathcal{T}}$ in (A.5), for every $T \in \mathcal{T}_h$, we find that

$$\begin{aligned}\fint_T \varphi_a\big(h_T^m |\nabla_h^m (v_h - \mathcal{J}_h^{av} v_h)|\big)\,\mathrm{d}x &\le \Delta_2(\varphi_a)^{\lceil \overline{c}_{av} c_{\mathcal{T}}^2 \rceil} \varphi_a\bigg(\frac{1}{\mathrm{card}(\mathcal{S}_h(T))} \sum_{S \in \mathcal{S}_h(T)} \fint_S |[\![v_h]\!]_S|\,\mathrm{d}s\bigg)\\ &\le c_\varphi^{\lceil \overline{c}_{av} c_{\mathcal{T}}^2 \rceil} \sum_{S \in \mathcal{S}_h(T)} \fint_S \varphi_a(|[\![v_h]\!]_S|)\,\mathrm{d}s. \end{aligned}$$

□

Corollary A.2. *Let $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an N–function such that $\varphi \in \Delta_2 \cap \nabla_2$. Then, for every $v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$, $m \in \{0,1,2\}$, $a \geq 0$ and $T \in \mathcal{T}_h$, we have that*

$$\begin{aligned}\fint_T \varphi_a(h_T^m |\nabla_h^m (v_h - \mathcal{J}_h^{av} v_h)|)\,\mathrm{d}x &\leq c_{av} \sum_{S \in \mathcal{S}_h(T)} \fint_S \varphi_a(h_S |[\![\nabla_h v_h]\!]_S|)\,\mathrm{d}s \\ &\leq \tilde{c}_{av} \fint_{\omega_T} \varphi_a(h_T |\nabla_h v_h|)\,\mathrm{d}x.\end{aligned}$$

where $\tilde{c}_{av} > 0$ only depends on $c_\varphi > 0$ and a constant $c_{\mathcal{T}} > 0$ that depends on geometry of the triangulations $\mathcal{T}_h$, $h > 0$, but not on their maximal, minimal or mean mesh-sizes.

Proof. Follows from Proposition A.1, if we exploit that $[\![v_h]\!]_S = [\![\nabla_h v_h]\!]_S \cdot (\mathrm{id}_{\mathbb{R}^d} - x_S)$ on S for all $S \in \mathcal{S}_h$ and $v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ and the discrete trace inequality [25, Lemma 12.8]. ☐

References

[1] M. Ainsworth and J. T. Oden, *A posteriori error estimation in finite element analysis*, *Pure and Applied Mathematics*, Wiley-Interscience, New York, 2000. doi:10.1002/9781118032824.

[2] A. C. Aitken, On Bernoulli's numerical solution of algebraic equations, *Proceedings of the Royal Society of Edinburgh* (1926), 280–305. doi:10.1017/S0370164600022070.

[3] I. Anjam and D. Pauly, An elementary method of deriving a posteriori error equalities and estimates for linear partial differential equations, *Comput. Methods Appl. Math.* **19** (2019), 311–322. doi:10.1515/cmam-2017-0060.

[4] S. Balay et al., PETSc Web page, `https://www.mcs.anl.gov/petsc`, 2019.

[5] J. W. Barrett and W. B. Liu, Finite element approximation of the p-Laplacian, *Math. Comp.* **61** (1993), 523–537.doi:10.2307/2153239.

[6] S. Bartels, *Numerical methods for nonlinear partial differential equations*, *Springer Series in Computational Mathematics* **47**, Springer, Cham, 2015. doi:10.1007/978-3-319-13797-1.

[7] S. Bartels, Nonconforming discretizations of convex minimization problems and precise relations to mixed methods, *Comput. Math. Appl.* **93** (2021), 214–229. doi:10.1016/j.camwa.2021.04.014.

[8] S. Bartels and A. Kaltenbach, *Error estimates for total-variation regularized minimization problems with singular dual solutions*, 2022. Available at `http://arxiv.org/abs/1905.13591`.

[9] S. Bartels and M. Milicevic, Primal-dual gap estimators for *a posteriori* error analysis of nonsmooth minimization problems, *ESAIM Math. Model. Numer. Anal.* **54** (2020), 1635–1660. doi:10.1051/m2an/2019074.

[10] S. Bartels and Z. Wang, Orthogonality relations of Crouzeix-Raviart and Raviart-Thomas finite element spaces, *Numer. Math.* **148** (2021), 127–139. doi:10.1007/s00211-021-01199-3.

[11] S. Bartels and C. Carstensen, A convergent adaptive finite element method for an optimal design problem, *Numer. Math.* **108** (2008), 359–385.doi:10.1007/s00211-007-0122-x.

[12] L. Belenki, L. Diening, and C. Kreuzer, Optimality of an adaptive finite element method for the p-Laplacian equation, *IMA J. Numer. Anal.* **32** (2012), 484–510. doi:10.1093/imanum/drr016.

[13] D. Braess and J. Schöberl, Equilibrated residual error estimator for edge elements, *Math. Comp.* **77** (2008), 651–672.doi:10.1090/S0025-5718-07-02080-7.

[14] D. Braess, *Finite elements*, third ed., Cambridge University Press, Cambridge, 2007, Theory, fast solvers, and applications in elasticity theory, Translated from the German by Larry L. Schumaker. doi:10.1017/CBO9780511618635.

[15] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, third ed., *Texts in Applied Mathematics* **15**, Springer, New York, 2008. doi:10.1007/978-0-387-75934-0.

[16] C. Carstensen and D. J. Liu, Nonconforming FEMs for an optimal design problem, *SIAM J. Numer. Anal.* **53** (2015), 874–894. doi:10.1137/130927103.

[17] J. Cascon, C. Kreuzer, R. Nochetto, and K. Siebert, Quasi-optimal convergence rate for an adaptive finite element method, *SIAM J. Numer. Anal.* **46** (2008), 2524–2550. doi:10.1137/07069047X.

[18] A. Chambolle and T. Pock, Crouzeix-Raviart approximation of the total variation on simplicial meshes, *J. Math. Imaging Vision* **62** (2020), 872–899. doi:10.1007/s10851-019-00939-3.
[19] A. Cherkaev, *Variational methods for structural optimization*, *Applied Mathematical Sciences* **140**, Springer-Verlag, New York, 2000. doi:10.1007/978-1-4612-1188-4.
[20] M. Crouzeix and P.-A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I, *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge* **7** (1973), 33–75.
[21] L. Diening and C. Kreuzer, Linear convergence of an adaptive finite element method for the p-Laplacian equation, *SIAM J. Numer. Anal.* **46** (2008), 614–638.doi:10.1137/070681508.
[22] W. Dörfler, A convergent adaptive algorithm for Poisson's equation, *SIAM J. Numer. Anal.* **33** (1996), 1106–1124. doi:10.1137/0733054.
[23] C. Ebmeyer and W. Liu, Quasi-norm interpolation error estimates for the piecewise linear finite element approximation of p-Laplacian problems, *Numer. Math.* **100** (2005), 233–258. doi:10.1007/s00211-005-0594-5.
[24] I. Ekeland and R. Témam, *Convex analysis and variational problems*, english ed., *Classics in Applied Mathematics* **28**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999, Translated from the French. doi:10.1137/1.9781611971088.
[25] A. Ern and J. L. Guermond, *Finite Elements I: Approximation and Interpolation*, *Texts in Applied Mathematics*, Springer International Publishing, 2021. doi:10.1007/978-3-030-56341-7.
[26] A. Ern and M. Vohralík, Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, *SIAM J. Numer. Anal.* **53** (2015), 1058–1081. doi:10.1137/130950100.
[27] T. Gudi, A new error analysis for discontinuous finite element methods for linear elliptic problems, *Math. Comp.* **79** (2010), 2169–2189. doi:10.1090/S0025-5718-10-02360-4.
[28] T. Gudi, Some nonstandard error analysis of discontinuous Galerkin methods for elliptic problems, *Calcolo* **47** (2010), 239–261. doi:10.1007/s10092-010-0022-9.
[29] A. Haberl, D. Praetorius, S. Schimanko, and M. Vohralík, Convergence and quasi-optimal cost of adaptive algorithms for nonlinear operators including iterative linearization and algebraic solver, *Numer. Math.* **147** (2021), 679–725. doi:10.1007/s00211-021-01176-w.
[30] W. Han, *A posteriori error analysis via duality theory*, *Advances in Mechanics and Mathematics* **8**, Springer-Verlag, New York, 2005.
[31] J. D. Hunter, Matplotlib: A 2d graphics environment, *Computing in Science & Engineering* **9** (2007), 90–95. doi:10.1109/MCSE.2007.55.
[32] D. J. Liu and Z. R. Chen, The adaptive finite element method for the P-Laplace problem, *Appl. Numer. Math.* **152** (2020), 323–337. doi:10.1016/j.apnum.2019.11.018.
[33] D. J. Liu, A. Q. Li, and Z. R. Chen, Nonconforming FEMs for the p-Laplace problem, *Adv. Appl. Math. Mech.* **10** (2018), 1365–1383.doi:10.4208/aamm.
[34] W. Liu and N. Yan, Quasi-norm a priori and a posteriori error estimates for the nonconforming approximation of p-Laplacian, *Numer. Math.* **89** (2001), 341–378. doi:10.1007/PL00005470.
[35] W. Liu and N. Yan, Quasi-norm local error estimators for p-Laplacian, *SIAM J. Numer. Anal.* **39** (2001), 100–127. doi:10.1137/S0036142999351613.
[36] W. Liu and N. Yan, On quasi-norm interpolation error estimation and a posteriori error estimates for p-Laplacian, *SIAM J. Numer. Anal.* **40** (2002), 1870–1895. doi:10.1137/S0036142901393589.
[37] A. Logg and G. N. Wells, Dolfin: Automated finite element computing, *ACM Transactions on Mathematical Software* **37** (2010). doi:10.1145/1731022.1731030.
[38] R. Luce and B. I. Wohlmuth, A local a posteriori error estimator based on equilibrated fluxes, *SIAM J. Numer. Anal.* **42** (2004), 1394–1414.doi:10.1137/S0036142903433790.
[39] L. D. Marini, An inexpensive method for the evaluation of the solution of the lowest order Raviart-Thomas mixed method, *SIAM J. Numer. Anal.* **22** (1985), 493–496. doi:10.1137/0722029.
[40] W. Prager and J. L. Synge, Approximations in elasticity based on the concept of function space, *Quart. Appl. Math.* **5** (1947), 241–269. doi:10.1090/qam/25902.
[41] P.-A. Raviart and J. M. Thomas, A mixed finite element method for 2nd order elliptic problems, in *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz.*

delle Ricerche (C.N.R.), Rome, 1975), 1977, pp. 292–315. Lecture Notes in Math., Vol. 606.

[42] S. I. REPIN, *A posteriori estimates for partial differential equations*, *Radon Series on Computational and Applied Mathematics* **4**, Walter de Gruyter GmbH & Co. KG, Berlin, 2008. doi:10.1515/9783110203042.

[43] S. I. REPIN, A posteriori error estimates for approximate solutions to variational problems with strongly convex functionals, *J. Math. Sci. (New York)* **97** (1999), 4311–4328, Problems of mathematical physics and function theory. doi:10.1007/BF02365047.

[44] S. I. REPIN and L. S. XANTHIS, A posteriori error estimation for nonlinear variational problems, *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), 1169–1174. doi:10.1016/S0764-4442(97)87906-2.

[45] S. I. REPIN, A posteriori error estimation for variational problems with uniformly convex functionals, *Math. Comp.* **69** (2000), 481–500. doi:10.1090/S0025-5718-99-01190-4.

[46] I. SMEARS and M. VOHRALÍK, Simple and robust equilibrated flux *a posteriori* estimates for singularly perturbed reaction-diffusion problems, *ESAIM Math. Model. Numer. Anal.* **54** (2020), 1951–1973. doi:10.1051/m2an/2020034.

[47] R. STEVENSON, Optimality of a standard adaptive finite element method, *Foundations of Computational Mathematics* **7** (2007), 245–269. doi:10.1007/s10208-005-0183-0.

[48] A. VEESER, Convergent adaptive finite elements for the nonlinear Laplacian, *Numer. Math.* **92** (2002), 743–770. doi:10.1007/s002110100377.