ORTHOGONALITY RELATIONS OF CROUZEIX–RAVIART AND RAVIART–THOMAS FINITE ELEMENT SPACES

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Abstract. Identities that relate projections of Raviart–Thomas finite element vector fields to discrete gradients of Crouzeix–Raviart finite element functions are derived under general conditions. Various implications such as discrete convex duality results and a characterization of the image of the projection of the Crouzeix–Ravaiart space onto elementwise constant functions are deduced.

1. Introduction

Recent developments in the numerical analysis of total variation regularized and related nonsmooth minimization problems show that nonconforming and discontinuous finite element methods lead to optimal convergence rates under suitable regularity conditions [CP19; Bar20b; Bar20a]. This is in contrast to standard conforming methods which often perform suboptimally [BNS15]. A key ingredient in the derivation of quasi-optimal error estimates are discrete convex duality results which exploit relations between Crouzeix–Raviart and Raviart–Thomas finite element spaces introduced in [CR73] and [RT77]. In particular, assume that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with a partitioning of the boundary into subsets $\Gamma_N, \Gamma_D \subset \partial\Omega$, and let $\mathcal{T}_h$ be a regular triangulation of Ω. For a function $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and a vector field $y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ we then have the integration-by-parts formula

$$\int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x = -\int_\Omega v_h \operatorname{div} y_h \,\mathrm{d}x.$$

Important aspects here are that despite the possible discontinuity of v_h and y_h no terms occur that are related to interelement sides and that the vector field y_h and the function v_h can be replaced by their elementwise averages on the left- and right-hand side, respectively. In combination with Fenchel's inequality this implies a weak discrete duality relation.

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The validity of a strong discrete duality principle has been established in [CP19; Bar20b] under certain differentiability or more generally approximability properties of minimization problems using the orthogonality relation

$$\big(\Pi_h \mathcal{RT}^0_N(\mathcal{T}_h)\big)^\perp = \nabla_h\big(\ker \Pi_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)}\big), \tag{1}$$

within the space of piecewise constant vector fields $\mathcal{L}^0(\mathcal{T}_h)^d$ equipped with the L^2 inner product and with ∇_h and Π_h denoting the elementwise application of the gradient and orthogonal projection onto $\mathcal{L}^0(\mathcal{T}_h)^d$, respectively, ker denotes the kernel of an operator. The identity implies that if a vector field $w_h \in \mathcal{L}^0(\mathcal{T}_h)^d$ satisfies

$$\int_\Omega w_h \cdot \nabla_h v_h \,\mathrm{d}x = 0$$

for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ with $\Pi_h v_h = 0$ then there exists a vector field $z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ such that

$$w_h = \Pi_h z_h.$$

Note that this is a stronger implication then the well known result that if w_h is orthogonal to discrete gradients of *all* Crouzeix–Raviart functions then it belongs to the Raviart–Thomas finite element space. Although strong duality is not required in the error analysis, it reveals a compatibility property of discretizations and indicates optimality of estimates. Moreover, it is related to postprocessing procedures that provide the solution of computationally expensive discretized dual problems via simple postprocessing procedures of numerical solutions of less expensive primal problems, cf. [Mar85; AB85; CL15; Bar20b].

The proof of (1) given in [CP19] makes use of a discrete Poincaré lemma which is valid if the Dirichlet boundary $\Gamma_D \subset \partial\Omega$ is empty or if $d = 2$ and Γ_D is connected. In this note we show that (1) can be established for general boundary partitions by avoiding the use of the discrete Poincaré lemma. The new proof is based on the surjectivity property of the discrete divergence operator

$$\mathrm{div} : \mathcal{RT}^0_N(\mathcal{T}_h) \to \mathcal{L}^0(\mathcal{T}_h).$$

This is a fundamental property for the use of the Raviart–Thomas method for discretizing saddle-point problems, cf. [RT77; BBF13]. It is an elementary consequence of a projection property of a quasi-interpolation operator $\mathcal{I}_{\mathcal{RT}} : H^s(\Omega;\mathbb{R}^d) \to \mathcal{RT}^0_N(\mathcal{T}_h)$ and the surjectivity of the divergence operator onto the space $L^2(\Omega)$.

Our arguments also provide a dual version of the orthogonality relation (1) which states that

$$\mathrm{div}\,\big(\ker \Pi_h|_{\mathcal{RT}^0_N(\mathcal{T}_h)}\big) = \big(\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h)\big)^\perp. \tag{2}$$

Unless $\Gamma_D = \partial\Omega$ we have that the left-hand side is trivial and hence the identity yields that

$$\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h) = \mathcal{L}^0(\mathcal{T}_h),$$

i.e., that the projection of Crouzeix–Raviart functions onto elementwise constant functions is a surjection. If $\Gamma_D = \partial\Omega$ then depending on the triangulation both equality or strict inclusion occur. This observation reveals that the discretizations of total-variation regularized problems devised in [CP19; Bar20b] can be seen as discretizations using elementwise constant functions with suitable nonconforming discretizations of the total variation functional.

The most important consequence of (2) is the strong duality relation for the discrete primal problem defined by minimizing the functional

$$I_h(u_h) = \int_\Omega \phi(\nabla_h u_h) + \psi_h(x, \Pi_h u_h)\,\mathrm{d}x$$

in the space $\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and the discrete dual problem consisting in maximizing the functional

$$D_h(z_h) = -\int_\Omega \phi^*(\Pi_h z_h) + \psi_h^*(x, \operatorname{div} z_h)\,\mathrm{d}x$$

in the space $\mathcal{RT}^0_N(\mathcal{T}_h)$. The functions ϕ and ψ_h are suitable convex functions with convex conjugates ϕ^* and ψ_h^*, we refer the reader to [Bar20b] for details.

This article is organized as follows. In Section 2 we define the required finite element spaces along with certain projection operators. Our main results are contained in Section 3, where we prove the identities (1) and (2) and deduce various corollaries. In the Appendix A we provide a proof of the discrete Poincaré lemma that leads to an alternative proof of the main result under certain restrictions.

2. Preliminaries

2.1. **Triangulations.** Throughout what follows we let $(\mathcal{T}_h)_{h>0}$ be a sequence of regular triangulations of the bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$, cf. [BS08; Cia78]. We let $P_k(T)$ denote the set of polynomials of maximal total degree k on $T \in \mathcal{T}_h$ and define the set of elementwise polynomial functions or vector fields

$$\mathcal{L}^k(\mathcal{T}_h)^\ell = \{w_h \in L^\infty(\Omega;\mathbb{R}^\ell) : w_h|_T \in P_k(T) \text{ for all } T \in \mathcal{T}_h\}.$$

The parameter $h > 0$ refers to the maximal mesh-size of the triangulation $\mathcal{T}_h$. The set of sides of elements is denoted by $\mathcal{S}_h$. We let x_S and x_T denote the midpoints (barycenters) of sides and elements, respectively. The L^2 projection onto piecewise constant functions or vector fields is denoted by

$$\Pi_h : L^1(\Omega;\mathbb{R}^\ell) \to \mathcal{L}^0(\mathcal{T}_h)^\ell.$$

For $v_h \in \mathcal{L}^1(\mathcal{T}_h)^\ell$ we have $\Pi_h v_h|_T = v_h(x_T)$ for all $T \in \mathcal{T}_h$. We repeatedly use that Π_h is self-adjoint, i.e.,

$$(\Pi_h v, w) = (v, \Pi_h w)$$

for all $v, w \in L^1(\Omega;\mathbb{R}^\ell)$ with the L^2 inner product $(\cdot,\cdot)$.

2.2. **Crouzeix–Raviart finite elements.** The Crouzeix–Raviart finite element space introduced in [CR73] consists of piecewise affine functions that are continuous at the midpoints of sides of elements, i.e.,

$$\mathcal{S}^{1,cr}(\mathcal{T}_h) = \{v_h \in \mathcal{L}^1(\mathcal{T}_h) : v_h \text{ continuous in } x_S \text{ for all } S \in \mathcal{S}_h\}.$$

The elementwise application of the gradient operator to a function $v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ defines an elementwise constant vector field $\nabla_h v_h$ via

$$\nabla_h v_h|_T = \nabla(v_h|_T)$$

for all $T \in \mathcal{T}_h$. For $v \in W^{1,1}(\Omega)$ we have $\nabla_h v = \nabla v$. Functions vanishing at midpoints of boundary sides on Γ_D are contained in

$$\mathcal{S}^{1,cr}_D(\mathcal{T}_h) = \{v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h) : v_h(x_S) = 0 \text{ for all } S \in \mathcal{S}_h \text{ with } S \subset \Gamma_D\}.$$

A basis of the space $\mathcal{S}^{1,cr}(\mathcal{T}_h)$ is given by the functions $\varphi_S \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$, $S \in \mathcal{S}_h$, satisfying the Kronecker property

$$\varphi_S(x_{S'}) = \delta_{S,S'}$$

for all $S, S' \in \mathcal{S}_h$. The function φ_S vanishes on elements that do not contain the side S and is continuous with value 1 along S. A quasi-interpolation operator is for $v \in W^{1,1}(\Omega)$ defined via

$$\mathcal{I}_{cr} v = \sum_{S \in \mathcal{S}_h} v_S \varphi_S, \quad v_S = |S|^{-1} \int_S v \,\mathrm{d}s,$$

We have that $\mathcal{I}_{cr}$ preserves averages of gradients, i.e.,

$$\nabla_h \mathcal{I}_{cr} v = \Pi_h \nabla v,$$

which follows from an integration by parts, cf. [BBF13; Bar16].

2.3. **Raviart–Thomas finite elements.** The Raviart–Thomas finite element space of [RT77] is defined as

$$\begin{aligned} \mathcal{R}T^0(\mathcal{T}_h) = \{y_h \in & H(\mathrm{div};\Omega) : y_h|_T(x) = a_T + b_T(x - x_T), \\ & a_T \in \mathbb{R}^d,\, b_T \in \mathbb{R} \text{ for all } T \in \mathcal{T}_h\}, \end{aligned}$$

where $H(\mathrm{div};\Omega)$ is the set of L^2 vector fields whose distributional divergence belongs to $L^2(\Omega)$. Vector fields in $\mathcal{R}T^0(\mathcal{T}_h)$ have continuous constant normal components on element sides. The subset of vector fields with vanishing normal component on the Neumann boundary Γ_N is defined as

$$\mathcal{R}T^0_N(\mathcal{T}_h) = \{y_h \in \mathcal{R}T^0(\mathcal{T}_h) : y_h \cdot n = 0 \text{ on } \Gamma_N\},$$

where n is the outer unit normal on $\partial\Omega$. A basis of the space $\mathcal{R}T^0(\mathcal{T}_h)$ is given by vector fields ψ_S associated with sides $S \in \mathcal{S}_h$. Each vector field ψ_S is supported on adjacent elements $T_\pm \in \mathcal{T}_h$ with

$$\psi_S(x) = \pm \frac{|S|}{d!|T_\pm|}(z_{S,T_\pm} - x) \tag{3}$$

for $x \in T_\pm$ with opposite vertex $z_{S,T_\pm}$ to $S \subset \partial T_\pm$. We have the Kronecker property

$$\psi_S|_{S'} \cdot n_{S'} = \delta_{S,S'}$$

for all sides S' with unit normal vector $n_{S'}$, if $S' = S$ we assume that n_S points from T_- into T_+. A quasi-interpolation operator is for vector fields $z \in W^{1,1}(\Omega;\mathbb{R}^d)$ given by

$$\mathcal{I}_{\mathcal{RT}} z = \sum_{S\in\mathcal{S}_h} z_S \psi_S, \quad z_S = |S|^{-1} \int_S z \cdot n_S \,\mathrm{d}s.$$

For the operator $\mathcal{I}_{\mathcal{RT}}$ we have the projection property

$$\operatorname{div} \mathcal{I}_{\mathcal{RT}} z = \Pi_h \operatorname{div} z,$$

which is a consequence of an integration by parts, cf. [BBF13; Bar16]. This identity implies that the divergence operator defines a surjection from $\mathcal{RT}^0_N(\mathcal{T}_h)$ into $\mathcal{L}^0(\mathcal{T}_h)$, provided that constants are eliminated from $\mathcal{L}^0(\mathcal{T}_h)$ if $\Gamma_D = \emptyset$.

2.4. **Integration by parts.** An elementwise integration by parts implies that for $v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}^0(\mathcal{T}_h)$ we have the integration-by-parts formula

$$\int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x + \int_\Omega v_h \operatorname{div} y_h \,\mathrm{d}x = \int_{\partial\Omega} v_h \, y_h \cdot n \,\mathrm{d}s. \tag{4}$$

Here we used that y_h has continuous constant normal components on inner element sides and that jumps of v_h have vanishing integral mean. If an elementwise constant vector field $w_h \in \mathcal{L}^0(\mathcal{T}_h)^d$ satisfies

$$\int_\Omega w_h \cdot \nabla_h v_h \,\mathrm{d}x = 0$$

for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ then by choosing $v_h = \varphi_S$ for $S \in \mathcal{S}_h \setminus \Gamma_D$ one finds that its normal components are continuous on inner element sides and vanish on the Γ_N, so that $w_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$. We thus have the decomposition

$$\mathcal{L}^0(\mathcal{T}_h)^d = \ker(\operatorname{div}|_{\mathcal{RT}^0_N(\mathcal{T}_h)}) \oplus \nabla_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h),$$

where we used that $\ker(\operatorname{div}|_{\mathcal{RT}^0_N(\mathcal{T}_h)}) = \mathcal{L}^0(\mathcal{T}_h)^d \cap \mathcal{RT}^0_N(\mathcal{T}_h)$.

3. Orthogonality relations

The following identities and in particular their proofs and corollaries are the main contributions of this article.

Theorem 3.1 (Orthogonality relations)**.** *Within the sets of elementwise constant vector fields and functions $\mathcal{L}^0(\mathcal{T}_h)^\ell$ equipped with the L^2 inner product we have*

$$\big(\Pi_h \mathcal{RT}^0_N(\mathcal{T}_h)\big)^\perp = \nabla_h\big(\ker \Pi_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)}\big),$$

$$\operatorname{div}\big(\ker \Pi_h|_{\mathcal{RT}^0_N(\mathcal{T}_h)}\big) = \big(\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h)\big)^\perp.$$

Proof. (i) The integration-by-parts formula (4) implies

$$(\nabla_h v_h, \Pi_h y_h) = -(v_h, \operatorname{div} y_h) = -(\Pi_h v_h, \operatorname{div} y_h) = 0$$

if $\Pi_h v_h = 0$ and hence $\Pi_h \mathcal{RT}^0_N(\mathcal{T}_h) \subset \big[\nabla_h\big(\ker \Pi_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)}\big)\big]^\perp$. To prove the converse inclusion let $y_h \in \mathcal{L}^0(\mathcal{T}_h)^d$ be orthogonal to $\nabla_h\big(\ker \Pi_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)}\big)$. We show that there exists $\widetilde{y}_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ with $\Pi_h \widetilde{y}_h = y_h$. For this, let $Z_h = \big(\ker \Pi_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)}\big)^\perp \subset \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $r_h \in Z_h$ be the uniquely defined function with

$$(\Pi_h r_h, \Pi_h v_h) = (y_h, \nabla_h v_h)$$

for all $v_h \in Z_h$. The identity holds for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ since y_h is orthogonal to discrete gradients of functions $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ with $\Pi_h v_h = 0$. In particular, $\Pi_h r_h$ is orthogonal to constant functions if $\Gamma_D = \emptyset$. We choose $z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ with $-\operatorname{div} z_h = \Pi_h r_h$ and verify that

$$(y_h - z_h, \nabla_h v_h) = (\Pi_h r_h, \Pi_h v_h) + (\operatorname{div} z_h, v_h) = 0$$

for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$. We next define $\widetilde{y}_h|_T = y_h|_T + \operatorname{div} z_h|_T(x - x_T)/d$ for all $T \in \mathcal{T}_h$ and note that

$$(\widetilde{y}_h - z_h, \nabla_h v_h) = (y_h - z_h, \nabla_h v_h) = 0$$

for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$. Since $\widetilde{y}_h - z_h$ is elementwise constant, it follows that $\widetilde{y}_h - z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ and in particular $\widetilde{y}_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$. By definition of $\widetilde{y}_h$ we have $\Pi_h \widetilde{y}_h = y_h$ which proves the first asserted identity.
(ii) For the second statement we first note that if $\Pi_h y_h = 0$ for $y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ then

$$(\Pi_h v_h, \operatorname{div} y_h) = (v_h, \operatorname{div} y_h) = -(\nabla_h v_h, \Pi_h y_h) = 0$$

for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and hence $\operatorname{div} y_h \in \big(\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h)\big)^\perp$. It remains to show that

$$\big(\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h)\big)^\perp \subset \operatorname{div}\big(\ker \Pi_h|_{\mathcal{RT}^0_N(\mathcal{T}_h)}\big).$$

If $w_h \in \mathcal{L}^0(\mathcal{T}_h)$ is orthogonal to $\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ we choose $z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ with $\operatorname{div} z_h = w_h$ and note that

$$(\Pi_h z_h, \nabla_h v_h) = (z_h, \nabla_h v_h) = -(w_h, v_h) = -(w_h, \Pi_h v_h) = 0$$

for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$. This implies that $\Pi_h z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ and hence also $y_h = z_h - \Pi_h z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$. Since $\Pi_h y_h = 0$ and $\operatorname{div} y_h = w_h$ we deduce the second identity. □

An implication is a surjectivity property of the mapping $\Pi_h : \mathcal{S}^{1,cr}_D(\mathcal{T}_h) \to \mathcal{L}^0(\mathcal{T}_h)$ if $\Gamma_D \neq \partial\Omega$.

Corollary 3.2 (Surjectivity). *If $\Gamma_D \neq \partial\Omega$ then we have*

$$\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h) = \mathcal{L}^0(\mathcal{T}_h).$$

Otherwise, the subspace $\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h) \subset \mathcal{L}^0(\mathcal{T}_h)$ has codimension at most one.

Proof. (i) From Theorem 3.1 we deduce that the asserted identity holds if and only if $\operatorname{div}\ker \Pi_h|_{\mathcal{RT}^0_N(\mathcal{T}_h)} = \{0\}$. Since

$$\ker \Pi_h|_{\mathcal{RT}^0_N(\mathcal{T}_h)} = \{y_h \in \mathcal{RT}^0_N(\mathcal{T}_h) : y_h|_T = b_T(x - x_T) \text{ f.a. } T \in \mathcal{T}_h\},$$

the latter condition is equivalent to $\ker \Pi_h|_{\mathcal{RT}^0_N(\mathcal{T}_h)} = \{0\}$. Let $T \in \mathcal{T}_h$ such that a side $S_0 \subset \partial T$ belongs to Γ_N, i.e., we have $y_h|_T(x) = \sum_{j=0}^d \alpha_j (x - z_{S_j})$, where z_{S_j} is the vertex of T opposite to the side $S_j \subset \partial T$, and with $\alpha_0 = 0$. If $y_h(x_T) = 0$ then it follows that $\alpha_j = 0$ for $j = 1, \dots, d$ since the vectors $x_T - z_{S_j}$ are linearly independent. Starting from this element we may successively consider neighboring elements to deduce that $y_h|_T = 0$ for all $T \in \mathcal{T}_h$.
(ii) If $\Gamma_D = \partial\Omega$ we may argue as in (i) by removing one side $S \in \mathcal{S}_h \cap \Gamma_D$ from Γ_D, define $\Gamma_D{}' = \Gamma_D \setminus S$, and using the larger space $\mathcal{S}^{1,cr}_{D'}(\mathcal{T}_h)$. We then have $\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h) \subset \Pi_h \mathcal{S}^{1,cr}_{D'}(\mathcal{T}_h) = \mathcal{L}^0(\mathcal{T}_h)$. The difference is trivial if and only if $\Pi_h \varphi_S$ belongs to $\Pi_h \mathcal{S}^{1,cr}_{D'}(\mathcal{T}_h)$. □

The following examples show that both equality or strict inequality can occur if $\Gamma_D = \partial\Omega$.

Examples 3.3. *(i) Let $L \in \{1, 2\}$, $\mathcal{T}_h = \{T_1, \dots, T_L\}$, $\overline{\Omega} = T_1 \cup \dots \cup T_L$, $\Gamma_D = \partial\Omega$. Then $\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h) \simeq \mathbb{R}^{L-1}$ while $\mathcal{L}^0(\mathcal{T}_h) \simeq \mathbb{R}^L$.*
(ii) Let $\mathcal{T}_h = \{T_1, T_2, T_3\}$ be a triangulation consisting of the subtriangles obtained by connecting the vertices of a macro triangle T with its midpoint x_T. Let $\overline{\Omega} = T_1 \cup T_2 \cup T_3$ and $\Gamma_D = \partial\Omega$. We then have $\Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h) = \mathcal{L}^0(\mathcal{T}_h)$.

The second implication concerns discrete versions of convex duality relations. We let

$$\phi^*(s) = \sup_{r \in \mathbb{R}^\ell} s \cdot r - \phi(r)$$

be the convex conjugate of a given convex function $\phi \in C(\mathbb{R}^d)$.

Corollary 3.4 (Convex conjugation)**.** *Let $\overline{u}_h \in \Pi_h \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $\phi \in C(\mathbb{R}^d)$ be convex. We then have*

$$\begin{aligned} \inf\Big\{ \int_\Omega \phi(\nabla_h u_h) \,\mathrm{d}x : u_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h),\, \Pi_h u_h = \overline{u}_h \Big\} \\ \geq \sup \Big\{ - \int_\Omega \phi^*(\Pi_h z_h) \,\mathrm{d}x - (\overline{u}_h, \operatorname{div} z_h) : z_h \in \mathcal{RT}^0_N(\mathcal{T}_h) \Big\}. \end{aligned}$$

If $\phi \in C^1(\mathbb{R}^d)$ and the infimum is finite then equality holds.

Proof. An integration by parts and Fenchel's inequality show that

$$-(\Pi_h u_h, \operatorname{div} z_h) = (\nabla_h u_h, \Pi_h z_h) \leq \phi(\nabla_h u_h) + \phi^*(\Pi_h z_h). \tag{5}$$

This implies that the left-hand side is an upper bound for the right-hand side. If ϕ is differentiable $u_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ is optimal in the infimum then we have the optimality condition

$$\int_\Omega \phi'(\nabla_h u_h) \cdot \nabla_h v_h \,\mathrm{d}x = 0$$

for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ with $\Pi_h v_h = 0$. Theorem 3.1 yields that $\phi'(\nabla_h u_h) = \Pi_h z_h$ for some $z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$. This identity implies equality in (5) and hence

$$\int_\Omega \phi(\nabla_h u_h)\,\mathrm{d}x = -\int_\Omega \phi^*(\Pi_h z_h)\,\mathrm{d}x - (\overline{u}_h, \operatorname{div} z_h)$$

so that the asserted equality follows. □

Remark 3.5. *For nondifferentiable functions ϕ, the strong duality relation can be established if there exists a sequence of continuously differentiable functions ϕ_ε such that the corresponding discrete primal and dual problems $I_{h,\varepsilon}$ and $D_{h,\varepsilon}$ are Γ-convergent to I_h and D_h as $\varepsilon \to 0$, respectively. An example is the approximation of $\phi(s) = |s|$ by functions $\phi_\varepsilon(s) = \min\{|s| - \varepsilon/2, |s|^2/(2\varepsilon)\}$ for $\varepsilon > 0$.*

With the conjugation formula we obtain a canonical definition of a discrete dual variational problem.

Corollary 3.6 (Discrete duality)**.** *Assume that $\phi \in C(\mathbb{R}^d)$ is convex and $\psi_h : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is elementwise constant in the first argument and convex with respect to the second argument. For $u_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ define*

$$I_h(u_h) = \int_\Omega \phi(\nabla_h u_h) + \psi_h(x, \Pi_h u_h)\,\mathrm{d}x,$$

$$D_h(z_h) = -\int_\Omega \phi^*(\Pi_h z_h) + \psi_h^*(x, \operatorname{div} z_h)\,\mathrm{d}x.$$

We then have

$$\inf_{u_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} I_h(u_h) \geq \sup_{z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)} D_h(z_h).$$

Proof. Using the result of Corollary 3.4 and exchanging the order of the extrema we find that

$$\begin{aligned}
\inf_{u_h} I_h(u_h) &\geq \inf_{u_h} \sup_{z_h} -\int_\Omega \phi^*(\Pi_h z_h)\,\mathrm{d}x - (\Pi_h u_h, \operatorname{div} z_h) + \int_\Omega \psi_h(x, \Pi_h u_h)\,\mathrm{d}x \\
&\geq \sup_{z_h} -\int_\Omega \phi^*(\Pi_h z_h)\,\mathrm{d}x + \inf_{u_h} -(\Pi_h u_h, \operatorname{div} z_h) + \int_\Omega \psi_h(x, \Pi_h u_h)\,\mathrm{d}x \\
&= \sup_{z_h} -\int_\Omega \phi^*(\Pi_h z_h)\,\mathrm{d}x - \sup_{u_h} (\Pi_h u_h, \operatorname{div} z_h) - \int_\Omega \psi_h(x, \Pi_h u_h)\,\mathrm{d}x \\
&\geq \sup_{z_h} -\int_\Omega \phi^*(\Pi_h z_h)\,\mathrm{d}x - \int_\Omega \psi_h^*(x, \Pi_h u_h)\,\mathrm{d}x \\
&= \sup_{z_h} D_h(z_h).
\end{aligned}$$

This proves the asserted inequality. □

The fourth implication concerns the postprocessing of solutions of the primal problem to obtain a solution of the dual problem. This also implies a strong discrete duality relation.

Corollary 3.7 (Strong discrete duality)**.** *In addition to the conditions of Corollary 3.6 assume that $\phi \in C^1(\mathbb{R}^d)$ and $\psi_h : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is finite and differentiable with respect to the second argument. If $u_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ is minimal for I_h then the vector field*

$$z_h = \phi'(\nabla_h u_h) + \psi_h'(x, \Pi_h u_h) d^{-1}(1 - \Pi_h)\,\mathrm{id}$$

is maximal for D_h with $I_h(u_h) = D_h(z_h)$.

Proof. The optimal $u_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ solves the optimality condition

$$\int_\Omega \phi'(\nabla_h u_h) \cdot \nabla_h v_h + \psi_h'(\cdot, \Pi_h u_h)\Pi_h v_h \,\mathrm{d}x = 0 \tag{6}$$

for all $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$. By restricting to functions satisfying $\Pi_h v_h = 0$ we deduce with Theorem 3.1 that there exists $z_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ with

$$\Pi_h z_h = \phi'(\nabla_h u_h).$$

The optimality condition (6) implies that $\operatorname{div} z_h = \psi_h'(\cdot, \Pi_h u_h)$. Hence, z_h satisfies the asserted identity. With the resulting Fenchel identities

$$\nabla_h u_h \cdot \Pi_h z_h = \phi(\nabla_h u_h) + \phi^*(\Pi_h z_h),$$
$$\Pi_h u_h \cdot \operatorname{div} z_h = \psi_h(\cdot, \Pi_h u_h) + \psi_h^*(\cdot, \operatorname{div} z_h),$$

and by choosing $v_h = u_h$ in (6) we find that

$$I_h(u_h) = D_h(z_h)$$

which in view of the weak duality relation $\inf_{u_h} I_h(u_h) \ge \sup_{z_h} D_h(z_h)$ implies that z_h is optimal. □

Appendix A. Discrete Poincaré lemma

For completeness we provide a derivation of (1) based on a discrete Poincaré lemma. We say that Γ_D is connnected if its relative interior has at most one connectivity component.

Proposition A.1 (Discrete Poincaré lemma)**.** *Assume that $\Gamma_D = \emptyset$ or $d = 2$ and Γ_D is connected. A vector field $w_h \in \mathcal{L}^0(\mathcal{T}_h)^d$ satisfies $w_h = \nabla_h v_h$ for a function $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ if and only if*

$$\int_\Omega w_h \cdot y_h \,\mathrm{d}x = 0$$

for all $y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ with $\operatorname{div} y_h = 0$.

Proof. If $w_h = \nabla_h v_h$ then the orthogonality relation follows from the integration-by-parts formula (4). Conversely, if w_h is orthogonal to vector fields $y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ with vanishing divergence then we can construct a function v_h by integrating w_h along a path connecting midpoints of sides, i.e., choosing a side S_0 at which some value is assigned to v_h, e.g., $v_h(x_{S_0}) = 0$. If $\Gamma_D \neq \emptyset$ then we choose $S_0 \subset \Gamma_D$. The values at other sides are obtained via

$$v_h(x_S) = v_h(x_{S_0}) + \sum_{j=1}^{J} w_h|_{T_j} \cdot (x_{S_j} - x_{S_{j-1}}),$$

where $(T_j)_{j=1,\dots,J}$ is a chain of (unique) elements connecting $S_0 \subset T_1$ with $S = S_J \subset T_J$ via the shared sides $S_1, \dots, S_{J-1}$, i.e., $T_j \cap T_{j+1} = S_j$ for $j = 1, \dots, J-1$. To see that this is well defined it suffices to show that for every closed path with $S_J = S_0$ the sum equals zero. To verify this we define the Raviart–Thomas vector field

$$y_h = \sum_{j=1}^{J} \frac{(d-1)!}{|S_j|} \psi_{S_j},$$

where we assume that the plus sign in

$$\psi_S(x) = \pm \frac{|S|}{d!|T_\pm|}(z_{S,T_\pm} - x)$$

occurs for ψ_{S_j} occurs on T_{j+1} and the minus sign on T_j. For the element T_j we then have that

$$\begin{aligned}\int_{T_j} w_h \cdot y_h \,\mathrm{d}x &= w_h|_{T_j} \cdot (d-1)! \int_{T_j} \big(|S_{j-1}|^{-1}\psi_{S_{j-1}} + |S_j|^{-1}\psi_{S_j}\big) \,\mathrm{d}x \\ &= w_h|_{T_j} \cdot (d-1)!|T_j|\big(|S_{j-1}|^{-1}\psi_{S_{j-1}}(x_{T_j}) + |S_j|^{-1}\psi_{S_j}(x_{T_j})\big) \\ &= w_h|_{T_j} \cdot d^{-1}\big((z_{S_{j-1}}) - x_{T_j}) - (z_{S_j} - x_{T_j})\big) \\ &= w_h|_{T_j} \cdot (x_{S_j} - x_{S_{j-1}}),\end{aligned}$$

where we used that $(z_{S,T} - x_T) = d(x_T - x_S)$ for $S \subset \partial T$. Moreover, we have

$$\operatorname{div} y_h|_{T_j} = \operatorname{div}\big((d-1)!|S_{j-1}|^{-1}\psi_{S_{j-1}} + (d-1)!|S_j|^{-1}\psi_{S_j}\big) = 0.$$

This implies that $\operatorname{div} y_h = 0$ and hence by the assumed orthogonality

$$\sum_{j=1}^{J} w_h|_{T_j} \cdot (x_{S_j} - x_{S_{j-1}}) = \sum_{j=1}^{J} \int_{T_j} w_h \cdot y_h \,\mathrm{d}x = \int_\Omega w_h \cdot y_h \,\mathrm{d}x = 0$$

for every closed path of elements. Hence, the function v_h is well defined with $\nabla_h v_h = w_h$. If $d = 2$ and Γ_D is connected then by letting $\varphi_z \in C(\overline{\Omega})$ be an elementwise affine nodal basis function associated with an inner node $z \in \mathcal{N}_h \cap \Gamma_D$, i.e., $z = S_1 \cap S_2$ for $S_1, S_2 \in \mathcal{S}_h \cap \Gamma_D$, and choosing $y_h =$

$(\nabla\varphi_z)^\perp \in \mathcal{RT}^0_N(\mathcal{T}_h)$, where $(a_1,a_2)^\perp = (-a_2,a_1)$, it follows that

$$\begin{aligned} 0 &= \int_\Omega \nabla_h v_h \cdot (\nabla\varphi_z)^\perp \,\mathrm{d}x = \int_{S_1\cup S_2} v_h (\nabla\varphi_z)^\perp \cdot n \,\mathrm{d}s \\ &= \pm\big(v_h(x_{S_2}) - v_h(x_{S_1})\big), \end{aligned}$$

i.e., that v_h is constant on Γ_D. Here we used that $(\nabla\varphi_z)^\perp \cdot n$ is the tangential derivative on Γ_D given by $\pm 1/|S_j|$ for $j = 1, 2$. □

To deduce (1) from the proposition we argue as in [CP19] and let $w_h \in \mathcal{L}^0(\mathcal{T}_h)^d$ be orthogonal to $\Pi_h \mathcal{RT}^0_N(\mathcal{T}_h)$ and hence also to $\mathcal{RT}^0_N(\mathcal{T}_h)$. Proposition A.1 implies that $w_h = \nabla_h v_h$ and it remains to show that v_h has the same value at all element midpoints. This follows from

$$0 = \int_\Omega \nabla_h v_h \cdot \psi_S \,\mathrm{d}x = -\int_\Omega v_h \operatorname{div} \psi_S \,\mathrm{d}x = \frac{|S|}{(d-1)!}\big(v_h(x_{T_+}) - v_h(x_{T_-})\big).$$

Hence, $w_h \in \nabla_h\big(\ker \Pi_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)}\big)$. Note that a nontrivial v_h only exists on triangulations that can be partitioned by two colors, e.g., consisting of halved squares with the same diagonal. Conversely, if $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ with $\Pi_h v_h = 0$ then the integration-by-parts formula (4) yields that $\nabla_h v_h$ is orthogonal to $\mathcal{RT}^0_N(\mathcal{T}_h)$ and in particular to $\Pi_h \mathcal{RT}^0_N(\mathcal{T}_h)$.

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