NECESSARY AND SUFFICIENT CONDITIONS FOR AVOIDING BABUSKA'S PARADOX ON SIMPLICIAL MESHES

SÖREN BARTELS AND PHILIPP TSCHERNER

Abstract. It is shown that discretizations based on variational or weak formulations of the plate bending problem with simple support boundary conditions do not lead to failure of convergence when polygonal domain approximations are used and the imposed boundary conditions are compatible with the nodal interpolation of the restriction of certain regular functions to approximating domains. It is further shown that this is optimal in the sense that a full realization of the boundary conditions leads to failure of convergence for conforming methods. The abstract conditions imply that standard nonconforming and discontinuous Galerkin methods converge correctly while conforming methods require a suitable relaxation of the boundary condition. The results are confirmed by numerical experiments.

1. Introduction

The plate or Babuška paradox refers to the failure of convergence when a linear bending problem with simple support boundary conditions on a domain with curved boundary is approximated using a sequence of problems on approximating polygonal domains, cf. [3]. An explanation for this is an insufficient consistency in the approximation of the curvature of the boundary. Remarkably, numerical experiments show that typical nonconforming and discontinuous Galerkin methods converge correctly on sequences of simplicial meshes [24]. It is the goal of this article to identify criteria for numerical methods that avoid the occurrence of the paradox and explain the observed convergence. The main result is that a suitable relaxation in the treatment of the boundary conditions avoids the failure of convergence independently of regularity properties. While this is naturally satisfied by canonical realizations of nonconforming and discontinuous Galerkin methods, in the case of a conforming method, the boundary condition need to be restricted to the boundary vertices only, which is typically consistent with the nodal interpolation of certain regular functions.

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Small elastic deflections $u : \omega \to \mathbb{R}$ of a thin plate are described by a minimization of the energy functional

$$I(v) = \frac{\sigma}{2}\int_\omega |\Delta v|^2 \,\mathrm{d}x + \frac{1-\sigma}{2}\int_\omega |D^2 v|^2 \,\mathrm{d}x - \int_\omega f v \,\mathrm{d}x$$

in a set $V \subset H^2(\omega)$ whose definition involves appropriate boundary conditions. The material parameter σ is the Poisson ratio of the elastic material and we assume for simplicity that $0 \le \sigma < 1$. So called conditions of simple support prescribe the deflection on the boundary, in the simplest setting via imposing

$$u|_{\partial\omega} = 0,$$

so that $V = H^2(\omega) \cap H^1_0(\omega)$. Clamped boundary conditions additionally prescribe the normal of the deformed plate along the boundary, e.g., via $\nabla u = 0$ on $\partial\omega$, so that in this case we have $V = H^2_0(\omega)$. Because of the density of compactly supported smooth functions in $H^2_0(\omega)$ it is straightforward to show that domain approximations are not critical for clamped boundary conditions.

In the case of simple support boundary conditions, the unique minimizer $u \in V$ is characterized by the Euler–Lagrange equations

$$\Delta^2 u = f \text{ in } \omega, \quad u = \Delta u - (1-\sigma)\kappa\partial_n u = 0 \text{ on } \partial\omega,$$

where κ denotes the curvature of the boundary (positive for locally convex boundary parts), and $\partial_n u = \nabla u \cdot n$ is the outer normal derivative along the boundary. For approximating polygonal domains ω_m one has $\kappa = 0$ away from corner points in which discrete curvature is captured by Dirac contributions. Owing to the boundary condition $u = 0$ on $\partial\omega_m$ we have however $\nabla u = 0$ and hence $\partial_n u = 0$ in those points. Thus, formally the term involving κ in the natural boundary condition disappears for polygonal domain approximations. In convex domains ω_m the solutions u_m can then be efficiently computed using an operator splitting, i.e., by introducing the variables $w_m = -\Delta u_m$ and successively solving the problems

$$\begin{aligned} (a) \quad & -\Delta w_m = f \ \text{ in } \omega_m, \quad w_m = 0 \text{ on } \partial\omega_m, \\ (b) \quad & -\Delta u_m = w_m \text{ in } \omega_m, \quad u_m = 0 \text{ on } \partial\omega_m. \end{aligned}$$

Since the approximation of boundaries is not critical for Poisson problems with Dirichlet boundary conditions, it follows that limits (u_∞, w_∞) of the sequence (u_m, w_m) provide the solution of the problem

$$\Delta^2 u_\infty = f \text{ in } \omega, \quad u_\infty = \Delta u_\infty = 0 \text{ on } \partial\omega.$$

In particular, u_∞ is independent of σ and in general different from the true solution u. A rotationally symmetric example with $f = 1$ in the unit disk has been determined in [4], cf. Figure 1 and Section 5. This discrepancy is referred to as the plate or Babuška paradox. A comprehensive discussion of the analytical aspects of this and related phenomena can be found in [18, 16]. For domain approximations using piecewise quadratic boundary curves the piecewise curvature correctly approximates κ in the sense of functions and coefficients of Dirac contributions associated with corner points decay superlinearly and are therefore irrelevant. We refer the reader to [21, 10, 1] for related ideas and numerical methods. Other methods to avoid the paradox introduce certain Dirac contributions in boundary condition obtained via asymptotic expansions, cf. [17], are based on nonconforming methods [13, 11, 1], or impose the boundary condition in a modified or penalized form [22, 19, 23]. The framework of variational convergence adopted here avoids conditional results based on regularity properties, does not require extension operators, and leads to weak conditions on penalty parameters.

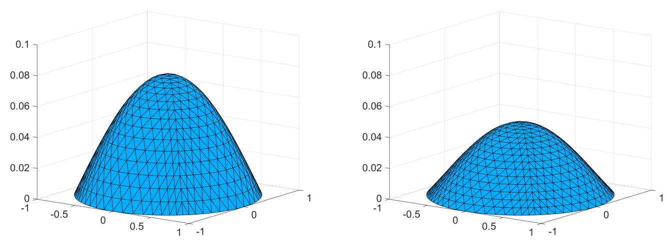

FIGURE 1. Interpolants of the solution of a plate bending problem with simple support boundary conditions (left) and of the incorrect solution obtained as a limit of problems on polygonal domain approximations (right).

The failure of convergence also occurs when working with the variational problems or weak formulations resulting from replacing ω by ω_m in the energy functional I, i.e., considering the minimization of

$$I_m(v) = \frac{\sigma}{2}\int_{\omega_m} |\Delta v|^2 \,\mathrm{d}x + \frac{1-\sigma}{2}\int_{\omega_m} |D^2 v|^2 \,\mathrm{d}x - \int_{\omega_m} f_m v \,\mathrm{d}x$$

in the set of functions $v_m \in V_m = H^1_0(\omega_m) \cap H^2(\omega_m)$. An integration by parts and the density of H^3 regular functions in V_m reveal boundary contributions that vanish for the functionals I_m but not for the original funtional I, cf. [11, 14]. Hence, variational convergence cannot hold. This argument also applies to finite element methods that are based on subspaces of V_m. It is thus a necessary condition that approximations are nonconforming.

To avoid the failure of convergence it suffices to introduce a nonconformity in the treatment of the boundary condition by reducing it to corner points or more generally to $\partial\omega_m \cap \partial\omega$, i.e., using the admissible space

$$\widetilde{V}_m = \big\{ v \in H^2(\omega_m) : v = 0 \text{ on } \partial\omega_m \cap \partial\omega \big\}.$$

To show that the minimization of I_m in $\widetilde{V}_m$ converges to the minimization of I on $V = H^2(\omega) \cap H^1_0(\omega)$ we use the concept of Γ-convergence which avoids imposing regularity conditions. We assume that $\omega_m \subset \omega$ such that corner points of $\partial\omega_m$ belong to $\partial\omega$, i.e., that ω is convex, and always extend functions and derivatives trivially by zero to ω. The first step consists in showing that accumulation points $v \in L^2(\omega)$ of sequences (v_m) satisfy the lim-inf inequality

$$I(v) \leq \liminf_{m\to\infty} I_m(v_m).$$

This is in fact straightforward since $\|v_m\|_{H^1(\omega_m)} \leq c_{21}\|D^2 v_m\|_{L^2(\omega_m)}$ and $D^2 v_m \rightharpoonup D^2 v$ after selection of a subsequence. By carrying out a nodal interpolation $\mathcal{I}_h v \in H^1_0(\omega_m)$ on triangulations $\mathcal{T}_m$ of ω_m it follows that the boundary conditions are correctly satisfied in the limit, i.e., $v|_{\partial\omega} = 0$ so that $v \in V$. The second step requires showing that the lower bound is attained for every $v \in V$. For this, it suffices to realize that owing to the definition of $\widetilde{V}_m$ the restrictions $v_m = v|_{\omega_m}$ belong to $\widetilde{V}_m$, and satisfy $D^2 v_m \to D^2 v$ in $L^2(\omega)$ since $\|D^2 v_m\|_{L^2(\omega\setminus\omega_m)} \to 0$ as well as

$$I(v) = \lim_{m\to\infty} I_m(v_m).$$

This establishes Γ convergence of I_m to I with respect to strong convergence in $L^2(\omega)$. An immediate consequence is that minimizers for I_m converge to the minimizer of I. It is straightforward to check that the convergence is strong in H^1 on compact subsets of ω.

The remarkable difference between finite element approximations with boundary conditions imposed on the entire boundary and the corner points of a pentagon is illustrated in Figure 2. While the first one is close to the restriction of the exact solution to the pentagon, the second one misses the exact maximal value by a factor ten.

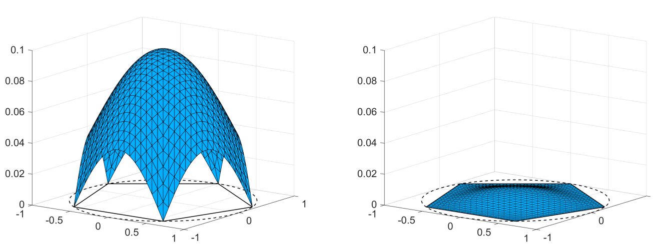

FIGURE 2. Finite element solutions on a pentagon imposing the boundary conditions at the corner points (left) and along the entire boundary (right).

Under mild additional integrability conditions on f and D^2u, error estimates can be derived for the difference of u_m and $u|_{\omega_m}$. Letting a_m and a denote the bilinear forms associated with the energy functionals I_m and I, noting that $u|_{\omega_m}$ is admissible in the approximating problem, and assuming that $E_m : H^2(\omega_m) \to H^2(\omega) \cap H^1_0(\omega)$ are uniformly bounded extension operators, straightforward calculations lead to, e.g., if $\sigma = 0$,

$$\begin{aligned}\|D^2(u-u_m)\|^2_{L^2(\omega_m)} &= a_m(u-u_m, u-u_m)\\ &= \int_{\omega\setminus\omega_m} f(u-E_m u_m)\,\mathrm{d}x - \int_{\omega\setminus\omega_m} D^2 u : D^2(u-E_m u_m)\,\mathrm{d}x\\ &\le c_E\,|\omega\setminus\omega_m|^{1/2}\big(\|f\|_{L^\infty(\omega)} + \|D^2 u\|_{L^\infty(\omega)}\big)\|D^2(u-E_m u_m)\|.\end{aligned}$$

For simplicial triangulations $\mathcal{T}_{h_m}$ with maximal mesh-size $h_m > 0$ that define the subdomains ω_m of the domain ω with piecewise C^2 boundary, we have the area difference estimate $|\omega\setminus\omega_m| \le ch^2$, so that the domain approximation leads to an error contribution of at least linear order. We refer the reader to [19] for related estimates based on the use of a Strang lemma.

An important aspect in the transfer of convergence proofs to finite element settings is the construction of a recovery sequence via the interpolation of the restriction of functions in $H^3(\omega)\cap V$ to ω_m. In this restriction only the zero boundary values at the corners are captured and only these information are seen in the interpolation process. The discrete admissible set thus has to be appropriately defined to ensure that the interpolants belong to it. The convergence proof given above serves as a template to derive an abstract convergence theory that can be applied to various finite element methods. The sufficient conditions are that (1) the approximating problems are uniformly coercive in $H^1_0(\omega)$, (2) possible discretizations of second order derivatives are stable, and (3) that interpolation operators map functions from $H^3(\omega)\cap V$ into the discrete admissible sets.

As an alternative to imposing the boundary condition in the corner points, one may impose it via penalty terms with suitably chosen penalty parameter. Letting

$$I_{m,\varepsilon}(v) = I_m(v) + \frac{1}{2\varepsilon}\int_{\partial\omega_m} v^2\,\mathrm{d}s$$

one establishes a Γ-convergence result if there exists a family of bounded linear operators $F_m : H^1(\omega_m) \to H^1(\omega)$ that map traces $v|_{\partial\omega_m}$ boundedly to traces $F_m(v)|_{\partial\omega}$. The condition implies that limits of sequences of functions with uniformly bounded energies have vanishing traces on $\partial\omega$. The restrictions $v_m = v|_{\omega_m}$ define a recovery sequence for a function $v \in H^3(\omega)\cap H^1_0(\omega)$ if ε is chosen such that $h_m^4\varepsilon \to 0$ as $h_m \to 0$ since $\|v_m\|_{L^\infty(\partial\omega_m)} \le h_m^2$. If quadrature is used in the penalty term it realizes a penalized variant of imposing the boundary condition in the corner points and in fact no restrictions on the parameters are required. These observations explain why discontinuous Galerkin methods, that impose the boundary conditions via penalty terms, converge correctly on sequences of polygonal domain approximations.

The outline of this article is as follows. Some auxiliary results are collected in Section 2. The abstract convergence theory and a result about failure of convergence are stated in Section 3. The application of the framework to conforming, nonconforming, and discontinuous Galerkin methods is discussed in Section 4. Numerical experiments that confirm the theoretical results are reported in Section 5.

2. Auxiliary results

We cite in this section an important density result from [14] together with a boundary representation formula and collect some basic operators and estimates related to finite element methods. Throughout, we use standard notation for Sobolev and Lebesgue spaces. We let $(\cdot,\cdot)_A$ denote the L^2 inner product on a set A and occasionally abbreviate

$$\|v\| = \|v\|_{L^2(\omega)}, \quad (v,w) = (v,w)_\omega$$

for functions $v, w \in L^2(\omega)$ possibly obtained as trivial extensions of functions defined in subsets $\omega' \subset \omega$. We always assume that ω is a convex and bounded Lipschitz domain with piecewise $C^{2,1}$ boundary and finitely many corner points. We note that we have the Poincaré inequality

$$\|\nabla v\| \le c_{\mathrm{P}} \|\Delta v\| \tag{1}$$

for all $v \in H^2(\omega) \cap H^1_0(\omega)$ with a constant $c_{\mathrm{P}} > 0$ that is uniformly bounded for families of convex domains whose inner and outer diameters are uniformly bounded. We occasionally use the symbol $\lesssim$ to express an inequality that holds up to a generic constant factor.

2.1. Bending energy.

Crucial for our analysis is a density result for certain regular functions in the set $H^2(\omega) \cap H^1_0(\omega)$, cf. [14, Thm. B.5].

Theorem 2.1 (Density). *The set $H^3(\omega) \cap H^1_0(\omega)$ is dense in $H^2(\omega) \cap H^1_0(\omega)$.*

An important consequence of this result is the following formula from [12] and [14, Lemma 3.8], that provides a representation of the total curvature of the graph of a function by a boundary integral.

Lemma 2.2 (Boundary representation). *For $v \in H^2(\omega) \cap H^1_0(\omega)$ we have that*

$$-\int_\omega \det D^2 v \,\mathrm{d}x = \frac{1}{2} \int_{\partial\omega} \kappa (\partial_n v)^2 \,\mathrm{d}s.$$

Proof (sketched). Integration by parts shows for $v \in H^3(\omega) \cap H^1_0(\omega)$ that

$$-\int_\omega \det D^2 v \,\mathrm{d}x = \frac{1}{2} \int_\omega \operatorname{div}(J D^2 v J \nabla v) \,\mathrm{d}x = \frac{1}{2} \int_{\partial\omega} (J D^2 v J \nabla v) \cdot n \,\mathrm{d}s,$$

where J denotes the clockwise rotation by $\pi/2$. If $p : (\alpha, \beta) \to \mathbb{R}^2$ is an arclength parametrization of a boundary segment, we have that

$$0 = \frac{d^2}{ds^2}(v \circ p) = (p')^{\mathsf{T}} (D^2 v \circ p) p' + (\nabla v \circ p) \cdot p''.$$

Using $p' = Jn$, $p'' = \kappa n$, and $J \nabla v = (\partial_n v) J n$ yields the formula. $\square$

The elementary relation $|D^2 v|^2 = (\Delta v)^2 - 2\det D^2 v$ and the lemma imply that the functionals I defined on ω and I_m defined on polygonal domains ω_m can be represented via

$$I(v) = \frac{1}{2}\int_\omega |\Delta v|^2 \,\mathrm{d}x + \frac{1-\sigma}{2}\int_{\partial\omega} \kappa (\partial_n v)^2 \,\mathrm{d}x \tag{2}$$

for $v \in H^2(\omega) \cap H^1_0(\omega)$ and

$$I_m(v) = \frac{1}{2}\int_{\omega_m} |\Delta v|^2 \,\mathrm{d}x \tag{3}$$

for $v \in H^2(\omega_m) \cap H^1_0(\omega_m)$ and polygonal domains ω_m. To establish the failure of convergence of the functionals I_m and the correct convergence of the modified functionals $\widetilde{I}_m$, we use the framework of Γ-convergence, cf. [2].

Definition 2.3 (Γ-convergence)**.** *The sequence of functionals $I_m : L^2(\omega) \to \mathbb{R} \cup \{+\infty\}$ is* Γ-convergent *to $I : L^2(\omega) \to \mathbb{R} \cup \{+\infty\}$ with respect to strong convergence in $L^2(\omega)$ if the following conditions hold:*
(i) If $v_m \to v$ in $L^2(\omega)$ then we have $I(v) \le \liminf_{m\to\infty} I_m(v_m)$.
(ii) For every $v \in L^2(\omega)$ there exists a sequence $(v_m)_{m\ge 0}$ with $v_m \to v$ and $I(v) = \lim_{m\to\infty} I_m(v_m)$.

If the sequence of functionals I_m is uniformly coercive, then it follows directly that (almost) minimizers for I_m converge to minimizers for I.

2.2. **Finite element spaces.** The finite element spaces considered below are defined on regular triangulations $\mathcal{T}_h$ consisting of triangles whose unions define bounded polygonal domains ω_h. The index $h > 0$ indicates a maximal mesh size that is assumed to converge to zero in a sequence $(\mathcal{T}_h)_{h>0}$. We further assume that the triangulations are uniformly shape regular, i.e., that the ratios of diameters and inner radii are uniformly bounded. Typical finite element spaces are subspaces of spline spaces

$$\mathcal{S}^{\ell,k}(\mathcal{T}_h) = \big\{ v_h \in C^k(\overline{\omega}_h) : v_h|_T \in P_\ell(T) \big\},$$

where $P_\ell(T)$ denotes the space of polynomials of total degree at most ℓ on T. If $k = 0$ then this superscript is omitted. An important space is the $P1$ finite element space with vanishing traces

$$\mathcal{S}^1_0(\mathcal{T}_h) = \mathcal{S}^1(\mathcal{T}_h) \cap H^1_0(\omega_h).$$

We let $\mathcal{N}_h$ denote the set of vertices in $\mathcal{T}_h$ and note that a function $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ satisfies $v_h|_{\partial\omega_h} = 0$ if and only if $v_h(z) = 0$ for all $z \in \mathcal{N}_h \cap \partial\omega_h$. The set of sides of elements in $\mathcal{T}_h$ is denoted by $\mathcal{S}_h$. We impose the following canonical conditions that relate the triangulations $\mathcal{T}_h$ to the domain ω:

- (T1) The boundary vertices of the triangulation $\mathcal{T}_h$ belong to the boundary of ω, i.e., $\mathcal{N}_h \cap \partial\omega_h \subset \partial\omega$.
- (T2) The corner points $c_0, c_1, \cdots, c_\ell$ of ω belong to the set of vertices, i.e., $c_0, c_1, \dots, c_\ell \subset \mathcal{N}_h$.

With the nodal basis $(\varphi_z)_{z\in\mathcal{N}_h}$ of $\mathcal{S}^1(\mathcal{T}_h)$ the nodal interpolation operator $\mathcal{I}_h^{\mathrm{p1}} : C(\overline{\omega}_h) \to \mathcal{S}^1(\mathcal{T}_h)$ is defined via

$$\mathcal{I}_h^{\mathrm{p1}} v = \sum_{z\in\mathcal{N}_h} v(z)\varphi_z.$$

Note that if $v \in H^3(\omega) \cap H^1_0(\omega)$ then we have that

$$\mathcal{I}_h^{\mathrm{p1}}(v|_{\omega_h}) \in \mathcal{S}^1_0(\mathcal{T}_h).$$

For all $v \in H^2(\omega_h)$ we have the interpolation estimate

$$\|v - \mathcal{I}_h^{\mathrm{p1}} v\|_{H^1(\omega_h)} \le c_{\mathcal{I}} h \|D^2 v\|_{L^2(\omega_h)}$$

with a constant $c_{\mathcal{I}}$ that remains bounded as $h \to 0$. A node averaging operator $\mathcal{J}_h^{\mathrm{av}} : \mathcal{L}^k(\mathcal{T}_h) \to \mathcal{S}^1_0(\mathcal{T}_h)$ defined on the space of elementwise polynomial functions of degree at most k is defined via the nodal values $v_z = 0$ for $z \in \mathcal{N}_h \cap \partial\omega_h$ and

$$v_z = \frac{1}{n_z} \sum_{T : z\in T} v_h|_T(z)$$

for inner nodes $z \in \mathcal{N}_h \setminus \partial\omega_h$ and the number n_z of elements containing z. The jumps of v_h are for sides $S \in \mathcal{S}_h$ defined by

$$[\![v_h]\!](x) = \begin{cases} v_h(x) & \text{if } S \subset \partial\omega_h, \\ \lim_{\varepsilon\to 0} v_h(x+\varepsilon n_S) - v_h(x - \varepsilon n_S) & \text{if } S \not\subset \partial\omega_h, \end{cases}$$

with fixed unit normal vectors n_S, $S \in \mathcal{S}_h$. We then have that

$$\|v_h - \mathcal{J}_h^{\mathrm{av}} v_h\|^2_{L^2(\omega_h)} \le c_k h^2 \sum_{S\in\mathcal{S}_h} h_S^{-1} \|[\![v_h]\!]\|^2_{L^2(S)},$$

cf., e.g., [15]. Averages of discontinuous functions v_h are on sides $S \in \mathcal{S}_h$ defined via

$$\{v_h\}(x) = \begin{cases} v_h(x) & \text{if } S \subset \partial\omega_h, \\ \lim_{\varepsilon\to 0} \big(v_h(x+\varepsilon n_S) + v_h(x - \varepsilon n_S)\big)/2 & \text{if } S \not\subset \partial\omega_h. \end{cases}$$

We make repeated use of inverse estimates, e.g.,

$$\|\nabla w_h\|_{L^2(T)} \le c_{\mathrm{inv}} h_T^{-1} \|w_h\|_{L^2(T)}$$

for a polynomial function w_h of bounded degree on an element $T \in \mathcal{T}_h$ with diameter $h_T > 0$. A scaled trace inequality asserts that for a side $S \in \mathcal{S}_h$ and an adjacent element $T_S \in \mathcal{T}_h$ we have

$$c_{\mathrm{tr}}^{-1} \|w\|^2_{L^2(S)} \le h_{T_S}^{-1} \|w\|^2_{L^2(T_S)} + h_{T_S} \|\nabla w\|^2_{L^2(T_S)}$$

for all $w \in H^1(T)$. For polynomial functions the second term on the right-hand side can be omitted at the expense of a larger constant c_{tr}.

3. Abstract results

We restrict to the most relevant cases $0 \le \sigma < 1$ and consider the plate bending problem defined by the energy functional

$$I(v) = \frac{\sigma}{2} \int_\omega |\Delta v|^2 \, \mathrm{d}x + \frac{1-\sigma}{2} \int_\omega |D^2 v|^2 \, \mathrm{d}x$$

on the space $V = H^2(\omega) \cap H^1_0(\omega)$. For a sequence of convex subdomains $\omega_h \subset \omega$ and function spaces $V_h \subset L^2(\omega_h)$ the approximating problems are defined via the functionals

$$I_h(v_h) = \frac{\sigma}{2} \int_{\omega_h} |\Delta_h v_h|^2 \, \mathrm{d}x + \frac{1-\sigma}{2} \int_{\omega_h} |D_h^2 v_h|^2 \, \mathrm{d}x,$$

with a linear operator $D_h^2 : V_h \to L^2(\omega_h)$ that approximates the Hessian in a sense specified below. The discrete Laplace operator Δ_h is assumed to be given by the trace of D_h^2. We assign the value $+\infty$ to the functionals I and I_h for functions in $L^2(\omega)$ not belonging to V and V_h, respectively. We always extend functions and derivatives defined in ω_h trivially to functions defined in ω and impose the following conditions on the approximating problems:

(S1) *Uniform H^1_0-coercivity:* There exists a family of operators $\mathcal{J}_h : V_h \to H^1_0(\omega_h)$ such that if $I_h(v_h)$ is bounded then $\mathcal{J}_h v_h$ is bounded in $H^1_0(\omega)$ and $v_h - \mathcal{J}_h v_h \to 0$ in $L^2(\omega)$.

(S2) *Stability of D_h^2:* If $v_h \rightharpoonup v$ in $L^2(\omega)$ and $D_h^2 v_h \rightharpoonup \psi$ in $L^2(\omega)$ then we have that $v \in H^2(\omega)$ with $D^2 v = \psi$.

(S3) *Interpolation in V_h:* There exist linear operators $\mathcal{I}_h : H^3(\omega) \cap H^1_0(\omega) \to V_h$ with $D_h^2 \mathcal{I}_h v \to D^2 v$ in $L^2(\omega)$ for every $v \in H^3(\omega) \cap H^1_0(\omega)$.

Proposition 3.1 (Sufficient conditions)**.** *Assume that conditions (S1)-(S3) are satisfied. We then have that $I_h \to I$ as $h \to 0$ in the sense of Γ convergence with respect to strong convergence in $L^2(\omega)$.*

Proof. (i) We first establish the lim-inf inequality. For this, let $(v_h)_{h>0} \subset L^2(\omega)$ be a sequence of functions $v_h \in V_h$ with $v_h \to v$ and, without loss of generality, $I_h(v_h) \le c$. By (S1) we then have, after extension by zero, that $(\mathcal{J}_h v_h)_{h>0}$, and $(D_h^2 v_h)_{h>0}$ are bounded sequences in $H^1_0(\omega)$ and $L^2(\omega)$ with weak limits $v \in H^1_0(\omega)$, and $\psi \in L^2(\omega; \mathbb{R}^{2\times 2})$ for a suitable subsequence. The assumed consistency properties in (S1) and (S2) yield that $\psi = D^2 v$ and $v_h \to v$. In particular, we have that $v \in V$, and by weak lower semicontinuity of quadratic functionals we find that

$$I(v) \le \liminf_{h\to 0} I_h(v_h).$$

(ii) To verify a lim-sup inequality we choose $v \in V$. By density of functions belonging to $V \cap H^3(\omega)$ in the set V stated in Theorem 2.1 and continuity of I, we may assume that $v \in H^3(\omega)$. Condition (S3) then guarantees that for $v_h = \mathcal{I}_h v \in V_h$ we have $D_h^2 v_h \to D^2 v$ in $L^2(\omega)$ and hence $I_h(v_h) \to I(v)$ as $h \to 0$. □

Remarks 3.2. *(i) If $V_h \subset C(\overline{\omega}_h)$ then one may choose the $P1$ nodal interpolation operator $\mathcal{I}_h^{p1} : C(\overline{\omega}_h) \to \mathcal{S}_0^1(\mathcal{T}_h)$ in condition (S1) provided that $v_h(z) = 0$ for every $v_h \in V_h$ and $z \in \mathcal{N}_h \cap \partial\omega$. For discontinuous methods node averaging or quasiinterpolation operators can be employed. The coercivity is then a discrete version of the Poincaré estimate* (1).
(ii) Condition (S2) is independent of boundary approximations and can be checked in subdomains compactly contained in ω.
(iii) The interpolation condition (S3) requires the discrete boundary condition to be compatible with the interpolation of restricted functions $v|_{\omega_h}$ whenever $v \in H^3(\omega) \cap H_0^1(\omega)$. In particular, conditions $v_h(z) = 0$ can only be imposed at vertices $z \in \mathcal{N}_h$ belonging to the boundary of ω and not, e.g., at midpoints of edges.
(iv) Our approach of imposing the boundary condition only in the corner points of ω_h coincides with the methods devised in [19]. One may incorporate conditions $\nabla u_h(c_i) \cdot t_i = 0$ with the exact tangents t_i in corner points c_i, $i = 0, 1, \dots, n_m$, as devised in [22] to improve the accuracy, although this may lead to difficulties in guaranteeing stability of the discrete problems.

For discontinuous Galerkin methods the discretization of the functional I typically involves stabilizing terms defined by positive semidefinite bilinear forms $s_h : V_h \times V_h \to \mathbb{R}$. The discrete funtionals are then given by

$$I_{h,\mathrm{stab}}(v_h) = I_h(v_h) + \frac{1}{2} s_h(v_h, v_h).$$

In this case an additional condition is needed:

(S4) *Stabilization:* For every $v \in H^3(\omega) \cap H_0^1(\omega)$ and the sequence $v_h = \mathcal{I}_h v$ with $\mathcal{I}_h$ from (S3) we have $s_h(v_h, v_h) \to 0$ as $h \to 0$.

Proposition 3.3 (Stabilization)**.** *Assume that conditions (S1)-(S4) are satisfied. We then have that $I_{h,\mathrm{stab}} \to I$ as $h \to 0$ in the sense of Γ convergence with respect to strong convergence in $L^2(\omega)$.*

Proof. Since $s_h(v_h, v_h) \geq 0$ the first part of the proof of Proposition 3.1 remains valid. The second part also holds by noting that condition (S4) applies to the sequence used in the proof. □

Fully conforming methods fail to converge so that an inconsistency in the approximation of the differential operator or the boundary conditions is necessary.

Proposition 3.4 (Necessary condition)**.** *Assume that $\partial\omega$ contains a curved part, that $V_h \subset H^2(\omega_h) \cap H_0^1(\omega_h)$, and $D_h^2 = D^2$ for all $h > 0$. Then the functionals I_h, $h > 0$, are not Γ-convergent to I as $h \to 0$.*

Proof. Assume that $I_h(v_h) \to I(v)$ for some $v \in V$ and a sequence $(v_h)_{h>0}$ with $v_h \in V_h$ and $v_h \to v$. Since I_h is quadratic in $D^2 v_h$, it then follows that $D^2 v_h \to D^2 v$ and $\Delta v_h \to \Delta v$ in $L^2(\omega)$. This however contradicts the convergence of the representations (3) of I_h to the representation (2) of I

with boundary integral terms provided by Lemma 2.2 unless the boundary of ω is piecewise straight or $\partial_n v = 0$ on $\partial\omega$. $\square$

The assumed convexity condition on ω simplifies certain arguments but can be avoided.

Remark 3.5. *If ω is not convex then by choosing a sufficiently large domain $\widehat{\omega}$ that contains the approximating domains ω_h one establishes convergence in $H^1_0(\widehat{\omega})$ and shows that limitting functions are supported in $\overline{\omega}$.*

4. Numerical methods

We apply the abstract results of the previous section to prototypical finite element methods for fourth order problems. Throughout this section we assume that the conditions (T1)-(T2) on the triangulations are satisfied. We always extend functionals by the value $+\infty$ to the entire set $L^2(\omega)$.

4.1. A conforming method.

For a triangle $T \in \mathcal{T}_h$ with vertices z_0, z_1, z_2 the Argyris element is given as the triple $(T, P_5(T), K_T)$ with the set of node functionals K_T containing the functionals $\chi_{i,\alpha}(v) = \partial^\alpha v(z_i)$ for $i = 0,1,2$, $\alpha \in \mathbb{N}_0^2$ with $\alpha_1 + \alpha_2 \le 2$, and $\chi_{i,n} = \nabla v(x_{S_i}) \cdot n_{S_i}$ associated with the sides S_i, $i = 0,1,2$, of T with midpoints x_{S_i} and outer normals n_{S_i}. The element is illustrated in Figure 3 and leads to the space

$$\mathcal{S}^{5,1}(\mathcal{T}_h) = \{v_h \in C^1(\overline{\omega}_h) : v_h|_T \in P_5(T) \text{ for all } T \in \mathcal{T}_h\}.$$

For subspaces V_h of $\mathcal{S}^{5,1}(\mathcal{T}_h)$ containing boundary conditions we consider the minimization of

$$I_h^{\mathrm{arg}}(v_h) = \frac{\sigma}{2}\int_{\omega_h} |\Delta v_h|^2 \,\mathrm{d}x + \frac{1-\sigma}{2}\int_{\omega_h} |D^2 v_h|^2 \,\mathrm{d}x$$

in the set of functions $v_h \in V_h$.

Proposition 4.1 (Failure of convergence)**.** *Let $V_h^{\mathrm{arg},o} = H^1_0(\omega_h) \cap \mathcal{S}^{5,1}(\mathcal{T}_h)$ and assume that $\partial\omega$ contains a curved part. Then the functionals $I_h^{\mathrm{arg}} : V_h^{\mathrm{arg},o} \to \mathbb{R}$ are not Γ-convergent to the functional I.*

Proof. The result is a direct consequence of Proposition 3.4. $\square$

Reducing the discrete boundary condition to the vertices on the boundary leads to correct convergence.

Proposition 4.2 (Correct convergence)**.** *Let*

$$V_h^{\mathrm{arg}} = \big\{v_h \in \mathcal{S}^{5,1}(\mathcal{T}_h) : v_h(z) = 0 \text{ for all } z \in \mathcal{N}_h \cap \partial\omega_h\big\}.$$

Then the functionals $I_h^{\mathrm{arg}} : V_h^{\mathrm{arg}} \to \mathbb{R}$ are Γ-convergent to I as $h \to 0$.

Proof. We verify the conditions of Proposition 3.1 to deduce the result.
(i) Given $v_h \in V_h^{\mathrm{arg}}$, we have that $\mathcal{I}_h^{\mathrm{p1}} v_h \in \mathcal{S}^1_0(\mathcal{T}_h)$ so that an integration by parts leads to

$$\|\nabla \mathcal{I}_h^{\mathrm{p1}} v_h\|^2 = -\int_{\omega_h} \mathcal{I}_h^{\mathrm{p1}} v_h \Delta v_h \,\mathrm{d}x + \int_{\omega_h} \nabla \mathcal{I}_h^{\mathrm{p1}} v_h \cdot \nabla(\mathcal{I}_h^{\mathrm{p1}} v_h - v_h)\,\mathrm{d}x.$$

A Poincaré inequality and a nodal interpolation estimate imply that

$$\|\nabla \mathcal{I}_h^{\mathrm{p1}} v_h\| \lesssim \|\Delta v_h\| + h\|D^2 v_h\|,$$

which guarantees the uniform coercivity property (S1).
(ii) Since $D_h^2 = D^2$ the stability requirement (S2) of the discrete Hessian is trivially satisfied.
(iii) A modification of the canonical interpolation operator $\mathcal{I}_h^{\mathrm{arg}} : H^4(\omega) \to \mathcal{S}^{5,1}(\mathcal{T}_h)$ defined by the node functionals is required to interpolate functions in $H^3(\omega)$. Given $v \in H^3(\omega) \cap H_0^1(\omega)$ we determine polynomials $p_T \in P_5(T)$ via the conditions $\chi_{i,\alpha}(p_T - v) = 0$, if $|\alpha| \le 1$, and $\chi_{i,n}(p_T - v) = 0$, and

$$\widetilde{\chi}_{i,\alpha}(p_T) = \frac{1}{|\omega_{z_i}|} \int_{\omega_{z_i}} \partial^\alpha v \,\mathrm{d}x,$$

if $|\alpha| = 2$ and for $i = 0, 1, 2$, where ω_{z_i} is the union of elements in $\mathcal{T}_h$ that contain z. The modified interpolant $\widetilde{\mathcal{I}}_h^{\mathrm{arg}} v \in V_h$ of v is then defined via $(\widetilde{\mathcal{I}}_h^{\mathrm{arg}} v)|_T = p_T$ for all $T \in \mathcal{T}_h$. Owing to $v|_{\partial\omega} = 0$ we have that $v_h(z) = 0$ for all $z \in \mathcal{N}_h \cap \partial\omega_h$, so that $\widetilde{\mathcal{I}}_h^{\mathrm{arg}} v \in V_h^{\mathrm{arg}}$. Since $\widetilde{\mathcal{I}}_h^{\mathrm{arg}}$ reproduces quadratic functions, an interpolation estimate follows from the Bramble–Hilbert lemma, which implies condition (S3). □

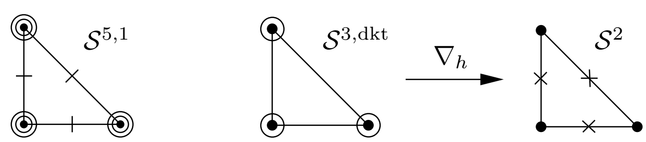

Figure 3. Schematical description of the Argyris element (left) and the discrete Kirchhoff triangle with cubic and quadratic polynomial spaces (right).

4.2. **A nonconforming element.** The discrete Kirchhoff triangle uses C^0 conforming scalar and vectorial spaces $\mathcal{S}^{3,\mathrm{dkt}}(\mathcal{T}_h) \subset H^1(\omega_h)$ and $\mathcal{S}^2(\mathcal{T}_h)^2 \subset H^1(\omega_h; \mathbb{R}^2)$, respectively, and a discrete gradient operator

$$\nabla_h : \mathcal{S}^{3,\mathrm{dkt}}(\mathcal{T}_h) \to \mathcal{S}^2(\mathcal{T}_h)^2,$$

cf. Figure 3, that approximates the weak gradient. With this, we define the operators $D_h^2 v_h = \nabla\nabla_h v_h$ and $\Delta_h v_h = \operatorname{div} \nabla_h v_h$ for $v_h \in \mathcal{S}^{3,\mathrm{dkt}}(\mathcal{T}_h)$ and the discrete functionals

$$I_h^{\mathrm{dkt}}(v_h) = \frac{\sigma}{2} \int_{\omega_h} |\Delta_h v_h|^2 \,\mathrm{d}x + \frac{1-\sigma}{2} \int_{\omega_h} |D_h^2 v_h|^2 \,\mathrm{d}x,$$

defined for $v_h \in V_h^{\mathrm{dkt}}$ with

$$V_h^{\mathrm{dkt}} = \big\{ v_h \in \mathcal{S}^{3,\mathrm{dkt}}(\mathcal{T}_h) : v_h(z) = 0 \text{ for all } z \in \mathcal{N}_h \cap \partial\omega_h \big\}.$$

Letting $\nabla_{\mathcal{T}}$ and $D^2_{\mathcal{T}}$ denote the elementwise application of the gradient and the Hessian, we have the equivalence of the seminorms $\|D^2_{\mathcal{T}} v_h\|$ and $\|D^2_h v_h\|$, and for all $v_h \in \mathcal{S}^{3,\mathrm{dkt}}(\mathcal{T}_h)$ and $v \in H^3(\omega)$ the estimates, cf. [9, 6],

$$\|\nabla_h \mathcal{I}_h^{3,\mathrm{dkt}} v - \nabla v\| + h\|D_h^2 \mathcal{I}_h^{3,\mathrm{dkt}} v - D^2 v\| \lesssim h^2 \|D^3 v\|,$$
$$\|\nabla_h v_h - \nabla v_h\| \lesssim h\|D^2_{\mathcal{T}} v_h\|.$$

The following proposition shows that imposing the boundary condition only in the boundary nodes is sufficient to avoid incorrect convergence.

Proposition 4.3 (Correct convergence)**.** *Let*

$$V_h^{\mathrm{dkt}} = \big\{ v_h \in \mathcal{S}^{3,\mathrm{dkt}}(\mathcal{T}_h) : v_h(z) = 0 \text{ for all } z \in \mathcal{N}_h \cap \partial\omega \big\}.$$

Then the functionals I_h^{dkt} restricted to V_h^{dkt} are Γ-convergent to I.

Proof. We verify the conditions of Proposition 3.1 to deduce the result.
(i) As in the proof of Proposition 4.2 we find that

$$\|\nabla \mathcal{I}_h^{\mathrm{p1}} v_h\|^2 = -\int_{\omega_h} \mathcal{I}_h^{\mathrm{p1}} v_h \cdot \Delta_h v_h \,\mathrm{d}x + \int_{\omega_h} \nabla \mathcal{I}_h^{\mathrm{p1}} v_h \cdot (\nabla \mathcal{I}_h^{\mathrm{p1}} v_h - \nabla_h v_h) \,\mathrm{d}x,$$

which implies the condition (S1) after application of a Poincaré inequality, i.e.,

$$\|\nabla \mathcal{I}_h^{\mathrm{p1}} v_h\| \lesssim \|\Delta_h v_h\| + h\|D^2_{\mathcal{T}} v_h\|.$$

(ii) Let $(v_h)_{h>0}$ be a sequence of functions $v_h \in V_h$ with $v_h \to v$ in $L^2(\omega)$ as $h \to 0$. For every compactly supported function $\phi \in C_0^\infty(\omega; \mathbb{R}^{2\times 2})$ we then find with the approximation properties of ∇_h that

$$\int_\omega D_h^2 v_h : \phi \,\mathrm{d}x = -\int_\omega \nabla_h v_h \cdot \operatorname{Div} \phi \,\mathrm{d}x \to -\int_\omega \nabla v \cdot \operatorname{Div} \phi \,\mathrm{d}x$$

as $h \to 0$, and hence $v \in H^2(\omega)$ with $D^2 v = \lim_{h\to 0} D_h^2 v_h$, which provides condition (S2).
(iii) The canonical nodal interpolation operator $\mathcal{I}_h^{\mathrm{dkt}} : H^2(\omega_h) \to \mathcal{S}^{3,\mathrm{dkt}}(\mathcal{T}_h)$ defined by $\mathcal{I}_h^{\mathrm{dkt}} v(z) = v(z)$ and $\nabla \mathcal{I}_h^{\mathrm{dkt}} v(z) = \nabla v(z)$ has the features needed to guarantee (S3). ☐

Remark 4.4. *By following the arguments of this subsection, convergence of discretizations based on the Morley element can be established, cf., e.g., [20].*

4.3. **A dG method.** We consider a simple discontinuous Galerkin method that does not fit exactly into the general framework provided by Proposition 3.1 but the convergence analysis only requires minor modifications. The method uses the space

$$V_h^{\mathrm{dg}} = \mathcal{L}^\ell(\mathcal{T}_h)$$

of elementwise polynomials of degree at most $\ell \geq 0$ and the functionals

$$I_h^{\mathrm{dg}}(v_h) = \frac{1}{2} a_h(v_h, v_h) + \frac{1}{2} s_h(v_h, v_h).$$

Here, a_h encodes a discretization of the weak form of the differential operator defined by I and s_h are stabilizing bilinear forms. For simplicity, we consider the case $\sigma = 0$ so that the elastic energy is a multiple of the integral of the squared norm of the Hessian. In the following and in contrast to the previous subsection, the symbols

$$\nabla_h,\ D_h^2,\ \Delta_h$$

denote here the elementwise application of the indicated differential operators. A bilinear form resulting from an elementwise integration by parts and a consistent symmetrization is given by

$$\begin{aligned} a_h(v_h, w_h) = (D_h^2 v_h, D_h^2 w_h) & \\ + (\{\partial_n \nabla_h v_h\}, [\![\nabla_h w_h]\!])_{\cup \mathcal{S}_h \setminus \partial\omega_h} &+ (\{\partial_n \nabla_h w_h\}, [\![\nabla_h v_h]\!])_{\cup \mathcal{S}_h \setminus \partial\omega_h} \\ - (\{\partial_n \Delta_h v_h\}, [\![w_h]\!])_{\cup \mathcal{S}_h} &- (\{\partial_n \Delta_h w_h\}, [\![v_h]\!])_{\cup \mathcal{S}_h}. \end{aligned}$$

With factors $\gamma_0, \gamma_1 > 0$ and the length function $h_{\mathcal{S}}|_S = h_S$ of sides, the stabilizing bilinear form is given by

$$\begin{aligned} s_h(v_h, w_h) = \gamma_0 (h_{\mathcal{S}}^{-3} [\![v_h]\!], [\![w_h]\!])_{\cup \mathcal{S}_h} & \\ + \gamma_1 (h_{\mathcal{S}}^{-1} [\![\nabla_h v_h]\!], [\![\nabla_h w_h]\!])_{\cup \mathcal{S}_h \setminus \partial\omega_h}&. \end{aligned}$$

The convergence analysis of the functionals makes repeated use of the scaled trace inequality in the form $\|h_{\mathcal{S}}^{1/2} v_h\|_{L^2(\cup\mathcal{S}_h)} \lesssim \|v_h\|$ for every $v_h \in V_h^{\rm dg}$.

Equicoercivity in $H_0^1(\omega)$. We define the discrete norm

$$\|v_h\|_{\rm dg}^2 = \|D_h^2 v_h\|^2 + s_h(v_h, v_h)$$

and note that with trace inequalities and inverse estimates one obtains for $\gamma_0, \gamma_1 > 0$ sufficiently large that there exists $\alpha > 0$ with

$$a_h(v_h, v_h) \ge \alpha \|v_h\|_{\rm dg}^2.$$

Given $v_h \in V_h^{\rm dg}$ we consider $\mathcal{J}_h^{\rm av} v_h \in H_0^1(\omega_h)$ and note via elementwise integration by parts, Hölder, Poincaré, and trace inequalities, that

$$\begin{aligned} \|\nabla \mathcal{J}_h^{\rm av} v_h\|^2 &= \int_{\omega_h} \nabla_h v_h \cdot \nabla \mathcal{J}_h^{\rm av} v_h \,\mathrm{d}x + \int_{\omega_h} \nabla_h (\mathcal{J}_h^{\rm av} v_h - v_h) \cdot \nabla \mathcal{J}_h^{\rm av} v_h \,\mathrm{d}x \\ &\lesssim \big(\|\Delta v_h\| + \|h_{\mathcal{S}}^{-1/2} [\nabla_h v_h]\|_{\cup\mathcal{S}_h} + \|\nabla_h (\mathcal{J}_h^{\rm av} v_h - v_h)\|\big) \|\nabla \mathcal{J}_h^{\rm av} v_h\|. \end{aligned}$$

Incorporating the estimate for the node averaging operator, we deduce that

$$\|\nabla \mathcal{J}_h^{\rm av} v_h\| \lesssim \|\Delta v_h\| + \|h_{\mathcal{S}}^{-1/2} [\![\nabla_h v_h]\!]\|_{\cup\mathcal{S}_h} + \|h_{\mathcal{S}}^{-1/2} [\![v_h]\!]\| \lesssim \|v_h\|_{\rm dg}.$$

This establishes the uniform coercivity in $H_0^1(\omega)$.

Interpolation and stabilization in V_h. Given a function $v \in H^3(\omega) \cap H_0^1(\omega)$ we consider the quadratic Lagrange interpolant $v_h = \mathcal{I}_h^{2,0} v \in V_h^{\rm dg} \cap C(\overline{\omega}_h)$, assuming that V_h contains quadratic polynomials, i.e., $\ell \ge 2$. For every $T \in \mathcal{T}_h$ we have for $r = 0, 1, 2$ that

$$\|D^r (v_h - v)\|_{L^2(T)} \lesssim h_T^{3-r} \|D^3 v\|_{L^2(T)}.$$

In combination with the trace inequality one obtains that

$$\|h_{\mathcal{S}}^{-1/2} [\![\nabla_h v_h]\!]\|_{L^2(\mathcal{S}_h \setminus \partial\omega_h)} = \|h_{\mathcal{S}}^{-1/2} [\![\nabla_h (v_h - v)]\!]\|_{L^2(\mathcal{S}_h \setminus \partial\omega_h)} \lesssim h \|D^3 v\|.$$

On inner sides we have $[\![v_h]\!] = 0$ while for the union of boundary sides $S \subset \partial\omega_h$ we have

$$\|[\![v_h]\!]\|_{L^2(\partial\omega_h)} \lesssim \|v\|_{L^\infty(\partial\omega_h)} \lesssim h^2 \|\nabla v\|_{L^\infty(\omega)},$$

where we used that $v|_{\partial\omega} = 0$ and for every $x \in S$ there exists $x' \in \partial\omega$ with $|x - x'| \leq ch^2$. Finally, we note that

$$\begin{aligned}([\![\partial_n \nabla_h v_h]\!], [\![\nabla_h v_h]\!])_S &\leq \|D_h^2 v_h\|_S \|[\![\nabla_h v_h]\!]\|_S \\ &\lesssim h_S^{-1/2} \|D_h^2 v_h\|_{L^2(T_S)} h_S^{3/2} \|D^3 v\|_{L^2(T_S)} \\ &\lesssim h_S \|D^3 v\|_{L^2(T_S)} \|D_h^2 v_h\|_{L^2(T_S)}.\end{aligned}$$

Altogether, sums of squared terms related to sides in a_h vanish as $h \to 0$, we have $s_h(v_h, v_h) \to 0$, and in particular for $h \to 0$ that

$$I_h^{\mathrm{dg}}(v_h) \to I(v).$$

Hessian stability. To establish a liminf inequality, we follow the arguments of [8]. Therein, a discrete Hessian matrix $H_h(v_h)$ is constructed for every $v_h \in V_h$ via a lifting operation, which satisfies the relation

$$\begin{aligned}(H_h(v_h), D_h^2 w_h) &= (D_h^2 v_h, D_h^2 w_h) \\ &\quad - ([\![\nabla_h v_h]\!], \{\partial_n \nabla_h w_h\})_{\cup\mathcal{S}_h} + ([\![v_h]\!], \{\partial_n \Delta_h w_h\})_{\cup\mathcal{S}_h},\end{aligned}$$

and obeys the inequality

$$\|H_h(v_h)\| \leq a_h(v_h, v_h).$$

Moreover, if $(v_h)_{h>0}$ is a sequence of functions $v_h \in V_h$ such that $a_h(v_h, v_h)$ is uniformly bounded and $v_h \to v$ in $L^2(\omega)$, then there exists $v \in H^2(\omega)$ with

$$(H_h(v_h), \phi) \to (D^2 v, \phi)$$

for all $\phi \in C_0^\infty(\omega; \mathbb{R}^{2\times 2})$. This convergence is established in [8] for a fixed domain that is accurately triangulated. However, since test functions are compactly supported the result carries over to the case of approximated domains.

The equicoercivity in $H_0^1(\omega)$, the interpolation property, and the Hessian consistency imply the following Γ convergence result.

Proposition 4.5 (Correct convergence)**.** *Let V_h^{dg} contain the set of elementwise quadratic, globally continuous polynomials, and assume that $\gamma_0, \gamma_1 > 0$ are sufficiently large. Then the functionals $I_h^{\mathrm{dg}} : V_h^{\mathrm{dg}} \to \mathbb{R}$ are Γ-convergent to I.*

Remark 4.6. *Local discontinuous Galerkin methods as discussed in [7] replace the Hessian in the functional I by the approximation H_h and thereby fit into the abstract framework of Proposition 3.1 for arbitrary choices of the parameters $\gamma_0, \gamma_1 > 0$.*

5. Numerical experiments

We verify the theoretical results via numerical experiments for the setting considered in [4]. All experiments were carried out using elementary realizations in Matlab as in [6, 5].

Example 5.1 (Simple support on unit disk)**.** *Let* $\omega = B_1(0)$ *and* $f = 1$. *Then the minimizer* $u \in V = H^2(\omega) \cap H_0^1(\omega)$ *for the functional* I *satisfies* $\Delta^2 u = f$, $u = \Delta u - (1-\sigma)\partial_n u = 0$ *and is given by*

$$u(x) = \frac{(5+\sigma) - (6+2\sigma)|x|^2 + (1+\sigma)|x|^4}{64(1+\sigma)}.$$

The solution obtained as a limit of an operator splitting on polygonal domains solves $\Delta^2 u_\infty = f$ *and* $u_\infty = \Delta u_\infty = 0$ *in* $\partial\omega$ *and is given by*

$$u_\infty(x) = \frac{3}{64} - \frac{1}{16}|x|^2 + \frac{1}{64}|x|^4.$$

We always set $\sigma = 0$.

We test the methods analyzed above on triangulations that are obtained via uniform refinements of triangulations of a square via the correction $z \mapsto \sqrt{2}|z|_\infty z/|z|_2$, cf. Figure 4. Corresponding nodal interpolants of the functions u and u_∞ are shown in Figure 1.

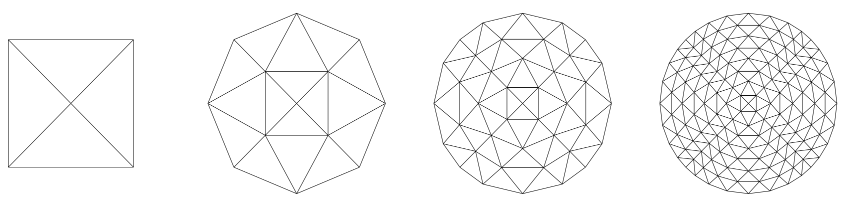

Figure 4. Triangulations obtained from refinements and subsequent corrections of triangulations of a square.

5.1. **Conforming method.**

Figure 5 shows finite element solutions using the Argyris element for a full realization of the boundary conditions as well as their reduced, nodal treatment as introduced in Section 4.1. We see that the maximal values at the center of the disk differ substantially and only the reduced treatment of the boundary conditions leads to correct approximations. The effect of the reduced treatment is visualized in Figure 6, where we observe that small arcs form along boundary sides that provide the right flexibility for the approximations to attain the correct maximal values.

5.2. **Nonconforming method.**

Approximating the solution of the model problem using the discrete Kirchhoff element as in Section 4.2 leads to the approximation shown in the left plot of Figure 7. The numerical solution accurately approximates the exact solution.

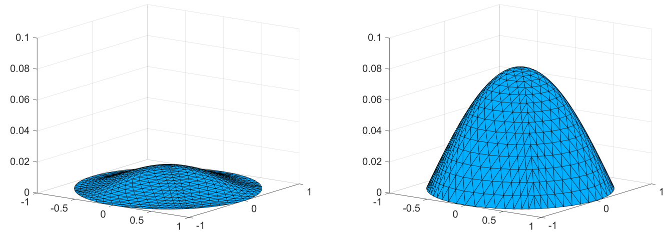

FIGURE 5. Finite element approximations obtained with the Argyris element and a full (left) and reduced (right) realization of the simple support boundary conditions.

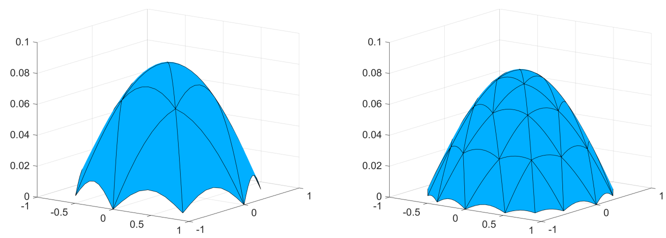

FIGURE 6. Coarse finite element approximations obtained with the Argyris element and simple support boundary conditions imposed in the corner points.

5.3. **DG method.** The discontinuous Galerkin method analyzed in Section 4.3 with parameters $\gamma_0 = \gamma_1 = 10$ provides the numerical solution shown in the right plot of Figure 7. As in the case of the nonconforming approximation, the maximal value accurately approximates the maximal value of the exact solution.

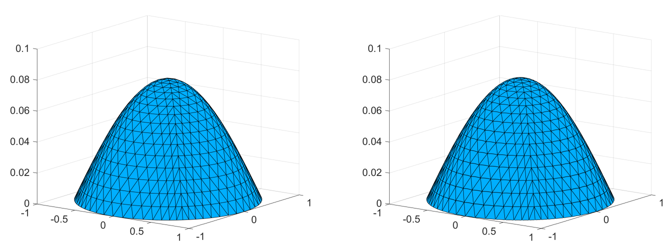

FIGURE 7. Finite element approximations obtained with the discrete Kirchhoff element (left) and a discontinuous Galerkin method (right).

5.4. **Convergence rates.** The plots shown in Figures 8 and 9 display the convergence behavior of the errors in approximating the midpoint value $u(0)$ and with respect to discrete H^2 norms, i.e., the error quantities

$$\delta_h^{\mathrm{mp}} = |u_h(0) - u(0)|, \quad \delta_h^{H^2} = \|u_h - \mathcal{I}_h u\|_{a_h},$$

where a_h is the discrete bilinear form defined by the different numerical methods and $\mathcal{I}_h$ is the corresponding nodal interpolant. We observe from Figure 8 that the approximations obtained with the Argyris method and reduced boundary condition (Argyris, nodal support), with the discrete Kirchhoff method (DKT), and discontinuous Galerkin methods with quadratic polynomials and linear (DG, $P1$ boundary) as well as isoparametric quadratic (DG, $P2$ boundary) approximations of the domain boundary provide highly accurate approximations of the exact value $u(0) = 5/64$. A $P1$ implementation of the operator splitting approach leads to the expected approximation of the incorrect value $u_\infty(0) = 3/64$. The Argyris element with simple support boundary along the entire boundary (Argyris, full support) indicates convergence to another, lower and incorrect, value. A critical conditioning of the system matrix defined by the Argyris element leads to spurious values for fine meshes.

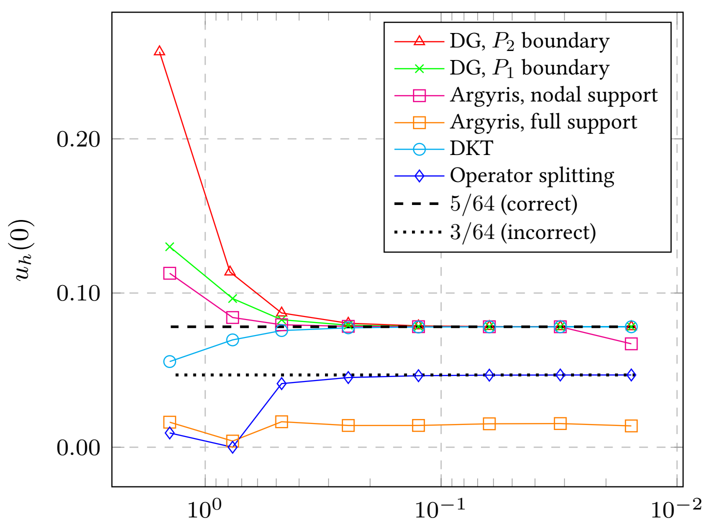

FIGURE 8. Midpoint values for different numerical methods and mesh sizes in Example 5.1. The Argyris method with simple support condition on the entire boundary and the operator splitting approach lead to incorrect approximations.

The convergence behavior of the H^2 errors shown in Figure 9 confirms that not only the midpoint values are correctly approximated but convergence to the exact solution takes place for the Argyris method with nodal boundary condition, the discrete Kirchhoff element, as well as for the affine and isoparametric variants of the quadratic discontinuous Galerkin methods. While all methods lead to positive convergence rates, the nonconforming method leads to a quadratic experimental convergence rate despite the involved domain approximations.

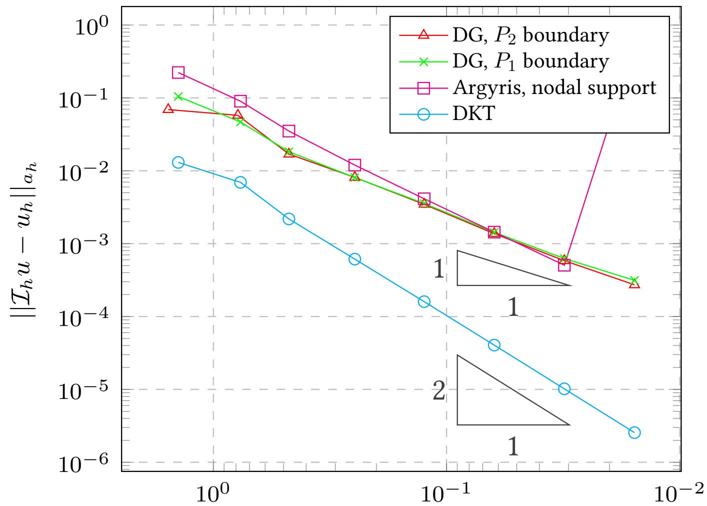

FIGURE 9. Experimental convergence behavior for the Argyris method with boundary conditions imposed at the corner points, the discrete Kirchhoff method, and quadratic discontinuous Galerkin methods.

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Abteilung für Angewandte Mathematik, Albert-Ludwigs-Universität Freiburg, Hermann-Herder-Str. 10, 79104 Freiburg i. Br., Germany
Email address: bartels@mathematik.uni-freiburg.de

Abteilung für Angewandte Mathematik, Albert-Ludwigs-Universität Freiburg, Hermann-Herder-Str. 10, 79104 Freiburg i. Br., Germany
Email address: philipp.tscherner@mathematik.uni-freiburg.de