CONVERGENCE OF FULLY DISCRETE IMPLICIT AND SEMI-IMPLICIT APPROXIMATIONS OF NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. The article addresses the convergence of implicit and semi-implicit, fully discrete approximations of a class of nonlinear parabolic evolution problems. Such schemes are popular in the numerical solution of evolutions defined with the p-Laplace operator since the latter lead to linear systems of equations in the time steps. The semi-implicit treatment of the operator requires introducing a regularization parameter that has to be suitably related to other discretization parameters. To avoid restrictive, unpractical conditions, a careful convergence analysis has to be carried out. The arguments presented in this article show that convergence holds under a moderate condition that relates the step size to the regularization parameter but which is independent of the spatial resolution.

1. INTRODUCTION

It has recently been shown in the article [BDN18] that the semi-implicit time stepping scheme for the p-Laplace gradient flow defined with an initial function u^0 via the recursion

$$d_\tau u^k = \operatorname{div} \frac{\nabla u^k}{|\nabla u^{k-1}|_\varepsilon^{2-p}} \tag{1.1}$$

with the regularized norm $|a|_\varepsilon = (|a|^2 + \varepsilon^2)^{1/2}$ and the backward difference quotient operator $d_\tau = (u^k - u^{k-1})/\tau$ is unconditionally energy stable. Specifically, this means that the estimate

$$E_{p,\varepsilon}[u^L] + \tau \sum_{k=1}^{L} \|d_\tau u^k\|_{L^2(\Omega)}^2 + \frac{\tau^2}{2} \sum_{k=1}^{L} \int_\Omega \frac{|\nabla d_\tau u^k|^2}{|\nabla u^{k-1}|_\varepsilon^{2-p}} \,\mathrm{d}x \le E_{p,\varepsilon}[u^0]$$

holds for all $\tau, \varepsilon > 0$ and $1 \le p \le 2$ and all $L \ge 1$ with the regularized p-Dirichlet energy

$$E_{p,\varepsilon}[u] = \frac{1}{p} \int_\Omega |\nabla u|_\varepsilon^p \,\mathrm{d}x.$$

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The energy estimate follows from testing (1.1) with $d_\tau u^k$ using special identities from finite difference calculus and certain monotonicity properties of the p-Laplace operator. An error analysis for a generic spatial discretization with mesh-size $h > 0$ of the scheme leads to an upper bound for the approximation error in $L^\infty(0,T;L^2(\Omega))$ involving the term

$$\tau^{1/2}(h\varepsilon)^{(p-2)/2}.$$

To deduce a convergence rate for the error the restrictive condition $\tau = o((h\varepsilon)^{2-p})$ has to be satisfied. The aim of this note is to show that the sequence of piecewise constant interpolants of the iterates $(u_h^k)_{k=0,\dots,K}$, $h > 0$, (weakly) converges to the solution of the continuous flow under the less restrictive condition $\tau = O(\varepsilon^{2-p})$ independently of the mesh-size $h > 0$ and even for a larger class of operators also including lower order contributions. To explain our ideas we interpret the iterates $(u^k)_{k=0,\dots,K}$ of the semi-implicit scheme as iterates of an implicit, unregularized scheme with discrepancy terms on the right-hand sides, i.e., with the L^2 inner product $(\cdot,\cdot)$ we have

$$(d_\tau u^k, v) + \big(|\nabla u^k|^{p-2}\nabla u^k, \nabla v\big) = (\mathcal{D}^k, \nabla v). \tag{1.2}$$

Using the operator

$$S_\varepsilon(a) = \frac{a}{|a|_\varepsilon^{2-p}}$$

we rewrite the discrepancy terms as

$$\begin{aligned}\mathcal{D}^k &= \big[|\nabla u^k|^{p-2} - |\nabla u^{k-1}|_\varepsilon^{p-2}\big]\nabla u^k \\ &= \big(S_0(\nabla u^k) - S_\varepsilon(\nabla u^k)\big) + \big(S_\varepsilon(\nabla u^k) - |\nabla u^{k-1}|_\varepsilon^{p-2}\nabla u^k\big) = E^k + F^k.\end{aligned}$$

The first term on the right-hand side is controlled using the uniform convergence property

$$|S_\varepsilon(a) - S_0(a)| \le (2-p)\varepsilon^{p-1}$$

which follows from the mean value estimate $\big||a|^{2-p} - |a|_\varepsilon^{2-p}\big| \le (2-p)|a|^{1-p}\varepsilon$ for $a \neq 0$. Therefore, we have

$$(E^k, \nabla v) \le (2-p)\varepsilon^{p-1}\|\nabla v\|_{L^1(\Omega)}.$$

To bound the second term on the right-hand side we use the estimate

$$|S_\varepsilon(a) - S_\varepsilon(b)| \le c_p|a-b|\big(\varepsilon^2 + |a|^2 + |b|^2\big)^{(p-2)/2},$$

cf. [DER07], which leads to

$$\begin{aligned}(F^k, \nabla v) &= \int_\Omega \big(S_\varepsilon(\nabla u^k) - S_\varepsilon(\nabla u^{k-1}) + |\nabla u^{k-1}|_\varepsilon^{p-2}\nabla[u^{k-1} - u^k]\big)\cdot\nabla v \,\mathrm{d}x \\ &\le \frac{(c_p+1)^2\tau\alpha_\varepsilon}{2}\int_\Omega \frac{|\nabla d_\tau u^k|^2}{|\nabla u^{k-1}|_\varepsilon^{2-p}}\,\mathrm{d}x + \frac{\tau\varepsilon^{p-2}}{2\alpha_\varepsilon}\int_\Omega |\nabla v|^2\,\mathrm{d}x.\end{aligned}$$

Letting $\overline{\mathcal{D}}$ be the piecewise constant interpolation of $\mathcal{D}^k$ and integrating the estimate for $\mathcal{D}^k$ over $(0,T)$ we thus obtain with the energy bound that

$$\int_0^T (\overline{\mathcal{D}}, \nabla v)\,\mathrm{d}t \le (2-p)\varepsilon^{p-1}\int_0^T \|\nabla v\|_{L^1(\Omega)}\,\mathrm{d}t + c_p^2\alpha_\varepsilon\tau^2\sum_{k=1}^K\int_\Omega \frac{|\nabla d_\tau u^k|^2}{|\nabla u^{k-1}|_\varepsilon^{2-p}}\,\mathrm{d}x + \frac{\tau\varepsilon^{p-2}}{2\alpha_\varepsilon}\int_0^T \|\nabla v\|_{L^2(\Omega)}^2\,\mathrm{d}t,$$

where $\alpha_\varepsilon > 0$ is arbitrary. Choosing, e.g., $\alpha_\varepsilon = (\tau\varepsilon^{p-2})^{1/2}$, and requiring $\tau = o(\varepsilon^{2-p})$ we find that the discrepancy term converges to zero whenever $v \in L^2(0,T;W_0^{1,2}(\Omega))$. If an implicit discretization of the p-Laplace gradient flow is known to converge to the exact solution then it follows that the iterates of the semi-implicit scheme (1.1) also converge to this object.

Surprisingly, a rigorous convergence analysis for the fully discrete, implicit scheme for the p-Laplace evolution does not seem to be available in the literature. Classical references such as [GGZ74], [Zei90b], [Sho97], and [Rou05] consider semi-discrete schemes, i.e., either Galerkin methods corresponding to a spatial discretization or Rothe methods realizing implicit time stepping schemes. Full discretizations lead to additional analytical difficulties as, e.g., the schemes only provide limited control on the time derivatives. To avoid the construction of a stable projection operator a generalized Aubin–Lions lemma has been established in [Rou05]. An alternative to this is based on the Hirano–Landes lemma, which ensures the convergence in the nonlinear operator provided an energy estimate can be established and a generalized condition (M) can be verified based on the approximate equations and the properties of the nonlinear operator (cf. [BR17] for previous versions of this approach). Another approach to establishing convergence of solutions can be based on the framework of subdifferential flows but this limits the analysis to convex energies and excludes other nonlinearities. In order to verify the generalized condition (M) we require in addition to the energy estimate stated above also bounds resulting from testing the scheme (1.1) by u^k.

Various error estimates are available for numerical approximations of p-Laplace evolutions and related equations, see, e.g., [BL94, Rul96, NSV00, FvOP05, DER07]. These are typically valid under certain regularity conditions, impose relations between discretization parameters, or consider only implicit time-stepping schemes. Here, we are interested in establishing convergence of the approximations obtained with the practical semi-implicit scheme (1.1) under moderate conditions on the relation between the step-size parameter τ and the regularization parameter ε. Therefore, we cannot resort to those results when we affiliate the convergence to the convergence of an implicit scheme with discrepancy terms.

To establish the convergence of the iterates $(u^k)_{k=0,\dots,K}$ of the semi-implicit scheme (1.1), even when a spatial discretization is carried out, we first consider the corresponding implicit scheme and prove that appropriate interpolants weakly accumulate at an exact solution. This result is the consequence of a general convergence result for a fully discrete implicit approximation proved in an abstract framework for evolution equations with pseudo-monotone operators. Typical examples of such operators are sums of a monotone and a compact operator. Only moderate assumptions will be made on the data and on the discretizations. A technical condition on the finite element spaces requires sequences of finite element spaces to be nested as the mesh-size tends to zero.
The outline of this article is as follows. In Subsection 1.1 we define a class of energy densities that lead to admissible operators to which our arguments apply. In Section 2 we derive a convergence result for approximations obtained with a fully discrete implicit scheme for general evolution equations with pseudo-monotone operators. This serves as a guideline to show that the approximations obtained with a semi-implicit, practical scheme generalizing (1.1) for a large class of monotone evolutions including lower order contributions converges to a solution.
Throughout this article we let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and use standard notation for Lebesgue and Sobolev spaces. Most results apply to bounded open sets Ω but in view of numerical discretizations we consider the slightly stronger condition. We denote the inner product in $L^2(\Omega)$ by $(\cdot,\cdot)$ and the duality pairing of a Banach space V with its dual V' which often extends the L^2 inner product by $\langle \cdot,\cdot\rangle_V$.

1.1. **Properties of the nonlinear operator.** For a given convex function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we consider energy functionals $E_\varphi : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined via

$$E_\varphi[u] = \int_\Omega \varphi(|\nabla u|)\,\mathrm{d}x.$$

We denote by $W^{1,\varphi}(\Omega)$ the set of weakly differentiable functions $u \in L^1(\Omega)$ for which we have $E_\varphi[u] < \infty$. We make the following assumptions on the energy density φ which define a class of sub-quadratic Orlicz functions.

Assumption 1.3 (Energy density). *Let $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belong to $C^0(\mathbb{R}_{\geq 0}) \cap C^1(\mathbb{R}_{>0})$. We assume that*

(C1) *$r \mapsto \varphi(r)$ is convex with $\varphi(0) = 0$.*
(C2) *$r \mapsto \varphi'(r)/r$ is positive and nonincreasing.*

Sometimes we additionally make the following assumption.

Assumption 1.4 (N-function). *Let $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belong to $C^1(\mathbb{R}_{\geq 0}) \cap C^2(\mathbb{R}_{>0})$. We assume that*

(C3) *The function φ is convex and positive on $(0,\infty)$, satisfies $\varphi(0) = 0$, and $\lim_{s\to 0} \varphi(s)/s = 0$ and $\lim_{s\to\infty} \varphi(s)/s = \infty$; moreover φ and its convex conjugate φ^* satisfy $\varphi(2s) \lesssim \varphi(s)$ and $\varphi^*(2r) \lesssim \varphi(r)$ for all*

$r,s \in \mathbb{R}_{\geq 0}$. Finally we assume that there exist constants $\kappa_0 \in (0,1]$, $\kappa_1 > 0$ such that for all $r \in \mathbb{R}_{>0}$

$$\kappa_0 \varphi'(r) \leq r\varphi''(r) \leq \kappa_1 \varphi'(r)\,.$$

For a given N-function φ we define the shifted N-functions $\{\varphi_\alpha\}_{\alpha\geq 0}$, cf. [DE08, DK08, RD07], for $t \geq 0$ by

$$\varphi_\alpha(t) := \int_0^t \varphi'_\alpha(s)\,ds \qquad \text{with} \qquad \varphi'_\alpha(t) := \varphi'(\alpha+t)\frac{t}{\alpha+t}. \tag{1.5}$$

If φ satisfies the conditions (C1), (C2) and (C3), than the family of shifted N-functions $\{\varphi_\alpha\}_{\alpha\geq 0}$ also satisfies conditions (C1), (C2) and (C3). The family of shifted N-functions $\{\varphi_\alpha\}_{\alpha\geq 0}$ induces operators $A_\alpha\colon \mathbb{R}^d \to \mathbb{R}^d$ with potential φ_α via

$$A_\alpha(a) := \frac{\varphi'_\alpha(|a|)}{|a|}a\,. \tag{1.6}$$

One easily checks the following relations (cf. [DE08, DK08, RD07]).

Lemma 1.7. *If φ satisfies (C3), then the following statements are valid:*
(i) For all $a, b \in \mathbb{R}^d$ and all $\alpha \geq 0$ we have with constants independent of α

$$\big(A_\alpha(a) - A_\alpha(b)\big)\cdot(a-b) \eqsim (\varphi_\alpha)_{|a|}(|a-b|), \tag{1.8}$$

$$\big|A_\alpha(a) - A_\alpha(b)\big| \eqsim (\varphi_\alpha)'_{|a|}(|a-b|), \tag{1.9}$$

and

$$(\varphi_\alpha)_{|a|}(|a-b|) \eqsim \frac{\varphi'_\alpha(|a|+|b|)}{|a|+|b|}|a-b|^2. \tag{1.10}$$

(ii) For all $\delta > 0$ there exists c_δ such that for all $\alpha, r, s \geq 0$ we have

$$\varphi'_\alpha(r)s \leq c_\delta \varphi_\alpha(r) + \delta\varphi_\alpha(s). \tag{1.11}$$

(iii) For all δ there exists c_δ such that for all $a, b \in \mathbb{R}^d$, and all $r \geq 0$

$$\begin{aligned} \varphi_{|a|}(r) &\leq c_\delta\, \varphi_{|b|}(r) + \delta\, \varphi_{|a|}(|a-b|),\\ (\varphi_{|a|})^*(r) &\leq c_\delta\, (\varphi_{|b|})^*(r) + \delta\, \varphi_{|a|}(|a-b|). \end{aligned} \tag{1.12}$$

Moreover, we have $\varphi_{|a|}(|a-b|) \eqsim \varphi_{|b|}(|a-b|)$

We need some further properties related to the function φ. In the same way as in [BDN18] one can prove the following inequality.

Lemma 1.13. *Under condition $(C2)$ we have for all $a, b \in \mathbb{R}^d$ and all $\varepsilon \geq 0$ that*

$$\frac{\varphi'_\varepsilon(|a|)}{|a|}b\cdot(b-a) \geq \varphi_\varepsilon(|b|) - \varphi_\varepsilon(|a|) + \frac{1}{2}\frac{\varphi'_\varepsilon(|a|)}{|a|}|b-a|^2.$$

To handle the difference between the implicit scheme and the semi-implicit scheme, the following estimate is useful.

Lemma 1.14. *If φ satisfies (C2), (C3), then we have for all $a, b \in \mathbb{R}^d$, $\varepsilon \geq 0$*

$$\Big|\Big(\frac{\varphi_\varepsilon'(|a|)}{|a|} - \frac{\varphi_\varepsilon'(|b|)}{|b|}\Big)a\Big| \lesssim \frac{\varphi_\varepsilon'(|b|)}{|b|}|a-b|\,.$$

Proof. We have

$$\begin{aligned}\Big|\Big(\frac{\varphi_\varepsilon'(|a|)}{|a|} - \frac{\varphi_\varepsilon'(|b|)}{|b|}\Big)a\Big| &= \Big|A_\varepsilon(a) - A_\varepsilon(b) + \frac{\varphi_\varepsilon'(|b|)}{|b|}(b-a)\Big| \\ &\lesssim (\varphi_\varepsilon')_{|a|}(|a-b|) + \frac{\varphi_\varepsilon'(|b|)}{|b|}|b-a| \\ &\lesssim \frac{\varphi_\varepsilon'(|b|+|b-a|)}{|b|+|b-a|}|b-a| + \frac{\varphi_\varepsilon'(|b|)}{|b|}|b-a| \\ &\leq 2\,\frac{\varphi_\varepsilon'(|b|)}{|b|}|b-a|\,,\end{aligned}$$

where we used that $|b| + |b-a| \eqsim |b| + |a|$ and condition (C2). □

We have a uniform convergence property for the operators A_ε.

Lemma 1.15. *If φ satisfies (C2), (C3), then we have for all $a \in \mathbb{R}^d$, $\varepsilon \geq 0$*

$$\big|A_\varepsilon(a) - A(a)\big| \leq (1-\kappa_0)\varphi'(\varepsilon)\,.$$

Proof. For $a = 0$ or $\varepsilon = 0$ the estimate is clear. Thus, we assume in the following $|a| > 0$ and $\varepsilon > 0$. Setting $f(t) := \frac{t}{\varphi'(t)}$, $t > 0$, we see from (C2) that f is nondecreasing. Moreover, from (C3) we obtain that $0 \leq f'(s) = \frac{1}{\varphi'(s)}\Big(1 - \frac{s\varphi''(s)}{\varphi'(s)}\Big) \leq \frac{1-\kappa_0}{\varphi'(s)}$. From the mean value theorem we get for all $t > 0$, $\varepsilon > 0$

$$\big|f(t+\varepsilon) - f(t)\big| = \varepsilon\, f'(\zeta) \leq \varepsilon\,\frac{1-\kappa_0}{\varphi'(\zeta)} \leq \varepsilon\,\frac{1-\kappa_0}{\varphi'(t)},$$

where we used that $\zeta \in (t, t+\varepsilon)$ and that φ' is increasing. Thus we get

$$\begin{aligned}\big|A_\varepsilon(a) - A(a)\big| &= \bigg|\frac{\varphi'(\varepsilon+|a|)}{\varepsilon+|a|} - \frac{\varphi'(|a|)}{|a|}\bigg|\,|a| \\ &= \bigg|\frac{f(|a|) - f(\varepsilon+|a|)}{f(|a|)\,f(\varepsilon+|a|)}\bigg|\,|a| \\ &\leq \varepsilon\,\frac{1-\kappa_0}{\varphi'(|a|)}\,\frac{\varphi'(|a|)\,\varphi'(\varepsilon+|a|)}{|a|\,(\varepsilon+|a|)}\,|a| \\ &\leq \varepsilon\,(1-\kappa_0)\frac{\varphi'(\varepsilon+|a|)}{\varepsilon+|a|} \leq (1-\kappa_0)\varphi'(\varepsilon)\,,\end{aligned}$$

where we used also (C2). □

Prototypical examples for functions φ satisfying the conditions (C1), (C2) and (C3) are N-functions with (p,δ)-structure. We say that an N-function

$\varphi \in C^1(\mathbb{R}_{\geq 0}) \cap C^2(\mathbb{R}_{>0})$ has (p,δ)-structure, with $p \in (1,\infty)$ and $\delta \geq 0$, if

$$\begin{aligned} \varphi(t) &\eqsim (\delta + t)^{p-2} t^2\,, && \text{uniformly in } t \geq 0\,, \\ \varphi''(t) &\eqsim (\delta + t)^{p-2}\,, && \text{uniformly in } t > 0\,. \end{aligned} \tag{1.16}$$

The constants in these equivalences and p are called characteristics of φ. A detailed discussion of N-functions with (p,δ)-structure can e.g. be found in [Růž13]. Using (1.16) and the change of shift (1.12) we easily see that for all $\varepsilon, \delta \geq 0$ we have uniformly in $t \geq 0$

$$\varphi_\varepsilon(t) + \varepsilon^p + \delta^p \eqsim t^p + \varepsilon^p + \delta^p \tag{1.17}$$

with constants only depending on p.

2. Convergence of an implicit scheme

In this section we study abstract evolution equations with pseudo-monotone operators. Concrete realizations of this situation will be discussed in the next section.

Let V be a Banach space. An operator $B\colon V \to V^*$ is said to be monotone if $\langle Bx - By, x - y\rangle_V \geq 0$ for all $x, y \in V$. The operator $B\colon V \to V^*$ is said to be pseudo-monotone if $x_n \rightharpoonup x$ in V and $\limsup_{n\to\infty} \langle Bx_n, x_n - x\rangle_V \leq 0$ implies

$$\langle Bx, x - y\rangle_V \leq \liminf_{n\to\infty} \langle Bx_n, x_n - y\rangle_V \qquad \text{for all } y \text{ in } V.$$

Let V be a separable, reflexive Banach space and H a Hilbert space. If the embedding $V \hookrightarrow H$ is dense, we call (V, H, V^*) a Gelfand-Triple. Using the Riesz representation theorem we obtain $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ where both embeddings are dense. In this situation there holds $(x,y)_H = \langle x, y\rangle_V = \langle y, x\rangle_V$ for all $x, y \in V$. We say that a function $u \in L^p(0,T;V)$ possesses a generalized derivative in $L^{p'}(0,T;V^*)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, if there is a function $w \in L^{p'}(0,T;V^*)$ such that

$$\int_0^T (u(t), v)_H\, \phi'(t)\, dt = -\int_0^T \langle w(t), v\rangle_V\, \phi(t)\, dt$$

for all $v \in V$ and all $\phi \in C_0^\infty((0,T))$. If such a function w exists, it is unique and we set $\frac{du}{dt} := w$. We define the Bochner–Sobolev space

$$W_p^1(0,T;V,H) := \big\{ u \in L^p(0,T;V) \,\big|\, \frac{du}{dt} \in L^{p'}(0,T;V^*) \big\}.$$

With the norm

$$\|u\|_{W_p^1(0,T;V,H)} := \|u\|_{L^p(0,T;V)} + \left\| \frac{du}{dt} \right\|_{L^{p'}(0,T;V^*)}$$

this space is a reflexive Banach space. Moreover, we have that $W_p^1(0,T;V,H)$ embeds continuously into $C(0,T;H)$ and the following integration by parts formula

$$(u(t),v(t))_H-(u(s),v(s))_H=\int_s^t\left\langle\frac{du}{dt}(\tau),v(\tau)\right\rangle_V+\left\langle\frac{dv}{dt}(\tau),u(\tau)\right\rangle_V\,d\tau$$

holds for any $u,v\in W^1_p(0,T;V,H)$ and arbitrary $0\le s,t\le T$ (cf. [Zei90a, Proposition 23.23]).

We study the following evolution equation with a pseudo-monotone operator B:

$$\begin{aligned}\frac{du}{dt}(t)+Bu(t)&=f(t)\quad\text{in } V^*\text{ for a.e. } t\in[0,T],\\ u(0)&=u^0\qquad\text{in } H.\end{aligned}\tag{2.1}$$

To establish the existence of solutions we make the following assumptions on the operator B.

Assumption 2.2 (Operator)**.** *Let (V,H,V^*) be a Gelfand triple and let $B\colon V\to V^*$ be an operator with the following properties:*

(A1) *B is pseudo-monotone.*

(A2) *There exist constants $c_1>0$, $c_2\ge 0$, $c_3\ge 0$, such that for all $x\in V$*

$$\langle Bx,x\rangle_V\ge c_1\,\|x\|_V^p-c_2\,\|x\|_H^2-c_3\,.$$

(A3) *There exists $0\le q<\infty$, as well as constants $c_4>0$, $c_5\ge 0$ and $c_6\ge 0$, such that for all $x\in V$*

$$\|Bx\|_{V^*}\le c_4\,\|x\|_V^{p-1}+c_5\,\|x\|_H^q\,\|x\|_V^{p-1}+c_6\,.$$

Under this assumption we have (cf. [Zei90b, Chapters 27, 30]):

Lemma 2.3. *Assume that the operator $B\colon V\to V^*$ satisfies Assumption 2.2. Then the induced operator $(Bu)(t):=Bu(t)$ maps the space $L^p(0,T;V)\cap L^\infty(0,T;H)$ into $L^{p'}(0,T;V^*)$ and is bounded.*

Previous existence results that we are aware of are based on either a Rothe approximation (cf. [Rou05]) or a Galerkin approximation (cf. [BR17]). We want to establish the existence of a solution of (2.1) with the help of a convergence proof of a Rothe-Galerkin scheme. To this end we introduce some notation. For each $K\in\mathbb{N}$ we set $\tau:=\frac{T}{K}$, $t_k=t_k^\tau:=k\tau$, $k=0,\dots,K$ and $I_k=I_k^\tau:=(t_{k-1},t_k]$, $k=1,\dots,K$. The backward difference quotient operator is defined as

$$d_\tau c^k:=\tau^{-1}(c^k-c^{k-1}).$$

For a given finite sequence $(c^k)_{k=0,\dots,K}$ we denote by $\bar c^\tau$ the piecewise constant interpolant and by $\hat c^\tau$ the piecewise affine interpolant, i.e. $\hat c^\tau(t)=\big(\frac{t}{\tau}-(k-1)\big)c^k+\big(k-\frac{t}{\tau}\big)c^{k-1}$, $\bar c^\tau(t)=c^k$, $t\in I_k$, $\bar c^\tau(0)=\hat c^\tau(0)=c^0$. Note that $\frac{d\hat c^\tau(t)}{dt}=d_\tau c^k$ for all $t\in(t_{k-1},t_k)$.

Assumption 2.4 (Data)**.** *Let (V,H,V^*) be a Gelfand triple. We assume that $u^0\in H$ and $f\in L^{p'}(0,T;V^*)$. Moreover, we assume that there exists an increasing sequence of finite dimensional subspaces V_M, $M\in\mathbb{N}$, such* *that $\bigcup_{M\in\mathbb{N}} V_M$ is dense in V. Finally, we assume that there exist $u_M^0 \in V_M$ such that $u_M^0 \to u^0$ in H, and that there exists a sequence $f_M \in C(0,T;V^*)$ such that $f_M \to f$ in $L^{p'}(0,T;V^*)$.*

For each $M \in \mathbb{N}$ and given $u_M^0 \in V_M$ the sequence of iterates $(u_M^k)_{k=0,\dots,K} \subseteq V_M$ is given via the implicit scheme

$$\langle d_\tau u_M^k, v_M\rangle_V + \langle B u_M^k, v_M\rangle_V = \langle f_M(t_k), v_M\rangle_V \qquad \forall v_M \in V_M. \tag{2.5}$$

Theorem 2.6 (Convergence of the implicit scheme)**.** *Let Assumption 2.2 and 2.4 be satisfied. Let $\bar{u}_n := \bar{u}_{M_n}^{\tau_n}$ be a sequence of piecewise constant interpolants generated by iterates $(u_{M_n}^k)_{k=0,\dots,K_n}$, $K_n = \frac{T}{\tau_n}$, solving (2.5) for some sequences $M_n \to \infty$, $\tau_n \to 0$. Then each weak* accumulation point u of the sequence $(\bar{u}_n)_{n\in\mathbb{N}}$ in the space $L^\infty(0,T;H) \cap L^p(0,T;V)$ belongs to the space $W_p^1(0,T;V,H)$ and is a solution of (2.1).*

The proof of this theorem is based on a generalization of Hirano's lemma (cf. [Shi97], [Rou05]) using ideas from [LM87], [Lan86]. The advantage of this generalization is that it avoids a technical assumption on the existence of suitable projections (cf. [BR17]).

Proposition 2.7 (Hirano, Landes)**.** *Let Assumption 2.2 be satisfied. Further assume that the sequence (u_n) is bounded in $L^p(0,T;V) \cap L^\infty(0,T;H)$ and satisfies*

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } L^p(0,T;V),\\ u_n &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H),\\ u_n(t) &\rightharpoonup u(t) \quad \text{in } H \text{ for almost all } t\in(0,T),\\ \limsup_{n\to\infty} &\langle Bu_n, u_n - u\rangle_{L^p(0,T;V)} \le 0. \end{aligned} \tag{2.8}$$

Then for any $z \in L^p(0,T;V)$ there holds

$$\langle Bu, u - z\rangle_{L^p(0,T;V)} \le \liminf_{n\to\infty} \langle Bu_n, u_n - z\rangle_{L^p(0,T;V)}. \tag{2.9}$$

Moreover, $Bu_n \rightharpoonup Bu$ in $L^p(0,T;V)^* = L^{p'}(0,T;V^*)$.

Proof. The proof is almost identical with the proof of [BR17, Lemma 4.2]. First note that from assumptions (A2), (A3) we can derive for all $x \in L^p(0,T;V) \cap L^\infty(0,T;H)$ with $\|x\|_{L^\infty(0,T;H)} \le K$, all $y \in L^p(0,T;V)$ and almost all $t \in (0,T)$

$$\langle Bx(t), x(t) - y(t)\rangle_{L^p(0,T;V)} \ge k_1 \|x(t)\|_V^p - k_2 \|y(t)\|_V^p - k_3\,,$$

with positive constants k_i, $i = 1,2,3$, depending on K and c_j, $j = 1,\dots,6$. The last inequality is exactly inequality (4.4) in [BR17], which is crucial for the proof of Lemma 4.2 there. Note, that assumption $(2.8)_3$ is not present in the formulation of [BR17, Lemma 4.2], but it is assumed instead that (u_n) is bounded in $L^{p'}(0,T;Z^*)$, for a certain separable, reflexive Banach space Z with $Z \hookrightarrow V$. This assumption is solely used to identify the pointwise limits $u_n(t) \rightharpoonup u(t)$ in Z^* for all $t \in [0,T]$ (cf. [BR17, equation (4.5)]). This identification together with the embedding $V \hookrightarrow Z^*$ implies for a certain subsequence $u_{n_k}(t) \rightharpoonup u(t)$ in V for almost all $t \in [0,T]$ (cf. [BR17, equation (4.8)]). This argument is replaced by our assumption $(2.8)_3$, that also identifies the pointwise limits of $(u_n(t))$ in H. This and the embedding $V \hookrightarrow H$ again yield that for a certain subsequence $u_{n_k}(t) \rightharpoonup u(t)$ in V for almost all $t \in [0,T]$. After that the proof can be finished in an identical manner as in [BR17]. ☐

We will also use a slight modification of the following compactness result of Landes, Mustonen [LM87], which is an alternative to the Aubin-Lions lemma in the case of Sobolev spaces.

Proposition 2.10. *Let $p, s \in (1,\infty)$, $q \in [1,p^*)$, where $p^* := \frac{dp}{d-p}$ if $p < d$, and $p^* := \infty$ if $p \geq d$. Let (u_n) be a bounded sequence in $L^\infty(0,T;L^1(\Omega))$ such that*

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } L^s(0,T;W_0^{1,p}(\Omega)), \\ u_n(t) &\rightharpoonup u(t) && \text{in } L^1(\Omega) \text{ for almost all } t \in (0,T), \end{aligned}$$

than $u_n \to u$ in $L^s(0,T;L^q(\Omega))$.

Proof. In [LM87] it is shown that from our assumptions follows $u_n \to u$ in $L^s(0,T;L^p(\Omega))$, which is the stated assertion if $q \leq p$. For $q \in (p,p^*)$ we use this convergence, the interpolation $\|v\|_q \leq \|v\|_p^{1-\lambda}\|\nabla v\|_p^\lambda$, for appropriate $\lambda \in (0,1)$ and Hölder's inequality after integration in time. ☐

Proof of Theorem 2.6. We want to use Proposition 2.7. Thus we have to verify all conditions in (2.8) for an appropriate sequence. To this end we proceed as follows: (i) existence of iterates and a priori estimates, (ii) identification of pointwise limit and (iii) verification of condition $(2.8)_4$.
(i) existence of iterates and a priori estimates: For each $M \in \mathbb{N}$ and each $\tau = \frac{T}{K}$, $K \in \mathbb{N}$, we obtain the existence of iterates $(u_M^k)_{k=0,\dots,K} \subseteq V_M$ solving (2.5) from Brouwer's fixed point theorem. Using $v_M = u_M^k$ in (2.5) we obtain in a standard manner the estimate

$$\begin{aligned} &\frac{1}{2}\|u_M^\ell\|_H^2 + \frac{c_1}{p'}\tau\sum_{k=1}^{\ell}\|u_M^k\|_V^p \\ &\quad \leq \frac{1}{2}\|u_M^0\|_H^2 + c_2\,\tau\sum_{k=1}^{\ell}\|u_M^k\|_H^2 + \frac{c_1^{\frac{-1}{p-1}}}{p'}\,\tau\sum_{k=1}^{\ell}\|f_M(t_k)\|_{V^*}^{p'} \end{aligned} \tag{2.11}$$

valid for all $\ell = 1,\dots,K$. Denoting by $\bar{f}_M^\tau, \hat{f}_M^\tau$ the interpolants generated by $(f_M(t_k))_{k=0,\dots,K}$, it follows from Assumption 2.4 that both $\bar{f}_M^\tau \to f$ and $\hat{f}_M^\tau \to f$ in $L^{p'}(0,T;V^*)$ as $M \to \infty$, $\tau \to 0$. Consequently we get that the first and the last term on the right-hand side in (2.11) are uniformly bounded with respect to $\ell \in \{1,\dots,K\}$, $M \in \mathbb{N}$ and $\tau \leq \tau_0$. From discrete Gronwall's inequality we deduce that the left-hand side of (2.11) is uniformly bounded with respect to $\ell \in \{1, \dots, K\}$, $M \in \mathbb{N}$ and $\tau \le \tau_0$. Thus the interpolants generated by $(u_M^k)_{k=0,\dots,K}$ satisfy for all $M \in \mathbb{N}$, $\tau \le \tau_0$

$$\begin{aligned} \|\bar{u}_M^\tau\|_{L^\infty(0,T;H)} + \|\bar{u}_M^\tau\|_{L^p(0,T;V)} &\le c(T, \|u^0\|_H, \|f\|_{L^{p'}(0,T;V^*)}) , \\ \|\hat{u}_M^\tau\|_{L^\infty(0,T;H)} &\le c(T, \|u^0\|_H, \|f\|_{L^{p'}(0,T;V^*)}) . \end{aligned} \tag{2.12}$$

This and Lemma 2.3 imply the existence of sequences $M_n \to \infty$, $\tau_n \to 0$ and elements $\bar{u} \in L^\infty(0,T;H) \cap L^p(0,T;V)$, $\hat{u} \in L^\infty(0,T;H)$, $u^* \in H$, $B^* \in L^{p'}(0,T;V^*)$ such that $\bar{u}_n := \bar{u}_{M_n}^{\tau_n}$, $\hat{u}_n := \hat{u}_{M_n}^{\tau_n}$ satisfy

$$\begin{aligned} \bar{u}_n &\rightharpoonup \bar{u} && \text{in } L^p(0,T;V), \\ \bar{u}_n &\overset{*}{\rightharpoonup} \bar{u} && \text{in } L^\infty(0,T;H), \\ B\bar{u}_n &\rightharpoonup B^* && \text{in } L^{p'}(0,T;V^*), \\ \hat{u}_n &\overset{*}{\rightharpoonup} \hat{u} && \text{in } L^\infty(0,T;H), \\ \bar{u}_n(T) = \hat{u}_n(T) &\rightharpoonup u^* && \text{in } H. \end{aligned} \tag{2.13}$$

We want to apply Proposition 2.7 to the sequence $(\bar{u}_n)_{n\in\mathbb{N}}$.
(ii) identification of pointwise limit: We have to verify that $\bar{u}_n(t) \rightharpoonup \bar{u}(t)$ in H for almost all $t \in (0,T)$. Let us first show that $\bar{u} = \hat{u}$ in $L^2(0,T;H)$. Note that linear combinations of functions of the form $\chi_{(s_1,s_2)}(t)v$, where $\chi_{(s_1,s_2)}$, $0 < s_1 < s_2 < T$, is the characteristic function of the intervall (s_1,s_2) and $v \in H$, are dense in $L^2(0,T;H)$. For $0 < s_1 < s_2 < T$ there exist $k_1^n, k_2^n \in \{1,\dots,K_n\}$, $\lambda_1^n, \lambda_2^n \in (0,1]$ such that $s_i = \tau_n(\lambda_1^n + k_i^n - 1) \in I_{k_i^n}^{\tau_n}$, $i = 1,2$. Using that $\hat{u}_n(t) - \bar{u}_n(t) = (u_{M_n}^k - u_{M_n}^{k-1})(\frac{t}{\tau_n} - k)$ on $I_k^{\tau_n}$ and (2.12) we easily see that

$$\begin{aligned} (\hat{u}_n - \bar{u}_n, \chi_{(s_1,s_2)}v)_{L^2(0,T;H)} &= \int_{s_1}^{s_2} (\hat{u}_n(t) - \bar{u}_n(t), v)_H \, dt \\ &\le 4\tau_n \|\bar{u}_n\|_{L^\infty(0,T;H)} \|v\|_H \to 0 \quad \text{for } n \to \infty. \end{aligned}$$

Thus $\hat{u}_n - \bar{u}_n \rightharpoonup 0$ in $L^2(0,T;H)$, which implies $\bar{u} = \hat{u}$ in $L^2(0,T;H)$, and thus also in $L^\infty(0,T;H)$.
Next, notice that (2.5) can for all $v \in V_{M_n}$ and almost all $t \in (0,T)$ be re-written as

$$\Big\langle \frac{d\hat{u}_n(t)}{dt}, v \Big\rangle_V + \langle B\bar{u}_n(t), v\rangle_V = \langle \bar{f}_n(t), v\rangle_V , \tag{2.14}$$

where $\bar{f}_n$ is the piecewise constant interpolant generated by $(f_{M_n}(t_k^{\tau_n}))_{k=0,\dots,K_n}$. For an arbitrary $s \in (0,T)$ let $\phi_s \in C_0^\infty(0,T)$ satisfy $0 \le \phi_s \le 1$ and $\phi_s \equiv 1$ in a neighborhood of s. Let $k \in \mathbb{N}$ and let $m, n \in \mathbb{N}$ be such that $M_n, M_m \ge k$. Multiplying (2.14) for an arbitrary $v \in V_k$ by ϕ_s, integrating over $(0,s)$ with respect to t, using the integration by parts formula and the properties of the Gelfand triple we obtain

$$\begin{aligned}(\hat u_n(s) - \hat u_m(s), v)_H &\qquad (2.15)\\ = \int_0^s (\hat u_n(t) - \hat u_m(t), v)_H \, \phi_s'(t)\, dt - \int_0^s \langle B\bar u_n(t) - B\bar u_m(t), v\rangle_V \, \phi_s(t)\, dt\\ + \int_0^s \langle \bar f_n(t) - \bar f_m(t), v\rangle_V \, \phi_s(t)\, dt\,.\end{aligned}$$

In view of (2.13) and $\bar f_n \to f$ in $L^{p'}(0,T;V^*)$ we see that the right-hand side converges to 0 for $n,m\to\infty$. Since $\bigcup_{k\in\mathbb{N}} V_k$ is dense in H, this shows that for every $s\in(0,T)$ the sequence $(\hat u_n(s))_{n\in\mathbb{N}}$ is a weak Cauchy sequence in H. Thus, for every $s\in(0,T)$ there exists $w(s)\in H$ such that $\hat u_n(s) \rightharpoonup w(s)$ in H. From this, (2.12) and the Lebesgue theorem on dominated convergence follows for all $\phi\in L^2(0,T;H)$

$$\lim_{n\to\infty} \int_0^T (\hat u_n(t), \phi(t))_H \, dt = \int_0^T (w(t), \phi(t))_H \, dt.$$

This together with $(2.13)_4$ implies $w = \hat u$ in $L^2(0,T;H)$. Since $\bar u = \hat u$ in $L^2(0,T;H)$ we proved for almost every $t\in(0,T)$

$$\hat u_n(t) \rightharpoonup \bar u(t) \qquad \text{in } H\,. \tag{2.16}$$

However we need to prove $\bar u_n(t) \rightharpoonup \bar u(t)$ in H for almost all $t\in(0,T)$. To this end we proceed as follows: For given $m\in\mathbb{N}$ let $n\ge m$ be arbitrary. Then we have, using that $\hat u_n(t) - \bar u_n(t) = d_\tau u^k_{M_n}(t-k\tau_n)$ on $I_k^{\tau_n}$

$$\begin{aligned}\|\hat u_n - \bar u_n\|^{p'}_{L^{p'}(0,T;V_m^*)} &\le \|\hat u_n - \bar u_n\|^{p'}_{L^{p'}(0,T;V_n^*)}\\ &= \sum_{k=1}^{K_n} \|d_\tau u^k_{M_n}\|^{p'}_{V_n^*} \int_{I_k^{\tau_n}} |t-k\tau_n|^{p'}\, dt\\ &= \frac{\tau_n^{p'}}{p'+1}\tau_n \sum_{k=1}^{K_n} \|d_\tau u^k_{M_n}\|^{p'}_{V_n^*}.\end{aligned}$$

The equations (2.5) yield

$$\|d_\tau u^k_{M_n}\|_{V_n^*} \le \|f_{M_n}(t_k^{\tau_n})\|_{V^*} + \|Bu^k_{M_n}\|_{V^*}\,,$$

and thus

$$\|\hat u_n - \bar u_n\|^{p'}_{L^{p'}(0,T;V_m^*)} \le \frac{\tau_n^{p'}}{p'+1}\big(\|\bar f_n\|^{p'}_{L^{p'}(0,T;V^*)} + \|B\bar u_n\|^{p'}_{L^{p'}(0,T;V^*)}\big)\,,$$

which converges to 0 in view of (2.13) and and $\bar f_n \to f$ in $L^{p'}(0,T;V^*)$. Applying a diagonal procedure we get for all $m\in\mathbb{N}$ and almost all $t\in(0,T)$ that

$$\hat u_n(t) - \bar u_n(t) \to 0 \qquad \text{in } V_m^*\,,$$

which together with (2.16), the properties of the Gelfand triple and the density of $\bigcup_{k\in\mathbb{N}} V_k$ in H yields

$$\bar{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,. \tag{2.17}$$

(iii) verification of condition $(2.8)_4$: From (2.14) and the integration by parts formula we obtain for all $\phi \in C_0^\infty(\mathbb{R})$ and all $v \in V_m$, where $M_n \geq m$

$$\begin{aligned}&(\hat{u}_n(T), v)_H \phi(T) - (\hat{u}_n(0), v)_H \phi(0)\\ &= \int_0^T (\hat{u}_n(t), v)_H\, \phi'(t) - \langle B\bar{u}_n(t), v\rangle_V\, \phi(t) + \langle \bar{f}_n(t), v\rangle_V\, \phi(t)\, dt\,.\end{aligned}$$

In view of (2.13), $\bar{f}_n \to f$ in $L^{p'}(0,T;V^*)$ the density of $\bigcup_{k\in\mathbb{N}} V_k$ in V and $\bar{u} = \hat{u}$ in $L^2(0,T;H)$ we obtain

$$\begin{aligned}&(u^*, v)_H \phi(T) - (u^0, v)_H \phi(0)\\ &= \int_0^T (\bar{u}(t), v)_H\, \phi'(t) - \langle B^*(t), v\rangle_V\, \phi(t) + \langle \bar{f}(t), v\rangle_V\, \phi(t)\, dt\end{aligned}$$

for all $\phi \in C_0^\infty(\mathbb{R})$ and all $v \in V$. For $\phi \in C_0^\infty(0,T)$ this and the definition of the generalized time derivative imply

$$\frac{d\bar{u}}{dt} = f - B^* \qquad \text{in } L^{p'}(0,T;V^*). \tag{2.18}$$

Moreover, by standard arguments we get $\bar{u} \in C(\bar{I};H)$, $u^* = \bar{u}(T)$, and $\hat{u}_n(T) = \bar{u}_n(T) \rightharpoonup \bar{u}(T)$ in H. Using (2.14) for $v = \bar{u}_n(t)$ and

$$\Big\langle \frac{d\hat{u}_n}{dt}, \bar{u}_n \Big\rangle_{L^p(0,T;V)} = \tau_n \sum_{k=1}^{K_n} (d_\tau u^k_{M_n}, u^k_{M_n})_H \geq \frac{1}{2}\|\bar{u}_n(T)\|_H^2 - \frac{1}{2}\|u_n^0\|_H^2$$

we obtain

$$\begin{aligned}\langle B\bar{u}_n, \bar{u}_n\rangle_{L^p(0,T;V)} &= \langle \bar{f}_n, \bar{u}_n\rangle_{L^p(0,T;V)} - \Big\langle \frac{d\hat{u}_n}{dt}, \bar{u}_n \Big\rangle_{L^p(0,T;V)}\\ &\leq \langle \bar{f}_n, \bar{u}_n\rangle_{L^p(0,T;V)} + \frac{1}{2}\|u_n^0\|_H^2 - \frac{1}{2}\|\bar{u}_n(T)\|_H^2.\end{aligned}$$

Thus (2.13), $\bar{f}_n \to f$ in $L^{p'}(0,T;V^*)$ and the lower weak semicontinuity of the norm imply

$$\limsup_{n\to\infty} \langle B\bar{u}_n, \bar{u}_n\rangle_{L^p(0,T;V)} \leq \langle f, \bar{u}\rangle_{L^p(0,T;V)} + \frac{1}{2}\|u^0\|_H^2 - \frac{1}{2}\|\bar{u}(T)\|_H^2.$$

From (2.18), the integration by parts formula and (2.13) we get

$$\langle f, \bar{u}\rangle_{L^p(0,T;V)} = \frac{1}{2}\|\bar{u}(T)\|_H^2 - \frac{1}{2}\|u^0\|_H^2 + \lim_{n\to\infty} \langle B\bar{u}_n, \bar{u}\rangle_{L^p(0,T;V)}\,.$$

The last two inequalities imply that also condition $(2.8)_4$ is satisfied.

Thus, we have verified all conditions in (2.8) and consequently Proposition 2.7 together with (2.13) implies $B^* = B\bar{u}$ in $L^{p'}(0,T;V^*)$. This and (2.18) yield

$$\frac{d\bar{u}}{dt} + B\bar{u} = f \qquad \text{in } L^{p'}(0,T;V^*),$$

i.e. $\bar{u}$ is a solution of (2.1). □

3. Convergence of a semi-implicit scheme

For a given N-function φ having (p,δ)-structure we address the following evolution problem

$$\begin{aligned} \frac{du}{dt}(t) - \operatorname{div} A_0(\nabla u(t)) + g(u(t)) = f \quad &\text{in } V^* \text{ for a.e. } t \in [0,T] \\ u(0) = u^0 \quad &\text{in } H, \end{aligned} \tag{3.1}$$

where A_0 is given by (1.6) for $\alpha = 0$ and $g\colon \mathbb{R} \to \mathbb{R}$ is a given function. Concerning the function g we make the following assumption:

Assumption 3.2 (Nonlinearity)**.** *Let the function $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(s) := d(s)\, s$, $s \in \mathbb{R}$, with a continuous function $d\colon \mathbb{R} \to \mathbb{R}$ that satisfies:*

(H1) *There exists a constant $c_7 > 0$ such that for all $s \in \mathbb{R}$*

$$d(s) \geq -c_7\,.$$

(H2) *There exists $r \in (2,\infty)$ and a constant $c_8 > 0$ such that for all $s \in \mathbb{R}$*

$$|d(s)| \leq c_8\big(1 + |s|^{r-2}\big)\,.$$

Note that (H2) implies that there exists a constant $c_9 = c_9(r, c_8) > 0$ such that for all $s \in \mathbb{R}$

$$|g(s)| \leq c_9\big(1 + |s|^{r-1}\big)\,. \tag{3.3}$$

In what follows we abbreviate

$$V := W_0^{1,p}(\Omega) \qquad \text{and} \qquad H := L^2(\Omega).$$

The N-function φ and the functions g, d induce operators $A\colon V \to V^*$, $G\colon L^q(\Omega) \to L^{\frac{q}{r-1}}(\Omega)$, $q \in [1,\infty)$, and $D\colon L^q(\Omega) \to L^{\frac{q}{r-2}}(\Omega)$, $q \in [\max\{1, r-2\},\infty)$ via

$$\begin{aligned} \langle Au, v\rangle_V &:= \int_\Omega A_0(\nabla u)\cdot \nabla v \,\mathrm{d}x\,, \\ (Gu)(x) &:= g(u(x))\,, \\ (Du)(x) &:= d(u(x))\,. \end{aligned} \tag{3.4}$$

Lemma 3.5. *Let φ have (p,δ)-structure for some $p \in (1,\infty)$ and $\delta \geq 0$ and let the Assumption 3.2 be satisfied. Then the operators $A\colon V \to V^*$, $D\colon L^q(\Omega) \to L^{\frac{q}{r-2}}(\Omega)$, $q \in [\max\{1, r-2\},\infty)$, and $G\colon L^q(\Omega) \to L^{\frac{q}{r-1}}(\Omega)$, $q \in [1,\infty)$ defined in (3.4) are continuous and bounded. Moreover, the operator A is strictly monotone and coercive. In particular, the operator*

$B\colon V \to V^*$ defined via $Bu := Au + Gu$ satisfies Assumption 2.2 if $p > \frac{2d}{d+2}$ and $r \in (2, p\frac{d+2}{d}]$.

Proof. Since $V \eqsim W_0^{1,\varphi}(\Omega)$ the properties of A follow from the properties of φ in a standard manner. Thus the operator A satisfies Assumption 2.2 with constants $c_1 = c_1(p)$, $c_3 = c_3(p)\delta^p$, $c_6 = c_6(p, |\Omega|)\delta^{p-1}$ and $c_4 = c_4(p)$, $c_2 = c_5 = 0$. From Assumption 3.2 we deduce that H, G are Nemyckii operators, for which the stated properties follow in a standard way. Moreover, for $r \in (2, p^*)$, recall that $p^* = \frac{dp}{d-p}$ if $p < d$, and $p^* = \infty$ if $p \geq d$, the operator $G\colon V \to V^*$ is compact, since the embedding $V \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact. Thus we get that the operator B is pseudomonotone. For $p > \frac{2d}{d+2}$ we get that (V, H, V^*) forms a Gelfand triple and that $p\frac{d+2}{d} < p^*$. The Assumption 3.2, Hölder's inequality, interpolation, embeddings and $r \leq p\frac{d+2}{d}$ (cf. [BR17] for more details) imply that G satisfies (A2), (A3) with constants $c_2 = c_7$, $c_4 = c_9$, $c_1 = c_3 = c_4 = c_6 = 0$. Consequently, B satisfies Assumption 2.2. □

In view of this lemma we can apply Theorem 2.6 to the present situation if we make analogous assumptions on the data to Assumption 2.4. The assumption applies to standard finite element methods on polyhedral Lipschitz domains (cf. [BS08]).

Assumption 3.6 (Data I)**.** *Let $p > \frac{2d}{d+2}$ and let $u^0 \in H$ and $f \in L^{p'}(0, T; V^*)$ be given. Let $V_h \subset W_0^{1,\infty}(\Omega)$, $h > 0$, be conforming finite element spaces, corresponding to shape regular triangulations $\mathcal{T}_h$. We equip V_h with the V-norm and assume that $V_{h/2} \subset V_h$ and that $\bigcup_{m\in\mathbb{N}} V_{h2^{-m}}$ is dense in V. We assume that there exists a sequence $(u_h^0) \subset V_h$ with $u_h^0 \to u^0$ in H. For each $\varepsilon > 0$ we set ${}^{\varepsilon}u_h^0 := u_h^0$. We further assume that there exists a sequence $(f_h) \subset C(0, T; V^*)$ such that $f_h \to f$ in $L^{p'}(0, T; V^*)$.*

Let us first study an implicit scheme. Let $\varepsilon \in [0, 1)$. For given $h > 0$ and ${}^{\varepsilon}u_h^0 \in V_h$ the sequence of iterates $({}^{\varepsilon}u_h^k)_{k=0,\dots,K} \subseteq V_h$ is given via

$$\begin{aligned} \big(d_\tau\, {}^{\varepsilon}u_h^k, v_h\big) &+ \Big(\frac{\varphi_\varepsilon'(|\nabla\, {}^{\varepsilon}u_h^k|)}{|\nabla\, {}^{\varepsilon}u_h^k|}\nabla\, {}^{\varepsilon}u_h^k, \nabla v_h\Big) \\ &+ (d({}^{\varepsilon}u_h^k)\, {}^{\varepsilon}u_h^k, v_h) = (f_h(t_k), v_h) \qquad \forall v_h \in V_h. \end{aligned} \tag{3.7}$$

Theorem 3.8 (Convergence of the implicit scheme)**.** *Let φ have (p, δ)-structure for some $p \in (\frac{2d}{d+2}, \infty)$ and $\delta \geq 0$, let Assumption 3.2 be satisfied for some $r \in (2, p\frac{d+2}{d}]$ and let Assumption 3.6 be satisfied. Let $\bar{u}_n := {}^{\varepsilon_n}\bar{u}_{h_n}^{\tau_n}$ be a sequence of piecewise constant interpolants generated by iterates $({}^{\varepsilon_n}u_{h_n}^k)_{k=0,\dots,K_n}$, $K_n = \frac{T}{\tau_n}$, solving (3.7) for some sequences $h_n \to 0$, $\tau_n \to 0$, $\varepsilon_n \to 0$. Then each weak* accumulation point u of the sequence $(\bar{u}_n)_{n\in\mathbb{N}}$ in the space $L^\infty(0, T; H) \cap L^p(0, T; V)$ belongs to the space $W_p^1(0, T; V, H)$ and is a solution of (3.1).*

Proof. In the case $\varepsilon = 0$ we choose $\varepsilon_n = 0$ and the statement of the theorem follows from Theorem 2.6. In the case $\varepsilon > 0$ we have to re-write the scheme (3.7) as

$$\begin{aligned} &\langle d_\tau\, {}^\varepsilon u_h^k, v_h\rangle_V + \big(A_0(\nabla\, {}^\varepsilon u_h^k), \nabla v_h\big) + (d({}^\varepsilon u_h^k)\, {}^\varepsilon u_h^k, v_h) \\ &= (f_h(t_k), v_h) + \big({}^\varepsilon E_h^k, \nabla v_h\big), \end{aligned} \tag{3.9}$$

where

$$\big({}^\varepsilon E_h^k, \nabla v_h\big) := \big(A_0(\nabla\, {}^\varepsilon u_h^k) - A_\varepsilon(\nabla\, {}^\varepsilon u_h^k), \nabla v_h\big)\,.$$

The proof of the assertion now follows along the lines of the proof of Theorem 2.6. The additional term ${}^\varepsilon E_h^k$ can be treated due to Lemma 1.15. We omit the details here, since they will be discussed in detail in the proof of Theorem 3.12, where the same term occurs. □

In the scheme (3.7) we still have to solve nonlinear equations. If we want to avoid this and only solve linear equations we can study the following semi-implicit scheme: Let $\varepsilon \in (0,1)$. For given $h > 0$ and ${}^\varepsilon u_h^0 \in V_h$ the sequence of iterates $({}^\varepsilon u_h^k)_{k=0,\dots,K} \subseteq V_h$ is given via

$$\begin{aligned} &\big(d_\tau\, {}^\varepsilon u_h^k, v_h\big) + \Big(\frac{\varphi_\varepsilon'(|\nabla\, {}^\varepsilon u_h^{k-1}|)}{|\nabla\, {}^\varepsilon u_h^{k-1}|}\nabla\, {}^\varepsilon u_h^k, \nabla v_h\Big) \\ &\quad + (d({}^\varepsilon u_h^{k-1})\, {}^\varepsilon u_h^k, v_h) = (f_h(t_k), v_h) \qquad \forall v_h \in V_h. \end{aligned} \tag{3.10}$$

To show that also this scheme converges to a weak solution of (3.1) we have to make more restrictive assumptions on the data.

Assumption 3.11 (Data II)**.** *Let $p > \frac{2d}{d+2}$ and let $u^0 \in V$ and $f \in L^{p'}(0,T;H)$ be given. Let $V_h \subset W_0^{1,\infty}(\Omega)$, $h > 0$, be conforming finite element spaces, corresponding to shape regular triangulations $\mathcal{T}_h$. We equip V_h with the V-norm and assume that $V_{h/2} \subset V_h$ and that $\bigcup_{m\in\mathbb{N}} V_{h2^{-m}}$ is dense in V. We assume that there exists a sequence $(u_h^0) \subset V_h$ with $u_h^0 \to u^0$ in V. For each $\varepsilon > 0$ we set ${}^\varepsilon u_h^0 := u_h^0$. We assume that there exists a sequence $(f_h) \subset C(0,T;H)$ such that $f_h \to f$ in $L^{p'}(0,T;H)$.*

The following theorem excludes the special case $p = 2$ which is discussed in a subsequent remark.

Theorem 3.12 (Convergence of the semi-implicit scheme)**.** *Let φ have (p,δ)-structure for some $p \in (\frac{2d}{d+2}, 2)$ and $\delta \geq 0$, let Assumption 3.2 be satisfied for some $r \in (2, p\frac{d+2}{2d} + 1]$ and let Assumptions 3.6 be satisfied. Let $\bar u_n := {}^{\varepsilon_n}\bar u_{h_n}^{\tau_n}$ be a sequence of piecewise constant interpolants generated by iterates $({}^{\varepsilon_n} u_{h_n}^k)_{k=0,\dots,K_n}$, $K_n = \frac{T}{\tau_n}$, solving (3.10) for some sequences $h_n \to 0$, $\tau_n \to 0$, $\varepsilon_n \to 0$ satisfying $\tau_n = o(\varphi''(\varepsilon_n))$. Then each weak* accumulation point u of the sequence $(\bar u_n)_{n\in\mathbb{N}}$ in the space $L^\infty(0,T;V)$ belongs to the space $W_p^1(0,T;V,H) \cap L^\infty(0,T;V)$ and is a solution of (3.1).*

Proof. In order to adapt the arguments of the proof of Theorem 2.6 to the present situation we re-write (3.10) as an implicit scheme with resulting error terms on the right-hand side. The handling of these new terms in the verification of the conditions in (2.8) is possible due to a second a priori estimate, obtained by testing with the backward difference quotient of the solution. For the verification of the last condition in (2.8) we also use the compactness argument in Proposition 2.10.

(i) existence of iterates and a priori estimates: For each $h>0$, $\varepsilon\in(0,\varepsilon_0)$, where we assume without loss of generality that $\varepsilon_0=1$, and each $\tau=\frac{T}{K}$, $K\in\mathbb{N}$, the existence of iterates $({}^{\varepsilon}u_h^k)_{k=0,\dots,K}\subseteq V_h$ solving (3.10) is clear since these are linear equations. Using $v_h={}^{\varepsilon}u_h^k$ in (3.10) we obtain, also using the Assumption 3.2 and Young's inequality, the estimate

$$\begin{aligned}\frac{1}{2}\|{}^{\varepsilon}u_h^{\ell}\|_H^2+\tau\sum_{k=1}^{\ell}\int_\Omega\frac{\varphi_\varepsilon'(|\nabla{}^{\varepsilon}u_h^{k-1}|)}{|\nabla{}^{\varepsilon}u_h^{k-1}|}|\nabla{}^{\varepsilon}u_h^k|^2\,\mathrm{d}x\\ \le\frac{1}{2}\|u_h^0\|_H^2+(c_7+1)\,\tau\sum_{k=1}^{\ell}\|u_M^k\|_H^2+\tau\sum_{k=1}^{\ell}\|f_h(t_k)\|_H^2\end{aligned}\tag{3.13}$$

valid for all $\ell=1,\dots,K$. Due to Assumption 3.11 the first and last term on the right-hand side of (3.13) are uniformly bounded with respect to $h>0$, $\tau,\varepsilon\in(0,1)$ and $\ell\in\{1,\dots,K\}$. Thus discrete Gronwall's inequality yields that the left-hand side of (3.13) is uniformly bounded with respect to $h>0$, $\tau,\varepsilon\in(0,1)$ and $\ell\in\{1,\dots,K\}$. In particular we get that interpolants generated by $({}^{\varepsilon}u_h^k)_{k=0,\dots,K}$ satisfy for all $h>0$, $\tau,\varepsilon\in(0,1)$

$$\|{}^{\varepsilon}\bar u_h^{\tau}\|_{L^\infty(0,T;H)}+\|{}^{\varepsilon}\hat u_h^{\tau}\|_{L^\infty(0,T;H)}\le c(\|u^0\|_H,\|f\|_{L^2(0,T;H)})\,.\tag{3.14}$$

Using $v_h=d_\tau\,{}^{\varepsilon}u_h^k$ and Lemma 1.13 we obtain in the same way as in [BDN18], using also Young's inequality,

$$\begin{aligned}E_{\varphi_\varepsilon}[{}^{\varepsilon}u_h^{\ell}]+\frac{\tau}{2}\sum_{k=1}^{\ell}\|d_\tau\,{}^{\varepsilon}u_h^k\|_H^2+\frac{\tau^2}{2}\sum_{k=1}^{\ell}\int_\Omega\frac{\varphi_\varepsilon'(|\nabla{}^{\varepsilon}u_h^{k-1}|)}{|\nabla{}^{\varepsilon}u_h^{k-1}|}|d_\tau\nabla{}^{\varepsilon}u_h^k|^2\,\mathrm{d}x\\ \le E_{\varphi_\varepsilon}[u_h^0]+\tau\sum_{k=1}^{\ell}\|f_h(t_k)\|_H^2+\tau\sum_{k=1}^{\ell}\int_\Omega|d({}^{\varepsilon}u_h^{k-1})|^2\,|{}^{\varepsilon}u_h^k|^2\,\mathrm{d}x\,,\end{aligned}\tag{3.15}$$

valid for all $\ell=1,\dots,K$. Due to Assumption 3.11 the first two terms on the right-hand side are uniformly bounded with respect to $h>0$ and $\varepsilon,\tau\in(0,1)$. Moreover, using (1.17) we get

$$E_{\varphi_\varepsilon}[v]\ge c\left(\|v\|_V^p-\varepsilon^p-\delta^p\right).\tag{3.16}$$

The Assumption (H2), Young's inequality, the interpolation of $L^{2(r-1)}(\Omega)$ between H and V and (3.14) yield

$$\begin{aligned}\int_\Omega |d({}^\varepsilon u_h^{k-1})|^2\, |{}^\varepsilon u_h^k|^2 \,\mathrm{d}x &\le c\Big(\|\,{}^\varepsilon u_h^k\|_H^2 + \|\,{}^\varepsilon u_h^k\|_{2(r-1)}^{2(r-1)} + \|\,{}^\varepsilon u_h^{k-1}\|_{2(r-1)}^{2(r-1)}\Big)\\ &\le c\big(1 + \|\,{}^\varepsilon u_h^k\|_V^{p\frac{2d(r-2)}{p(d+2)-2d}} + \|\,{}^\varepsilon u_h^{k-1}\|_V^{p\frac{2d(r-2)}{p(d+2)-2d}}\big)\,. \end{aligned} \tag{3.17}$$

Requiring that $p\frac{2d(r-2)}{p(d+2)-2d} \le 1$ we get the restriction $r \le p\frac{d+2}{2d} + 1$. The last estimate together with (3.16), (3.14), (3.15) and discrete Gronwall's inequality yield that the interpolants generated by $({}^\varepsilon u_h^k)_{k=0,\dots,K}$ and the piecewise constant interpolant generated by $({}^\varepsilon u_h^{k-1})_{k=0,\dots,K}$, which we denote by ${}^\varepsilon\tilde u_h^\tau$, satisfy for all $h>0$, $\tau,\varepsilon \in (0,1)$

$$\begin{aligned}\|\,{}^\varepsilon\tilde u_h^\tau\|_{L^\infty(0,T;V)} + \|\,{}^\varepsilon\bar u_h^\tau\|_{L^\infty(0,T;V)} &\le c(\delta,p,T,|\Omega|,\|f\|_{L^2(0,T;H)},\|u^0\|_V)\,,\\ \Big\|\frac{d\,{}^\varepsilon\hat u_h^\tau}{dt}\Big\|_{L^2(0,T;H)} + \|\,{}^\varepsilon\hat u_h^\tau\|_{L^\infty(0,T;V)} &\le c(\delta,p,T,|\Omega|,\|f\|_{L^2(0,T;H)},\|u^0\|_V)\,. \end{aligned} \tag{3.18}$$

Using Assumptions (A3) and (H2) one can show (cf. Lemma 2.3) that the induced operators A,B,D,G are bounded operators in the following settings: $A\colon L^\infty(0,T;V) \to L^\infty(0,T;V^*)$, $B\colon L^\infty(0,T;V) \to L^\infty(0,T;V^*)$, $G\colon L^\infty(0,T;V) \to L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega))$, $D\colon L^\infty(0,T;V) \to L^\infty(0,T;L^{\frac{p^*}{r-2}}(\Omega))$. For later purposes we now choose $\tau = o(\varphi''(\varepsilon))$. Thus (3.18) and the last observation imply the existence of sequences $h_n \to 0$, $\tau_n \to 0$, $\varepsilon_n \to 0$ and elements $u^* \in H$, $\bar u \in L^\infty(0,T;V)$, $\hat u \in L^\infty(0,T;V)$, $\tilde u \in L^\infty(0,T;V)$, $A^* \in L^\infty(0,T;V^*)$, $D^* \in L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega))$ such that $\bar u_n := {}^{\varepsilon_n}\bar u_{h_n}^{\tau_n}$, $\hat u_n := {}^{\varepsilon_n}\hat u_{h_n}^{\tau_n}$, $\tilde u_n := {}^{\varepsilon_n}\tilde u_{h_n}^{\tau_n}$ satisfy

$$\begin{aligned} \bar u_n &\overset{*}{\rightharpoonup} \bar u && \text{in } L^\infty(0,T;V),\\ \hat u_n &\overset{*}{\rightharpoonup} \hat u && \text{in } L^\infty(0,T;V),\\ \tilde u_n &\overset{*}{\rightharpoonup} \tilde u && \text{in } L^\infty(0,T;V),\\ A\bar u_n &\overset{*}{\rightharpoonup} A^* && \text{in } L^\infty(0,T;V^*),\\ D(\tilde u_n)\bar u_n &\overset{*}{\rightharpoonup} D^* && \text{in } L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega)) \cap L^\infty(0,T;V^*),\\ \bar u_n(T) = \hat u_n(T) &\rightharpoonup u^* && \text{in } H. \end{aligned} \tag{3.19}$$

We want to apply Proposition 2.7 to the sequence $(\bar u_n)_{n\in\mathbb{N}}$ and the operator $B\colon V \to V^*$ defined via $Bv := Av + D(v)v$ (cf. Lemma 3.5).
(ii) perturbed implicite scheme: To adapt the arguments from the proof of Theorem 2.6 to the present situation, we re-write the scheme (3.10) for all

$v_h \in V_h$ as a perturbed implicite scheme

$$\begin{aligned}&\langle d_\tau \,{}^\varepsilon u_h^k, v_h\rangle_V + \big(A_0(\nabla \,{}^\varepsilon u_h^k), \nabla v_h\big) + (d({}^\varepsilon u_h^{k-1})\, {}^\varepsilon u_h^k, v_h)\\ &= (f_h(t_k), v_h) + \big({}^\varepsilon E_h^k, \nabla v_h\big) + \big({}^\varepsilon F_h^k, \nabla v_h\big),\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\big({}^\varepsilon E_h^k, \nabla v_h\big) &:= \big(A_0(\nabla \,{}^\varepsilon u_h^k) - A_\varepsilon(\nabla \,{}^\varepsilon u_h^k), \nabla v_h\big)\,,\\ \big({}^\varepsilon F_h^k, \nabla v_h\big) &:= \Big(A_\varepsilon(\nabla \,{}^\varepsilon u_h^k) - \frac{\varphi_\varepsilon'(|\nabla \,{}^\varepsilon u_h^{k-1}|)}{|\nabla \,{}^\varepsilon u_h^{k-1}|}\nabla \,{}^\varepsilon u_h^k, \nabla v_h\Big)\,.\end{aligned}$$

To verify the conditions (2.8) we proceed as in the proof of Theorem 2.6. In the following we concentrate on the treatment of the new terms.
(iii) identification of the pointwise limit: In view of (3.18) we can prove in the same way as in the proof of Theorem 2.6 that $\bar u = \hat u$ in $L^2(0,T;H)$, and thus also in $L^\infty(0,T;V)$. From (3.18) follows

$$\int_0^T \|\tilde u_n - \bar u_n\|_H^2 \,\mathrm{d}t = \tau_n^2 \Big\|\frac{d\,{}^{\varepsilon_n}\hat u_{h_n}^{\tau_n}}{dt}\Big\|_{L^2(0,T;H)}^2 \to 0\,, \tag{3.21}$$

which implies that also $\tilde u = \bar u$ in $L^\infty(0,T;V)$.
Next, notice that (3.20) can for all $v \in V_{h_n}$ and almost all $t \in (0,T)$ be re-written as

$$\begin{aligned}&\Big\langle \frac{d\hat u_n(t)}{dt}, v\Big\rangle_V + \langle A\bar u_n(t), v\rangle_V + (D(\tilde u_n)(t)\bar u_n(t), v)_H\\ &= (\bar f_n(t), v)_H + \langle E_n(t), v\rangle_V + \langle F_n(t), v\rangle_V,\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}\langle E_n(t), v\rangle_V &:= \big(A_0(\nabla \bar u_n(t)) - A_{\varepsilon_n}(\nabla \bar u_n(t)), \nabla v\big)_H\\ \langle F_n(t), v\rangle_V &:= \Big(\frac{\varphi_{\varepsilon_n}'(|\nabla \bar u_n(t)|)}{|\nabla \bar u_n(t)|}\nabla \bar u_n(t) - \frac{\varphi_{\varepsilon_n}'(|\nabla \tilde u_n(t)|)}{|\nabla \tilde u_n(t)|}\nabla \bar u_n(t), \nabla v\Big)_H\,,\end{aligned}$$

where $\bar f_n$ is the piecewise constant interpolant generated by $(f_{h_n}(t_k^{\tau_n}))_{k=0,\dots,K_n}$. Similarly to the derivation of (2.15) we obtain for an arbitrary $s \in (0,T)$, an arbitrary $k \in \mathbb{N}$, $m, n \geq k$, and all $v \in V_{h_k}$, all $\phi_s \in C_0^\infty(0,T)$ satisfying

$\phi_s \equiv 1$ in a neighborhood of s

$$
\begin{aligned}
&(\hat{u}_n(s) - \hat{u}_m(s), v)_H \\
&= \int_0^s (\hat{u}_n(t) - \hat{u}_m(t), v)_H \,\phi_s'(t) - \langle A\bar{u}_n(t) - A\bar{u}_m(t), v\rangle_V \,\phi_s(t)\, dt \\
&\quad + \int_0^s \big(A_0(\nabla\bar{u}_n(t)) - A_{\varepsilon_n}(\nabla\bar{u}_n(t)), \nabla v\big)_H \,\phi_s(t)\, dt \\
&\quad - \int_0^s \big(A_0(\nabla\bar{u}_m(t)) - A_{\varepsilon_m}(\nabla\bar{u}_m(t)), \nabla v\big)_H \,\phi_s(t)\, dt \\
&\quad + \int_0^s \int_\Omega \Big(\frac{\varphi_{\varepsilon_n}'(|\nabla\bar{u}_n|)}{|\nabla\bar{u}_n|}\nabla\bar{u}_n - \frac{\varphi_{\varepsilon_n}'(|\nabla\tilde{u}_n|)}{|\nabla\tilde{u}_n|}\nabla\bar{u}_n\Big)\nabla v \,\mathrm{d}x \,\phi_s(t)\, dt \\
&\quad - \int_0^s \int_\Omega \Big(\frac{\varphi_{\varepsilon_m}'(|\nabla\bar{u}_m|)}{|\nabla\bar{u}_m|}\nabla\bar{u}_m - \frac{\varphi_{\varepsilon_m}'(|\nabla\tilde{u}_m|)}{|\nabla\tilde{u}_m|}\nabla\bar{u}_m\Big)\nabla v \,\mathrm{d}x \,\phi_s(t)\, dt \\
&\quad - \int_0^s (D(\tilde{u}_n)(t)\bar{u}_n(t) - D(\tilde{u}_m)(t)\bar{u}_m(t), v)_H \,\phi_s(t)\, dt \\
&\quad + \int_0^s (\bar{f}_n(t) - \bar{f}_m(t), v)_H \,\phi_s(t)\, dt \\
&=: I_1^{n,m} + I_2^{n,m} + I_3^n + I_4^m + I_5^n + I_6^m + I_7^{n,m} + I_8^{n,m}\,.
\end{aligned}
$$

Since $\phi_s(\cdot)v \in L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;L^{p^*}(\Omega))$ and $(p^*)' \le \frac{p^*}{r-1}$ due to $r \le p\frac{d+2}{2d} + 1$, we deduce from (3.19) and $\bar{f}_n \to f$ in $L^2(0,T;H)$ that $I_1^{n,m}$, $I_2^{n,m}$, $I_7^{n,m}$ and $I_8^{n,m}$ converge to zero for $n, m \to \infty$. Using Lemma 1.15 we get

$$
|I_3^n| \le c\varphi'(\varepsilon_n) \int_0^s \int_\Omega |\nabla v|\phi_s \,\mathrm{d}x\, dt \to 0 \qquad \text{for } n \to \infty. \tag{3.23}
$$

In the same way we get that I_4^m converges to zero for $m \to \infty$. There exists $\ell \in \mathbb{N}$ such that $(\ell-1)\tau_n < s \le \ell\tau_n$. Using the definition of $\bar{u}_u$, $\tilde{u}_n$, Lemma 1.14, $\max_{t\in(0,T)} |\phi_s(t)| \le 1$ and Young's inequality we get

$$
\begin{aligned}
|I_5^n| &\le c\,\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|} |\nabla d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k|\,|\nabla v| \,\mathrm{d}x \\
&\le \gamma(\varepsilon_n)\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|} |\nabla d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k|^2 \,\mathrm{d}x \\
&\quad + \frac{c}{\gamma(\varepsilon_n)}\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|} |\nabla v|^2 \,\mathrm{d}x \\
&\le \gamma(\varepsilon_n) E_{\varphi_{\varepsilon_n}}[u^0] + \frac{c\,\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)}\,\tau_n\, \|\nabla v\|_H^2 \\
&\le \gamma(\varepsilon_n) E_{\varphi}[u^0] + \frac{c\,\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)}\,\tau_n\, \|\nabla v\|_H^2,
\end{aligned}
\tag{3.24}
$$

where we also used $\frac{\varphi_\varepsilon'(t)}{t} \le \kappa_0^{-1}\varphi''(\varepsilon)$ due to (C2), (C3), and $\varphi_\varepsilon(t) \le \varphi(t)$. Since $v \in W_0^{1,\infty}(\Omega)$, the terms in the last line of the previous estimate converge to zero since $\tau_n = o(\varphi''(\varepsilon_n)^{-1})$ as then, e.g., $\gamma^2(\varepsilon_n) = \tau_n\varphi''(\varepsilon_n)$ satisfies $\gamma(\varepsilon_n) = o(1)$ and $\tau_n\varphi''(\varepsilon_n)/\gamma(\varepsilon_n) = o(1)$ as $n \to \infty$. The term I_6^m is treated analogously. Since $\bigcup_{k\in\mathbb{N}} V_{h_k}$ is dense in H, we have shown that for every $s \in (0,T)$ the sequence $(\hat{u}_n(s))_{n\in\mathbb{N}}$ is a weak Cauchy sequence in H. Thus, for every $s \in (0,T)$ there exists $w(s) \in H$ such that $\hat{u}_n(s) \rightharpoonup w(s)$ in H. From this we deduce as in the proof of Theorem 2.6 that for almost every $t \in (0,T)$

$$\hat{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,. \tag{3.25}$$

However we need to prove $\bar{u}_n(t) \rightharpoonup \bar{u}(t)$ in H for almost all $t \in (0,T)$. To this end we proceed as follows: We equip the set V_{h_n}, $n \in \mathbb{N}$, with the $W_0^{1,2}(\Omega)$-norm and denote this space by X_n. For given $m \in \mathbb{N}$ let $n \ge m$ be arbitrary. Then we get, using that $\hat{u}_n(t) - \bar{u}_n(t) = d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k(t - k\tau_n)$ on $I_k^{\tau_n}$

$$\begin{aligned}
\|\hat{u}_n - \bar{u}_n\|_{L^1(0,T;X_m^*)} &\le \|\hat{u}_n - \bar{u}_n\|_{L^1(0,T;X_n^*)} \\
&= \sum_{k=1}^{K_n} \|d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k\|_{X_n^*} \int_{I_k^{\tau_n}} |t - k\tau_n|\, dt \\
&= \frac{\tau_n}{2}\tau_n \sum_{k=1}^{K_n} \|d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k\|_{X_n^*}.
\end{aligned}$$

Since (V, H, V^*) and $(W_0^{1,2}(\Omega), H, (W_0^{1,2}(\Omega))^*)$ are Gelfand triples we get $\big(d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k, v\big)_H = \langle d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k, v\rangle_V = \langle d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k, v\rangle_{W_0^{1,2}(\Omega)}$ for $v \in W_0^{1,\infty}(\Omega)$. This and (3.20) yields

$$\begin{aligned}
\|d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k\|_{X_n^*} &= \sup_{\substack{v\in X_n \\ \|v\|_{W_0^{1,2}(\Omega)}\le 1}} \big(d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k, v\big) \\
&= \sup_{\substack{v\in X_n \\ \|v\|_{W_0^{1,2}(\Omega)}\le 1}} \Big[-\big(A_{\varepsilon_n}(\nabla\, {}^{\varepsilon_n}u_{h_n}^k), \nabla v\big) - (d({}^{\varepsilon_n}u_{h_n}^{k-1})\, {}^{\varepsilon_n}u_{h_n}^k, v) \\
&\qquad + (f_h(t_k), v) + \big({}^{\varepsilon}F_h^k, \nabla v\big)\Big].
\end{aligned}$$

Using Hölder's inequality, $\frac{\varphi_\varepsilon'(t)}{t} \le \kappa_0^{-1}\varphi''(\varepsilon)$, the properties of φ and Young's inequality we obtain

$$\begin{aligned}
\big|(A_{\varepsilon_n}(\nabla\, {}^{\varepsilon_n}u_{h_n}^k), \nabla v)\big| &\le \Big(\int_\Omega \frac{\big(\varphi_{\varepsilon_n}'(|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|)\big)^2}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|^2} |\nabla\, {}^{\varepsilon_n}u_{h_n}^k|^2\, \mathrm{d}x\Big)^{\frac{1}{2}} \|\nabla v\|_H \\
&\le c\,\varphi''(\varepsilon_n)\, \|\nabla v\|_H^2 + c\int_\Omega \varphi_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|)\, \mathrm{d}x\,.
\end{aligned}$$

Similarly as in (3.17) we get

$$|(d({}^{\varepsilon_n}u_{h_n}^{k-1})\ {}^{\varepsilon_n}u_{h_n}^{k}, v)| \le c\big(1 + \|v\|_H^2 + \|\,{}^{\varepsilon_n}u_{h_n}^{k-1}\|_V^p + \|\,{}^{\varepsilon_n}u_{h_n}^{k}\|_V^p\big)\,.$$

From Assumption 3.11 we conclude

$$|(f_h(t_k), v)| \le \|v\|_H^2 + \|f_h(t_k)\|_H^2\,.$$

Using Lemma 1.14, Young's inequality and $\frac{\varphi_\varepsilon'(t)}{t} \le \kappa_0^{-1}\varphi''(\varepsilon)$ we get

$$\begin{aligned}
\big|({}^{\varepsilon}F_h^k, \nabla v)\big| &\le c\,\tau_n \int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\ {}^{\varepsilon_n}u_{h_n}^{k}|\,|\nabla v|\,\mathrm{d}x \\
&\le \gamma(\varepsilon_n)\tau_n \int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\ {}^{\varepsilon_n}u_{h_n}^{k}|^2\ \mathrm{d}x \\
&\quad + \frac{c}{\gamma(\varepsilon_n)}\tau_n \int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla v|^2\ \mathrm{d}x \\
&\le \gamma(\varepsilon_n)\tau_n \int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\ {}^{\varepsilon_n}u_{h_n}^{k}|^2\ \mathrm{d}x + \frac{c\,\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)}\tau_n\|\nabla v\|_H^2.
\end{aligned}$$

Consequently, we proved

$$\begin{aligned}
\|d_{\tau_n}\ {}^{\varepsilon_n}u_{h_n}^{k}\|_{X_n^*} &\le \gamma(\varepsilon_n)\tau_n \int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\ {}^{\varepsilon_n}u_{h_n}^{k}|^2\ \mathrm{d}x + \frac{c\,\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)}\tau_n \\
&\quad + c\,\varphi''(\varepsilon_n) + c\int_\Omega \varphi_{\varepsilon_n}(|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k}|)\,\mathrm{d}x + c\,\|\,{}^{\varepsilon_n}u_{h_n}^{k-1}\|_V^p \\
&\quad + c\,\|\,{}^{\varepsilon_n}u_{h_n}^{k}\|_V^p + \|f_h(t_k)\|_H^2 + c
\end{aligned}$$

and thus

$$\begin{aligned}
&\|\hat{u}_n - \bar{u}_n\|_{L^1(0,T;X_m^*)} \\
&\le c\tau_n\,\tau_n\sum_{k=1}^{K_n}\int_\Omega \varphi_{\varepsilon_n}(|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k}|)\,\mathrm{d}x + c\tau_n\,\tau_n\sum_{k=1}^{K_n}\varphi''(\varepsilon_n) + c\tau_n\,\tau_n\sum_{k=1}^{K_n}\frac{\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)}\tau_n \\
&\quad + c\tau_n\,\gamma(\varepsilon_n)\,\tau_n^2\sum_{k=1}^{K_n}\int_\Omega \frac{\varphi_{\varepsilon_n}'(|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\,{}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\ {}^{\varepsilon_n}u_{h_n}^{k}|^2\ \mathrm{d}x + c\tau_n\,\tau_n\sum_{k=1}^{K_n}1 \\
&\quad + c\tau_n\,\tau_n\sum_{k=1}^{K_n}\|\,{}^{\varepsilon_n}u_{h_n}^{k-1}\|_V^p + c\tau_n\,\tau_n\sum_{k=1}^{K_n}\|\,{}^{\varepsilon_n}u_{h_n}^{k}\|_V^p + \tau_n\,\tau_n\sum_{k=1}^{K_n}\|f_h(t_k)\|_H^2\,.
\end{aligned}$$

Using $\tau_n = o(\varphi''(\varepsilon_n))$, the estimates (3.15) and (3.18) as well as Assumption 3.11 we see that all terms on the right-hand side converge to zero for $n \to \infty$. A diagonal procedure implies for all $m \in \mathbb{N}$ and almost all $t \in (0,T)$

$$\hat{u}_n(t) - \bar{u}_n(t) \to 0 \qquad \text{in } X_m^*\,,$$

which together with (3.25), the properties of the Gelfand triple with the spaces $W_0^{1,2}(\Omega)$, $L^2(\Omega)$, and $(W_0^{1,2}(\Omega))^*$ and the density of $\bigcup_{k\in\mathbb{N}} X_k$ in H yields

$$\bar{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,. \tag{3.26}$$

This and (3.21) implies

$$\tilde{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,. \tag{3.27}$$

(iv) verification of condition $(2.8)_4$: We first show that $D^* = D(\bar{u})\bar{u}$ in $L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega))$. In view of (3.18), (3.26) and (3.27) Proposition 2.10 yields for all $s\in[1,\infty)$, $q\in[1,p^*)$

$$\bar{u}_n, \tilde{u}_n \to \bar{u} \qquad \text{in } L^s(0,T;L^q(\Omega))\,. \tag{3.28}$$

Condition (H2) and the theory of Nemyckii operators yields (cf. Lemma 3.5) that $D\colon L^q(0,T;L^q(\Omega)) \to L^{\frac{q}{r-2}}(0,T;L^{\frac{q}{r-2}}(\Omega))$, $q\geq \max\{1,r-2\}$, is bounded and continuous. This and (3.28) yields for all $q\in[\max\{1,r-2\},p^*)$

$$D(\bar{u}_n), D(\tilde{u}_n) \to D(\bar{u}) \qquad \text{in } L^{\frac{q}{r-2}}(0,T;L^{\frac{q}{r-2}}(\Omega))\,. \tag{3.29}$$

From this and (3.28) follows for all $q\in[\max\{1,r-1\},p^*)$

$$D(\bar{u}_n)\bar{u}_n, D(\tilde{u}_n)\bar{u}_n \to D(\bar{u})\bar{u} \qquad \text{in } L^{\frac{q}{r-1}}(0,T;L^{\frac{q}{r-1}}(\Omega))\,, \tag{3.30}$$

which together with (3.19) proves $D^* = D(\bar{u})\bar{u}$ in $L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega))$. Using (3.22), the integration by parts formula we obtain for all $\phi\in C_0^\infty(\mathbb{R})$ and all $v\in X_{h_m}$, where $n\geq m$

$$\begin{aligned}
&(\hat{u}_n(T),v)_H\phi(T) - (\hat{u}_n(0),v)_H\phi(0)\\
&= \int_0^T (\hat{u}_n(t),v)_H\,\phi'(t) - \big(\langle A\bar{u}_n(t),v\rangle_V - \langle E_n(t),v\rangle_V - \langle F_n(t),v\rangle_V\big)\phi(t)\,dt\\
&\quad + \int_0^T \big((\bar{f}_n(t),v)_H - (D(\tilde{u}_n)(t)\bar{u}_n(t),v)_H\big)\phi(t)\,\mathrm{d}t\,.
\end{aligned}$$

Notice that the last two terms in the first line of the right-hand side converge to zero by similar arguments as in (3.23) and (3.24). Further we have $\phi(\cdot)v\in L^\infty(0,T;V)\hookrightarrow L^\infty(0,T;L^{p^*}(\Omega))$ and $(p^*)' < \frac{p^*}{r-1}$, which holds due to $r\leq p\frac{d+2}{2d}+1$ and $p>\frac{2d}{d+2}$. Thus, the convergences in (2.13) and (3.30), the convergence $\bar{f}_n\to f$ in $L^{p'}(0,T;H)$, the identity of sets $X_{h_k}=V_{h_k}$, the density of $\bigcup_{k\in\mathbb{N}} V_{h_k}$ in V and H, and $\bar{u}=\hat{u}$ in $L^2(0,T;H)$ yield

$$\begin{aligned}
&(u^*,v)_H\phi(T) - (u^0,v)_H\phi(0)\\
&= \int_0^T (\bar{u}(t),v)_H\,\phi'(t) + \big((f(t),v)_H - \langle A^*(t),v\rangle_V - (D(\bar{u}(t))\bar{u}(t),v)_H\big)\,\phi(t)\,dt
\end{aligned}$$

for all $\phi \in C_0^\infty(\mathbb{R})$ and all $v \in V$. For $\phi \in C_0^\infty(0,T)$ this and the definition of the generalized time derivative together with $H \hookrightarrow V^*$ imply

$$\frac{d\bar{u}}{dt} = f - A^* - D(\bar{u})\bar{u} \qquad \text{in } L^{p'}(0,T;V^*). \tag{3.31}$$

Moreover, by standard arguments we get $\bar{u} \in C(\bar{I};H)$, $u^* = \bar{u}(T)$, and $\hat{u}_n(T) = \bar{u}_n(T) \rightharpoonup \bar{u}(T)$ in H. Using (3.22) for $v = \bar{u}_n(t)$ and

$$\Big\langle \frac{d\hat{u}_n}{dt}, \bar{u}_n \Big\rangle_{L^p(0,T;V)} = \tau_n \sum_{k=1}^{K_n} (d_\tau u^k_{M_n}, u^k_{M_n})_H \geq \frac{1}{2}\|\bar{u}_n(T)\|_H^2 - \frac{1}{2}\|u_n^0\|_H^2$$

we obtain with $\langle G_n(t), v\rangle_V := \big((D(\bar{u})(t)\bar{u}(t) - (D(\tilde{u})(t)\bar{u}(t), v\big)_H$

$$\begin{aligned}
&\langle A\bar{u}_n + D(\bar{u}_n)\bar{u}_n, \bar{u}_n\rangle_{L^p(0,T;V)} \\
&= \langle \bar{f}_n, \bar{u}_n\rangle_{L^p(0,T;H)} + \langle E_n + F_n + G_n, \bar{u}_n\rangle_{L^p(0,T;V)} - \Big\langle \frac{d\hat{u}_n}{dt}, \bar{u}_n \Big\rangle_{L^p(0,T;V)} \\
&\leq \langle \bar{f}_n, \bar{u}_n\rangle_{L^p(0,T;H)} + \langle E_n + F_n + G_n, \bar{u}_n\rangle_{L^p(0,T;V)} + \frac{1}{2}\|u_n^0\|_H^2 - \frac{1}{2}\|\bar{u}_n(T)\|_H^2.
\end{aligned}$$

Similarly as in (3.23) and (3.24) we obtain

$$\begin{aligned}
&|\langle E_n, \bar{u}_n\rangle_{L^p(0,T;V)}| \leq c\varphi'(\varepsilon_n) \int_0^T \int_\Omega |\nabla \bar{u}_n| \,\mathrm{d}x\,dt \to 0 \qquad n \to \infty, \\
&|\langle F_n, \bar{u}_n\rangle_{L^p(0,T;V)}| \\
&\leq \gamma(\varepsilon_n)\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla{}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla{}^{\varepsilon_n}u_{h_n}^{k-1}|} |\nabla d_{\tau_n}{}^{\varepsilon_n}u_{h_n}^k|^2 \,\mathrm{d}x \\
&\quad + \frac{c}{\gamma(\varepsilon_n)}\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla{}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla{}^{\varepsilon_n}u_{h_n}^{k-1}|} |\nabla{}^{\varepsilon_n}u_{h_n}^k|^2 \,\mathrm{d}x \\
&\leq \gamma(\varepsilon_n)\Big(E_\varphi[u^0] + \|f\|^2_{L^2(0,t;H)} + (\varepsilon_n^p + \delta_n^p)T|\Omega| + o(1)\Big) \\
&\quad + \frac{c\,\tau_n}{\gamma(\varepsilon_n)}\Big(E_\varphi[u^0] + \|u^0\|_H + (\varepsilon_n^p + \delta_n^p)T|\Omega| + o(1)\Big) \to 0 \quad n \to \infty,
\end{aligned}$$

where we used that $L^p(0,T;V)$ embeds into $L^1(0,T;V)$; the properties of φ, (3.13), (3.15), the choice $\tau_n = o(\varphi''(\varepsilon_n)^{-1})$ and Assumption 3.11. In view of (3.30) and (3.19) we get $|\langle G_n, \bar{u}_n\rangle_{L^p(0,T;V)}| \to 0$. Thus (3.19), $\bar{f}_n \to f$ in $L^{p'}(0,T;H)$ and the lower weak semicontinuity of the norm imply

$$\limsup_{n\to\infty} \langle A\bar{u}_n(t) + D(\bar{u}_n)\bar{u}_n, \bar{u}_n\rangle_{L^p(0,T;V)} \leq \langle f, \bar{u}\rangle_{L^p(0,T;H)} + \frac{1}{2}\|u^0\|_H^2 - \frac{1}{2}\|\bar{u}(T)\|_H^2.$$

From (3.31), the integration by parts formula and (3.19), (3.30) we get

$$\langle f, \bar{u}\rangle_{L^p(0,T;H)} = \frac{1}{2}\|\bar{u}(T)\|_H^2 - \frac{1}{2}\|u^0\|_H^2 + \lim_{n\to\infty} \langle A\bar{u}_n + D(\bar{u}_n)\bar{u}_n, \bar{u}\rangle_{L^p(0,T;V)}\,.$$

The last two inequalities imply that also condition $(2.8)_4$ is satisfied.

Thus, we have verified all conditions in (2.8) and consequently Proposition 2.7 together with (3.19) implies $A^* + H^* = A\bar{u} + D(\bar{u})\bar{u}$ in $L^{p'}(0,T;V^*)$. This and (3.31) yield

$$\frac{d\bar{u}}{dt} + A\bar{u} + D(\bar{u})\bar{u} = f \qquad \text{in } L^{p'}(0,T;V^*),$$

i.e. $\bar{u}$ is a solution of (3.1). □

Remark 3.32. *For $p = 2$ we have to distinguish between the cases $d = 2$ and $d \geq 3$. In the latter one Theorem 3.12 holds as stated and also the proof is the same. If $d = 2$ the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^s(\Omega)$, $s \in [1,\infty)$ is different from the other cases we considered. Thus, estimate* (3.17) *has to be adapted and results in the restriction $r < 3$. Consequently, in Theorem 3.12 we have to require $r \in (2,3)$ if $p = 2$ and $d = 2$.*

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